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Boundedness criterion and global solvability for the three-species food chain model with taxis mechanisms

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Abstract. In this paper, we shall investigate a three-species food chain model with taxis mechanisms including prey-taxis and alarm-taxis in a smooth bounded domain $\Omega \subset \mathbb{R}^n (n \geq 1)$ with homogeneous Neumann boundary conditions. More precisely, we first establish the boundedness criterion for a general food chain model with various taxis mechanisms for arbitrary spatial dimensions by using the semigroup estimates and coupled energy estimates. With the boundedness criterion, we prove the global boundedness of the solution with the general functional response functions under some smallness assumptions on the taxis coefficients by using the weighted energy estimates. On the other hand, for some special functional response functions including Beddington–DeAngelis type, ratio-dependent type and Harrison type, we also obtain the global existence of the solution with uniform-in-time bound without any smallness assumptions on the taxis coefficients or initial data.

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1. Introduction and main results

To deeply understand the foundational principles governing ecosystem dynamics, energy transfer, and species interactions, various temporal food chain models have been proposed and studied [13, 16, 17, 24, 25, 29, 30, 36, 38, 43]. It has been shown that the temporal three-species food chain models exhibit rich dynamics such as chaos [16, 24, 30, 32, 34, 51], periodic orbits [31] and bistability [35] and so on. However, compared with the well-known results on the temporal food chain models mentioned above, few results are known for the food chain model with spatial movements, which actually plays an indispensable role for the population species to survive and thrive. As experimental observation [23], the spatial movements not only include the classical random movements (diffusion) but also the directed movements (taxis) such as prey-taxis or alarm-taxis, which refers to the ability of predators to detect and move towards areas of higher prey density. To gain a more comprehensive understanding of species' dispersion and migration patterns, we shall study the following three-species food chain model with spatial movements in a bounded domain $\Omega \subset \mathbb{R}^n$ with homogeneous Neumann boundary conditions

$$\begin{cases} u_t = \Delta u + u(1-u) - b_1 F_1(u,v)v, & x \in \Omega,\ t > 0, \\ v_t = \Delta v - \xi \nabla \cdot (v \nabla u) + F_1(u,v)v - b_2 F_2(v,w)w - \theta_1 v, & x \in \Omega,\ t > 0, \\ w_t = \Delta w - \chi \nabla \cdot [w \nabla \phi(u,v)] + F_2(v,w)w - \theta_2 w, & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), v(x,0) = v_0(x), w(x,0) = w_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where ν is the outward unit normal vector on $\partial\Omega$ and the homogeneous Neumann boundary conditions means that no individuals can cross the boundary. Here u, v, w represent the density of the prey species, primary and top predators respectively. For $i = 1, 2$, the parameters $b_i > 0$ denote consumption rates of the prey and the primary predators, and $\theta_i > 0$ represent the mortality rates of the primary and top predators, respectively. Here $F_i (i = 1, 2)$ are functional response functions (trophic functions), which describe the consumption rate of a predator varies with the density of its prey, and the classical forms include Holling type [18,24,30,31], ratio-dependent type [18], Beddington–DeAngelis type [4,9,34,51], Harrison type [14] and so on. The term $-\xi \nabla \cdot (v \nabla u)$ was used as prey-taxis mechanism [23] to describe the directional movement of primary predators toward prey density gradient. Similarly, the term $-\chi \nabla \cdot [w \nabla \phi(u,v)]$ describes that the top predators move toward to high gradient of the signal produced due to the interaction between the prey and primary predator.

Before presenting our main results, we first recall some relevant results on the system (1.1). If $w \equiv 0$, the system (1.1) becomes the following two species prey-taxis system

$$\begin{cases} u_t = \Delta u + u(1-u) - b_1 F_1(u,v)v, & x \in \Omega,\ t > 0, \\ v_t = \Delta v - \xi \nabla \cdot (v \nabla u) + F_1(u,v)v - \theta_1 v, & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.2}$$

which was proposed to interpret the heterogeneous aggregative patterns due to the area-restricted search strategy by Kareiva and Odell [23]. The solution behaviors of two species prey-taxis system (1.2) such as global boundedness and large time behavior as well as pattern formations have been extensively studied in the recent years (cf. [7,19,20,23,47,48] and references therein). Recently, some interesting results have been extensively established for two species predator-prey system with various taxis mechanisms such as the indirect prey-taxis mechanism [1,42,44], predator-taxis mechanism [49], dual-taxis mechanism [10,41], signal-dependent prey-taxis mechanism [20] and so on. However, to our knowledge, due to the more complex coupled structures than the two species predator-prey systems with various taxis mechanisms, few results are known for the three-species spatial food chain model (1.1) (i.e., $w \neq 0$).

As far as we know, the first result on the system (1.1) was established in [21]. More precisely, by assuming the functional response functions $F_i(i=1,2)$ are Holling type I and top predators move toward to high gradient of the signal produced by the primary predator, that is under the following assumptions

$$F_1(u,v)=u, \quad F_2(v,w)=v \quad \text{and} \quad \phi(u,v)=v, \tag{1.3}$$

the global boundedness and stabilization of solution for the system (1.1) have been established in two-dimensional bounded domains [21]. In fact, under the assumptions (1.3), we can view the system (1.1) as two different two-species predator–prey system and then use the nice entropy estimate found in [40] for the classical chemotaxis system with consumption of chemoattractant and developed for the prey-taxis system [19]. On the other hand, if we add the terms $-\alpha_1 v^2$ and $-\alpha_2 w^2$ in the second equation and third equation of system (1.1) (i.e., there exists intra-specific competition for v and w) respectively, and the functional response functions $F_i(i=1,2)$ and the signal intensity function $\phi(u,v)$ take the following form

$$F_1(u,v)=u, \quad F_2(v,w)=v \quad \text{and} \quad \phi(u,v)=uv, \tag{1.4}$$

the system (1.1) was first proposed in [15] to text the "burglar alarm" hypothesis (c.f. [6]): a prey species renders itself dangerous to a primary predator by generating an alarm call to attract a second predator at higher trophic levels in the food chain that prey on the primary predator. Due to the nonlinearity of the signal intensity function, the entropy inequality used in [21] for the food chain model with linear signal intensity function (i.e., $\phi(u,v)=v$) does not hold anymore. Hence the known results such as global boundedness and stabilization of solutions to the system (1.1) with $\phi(u,v)=uv$ are limited to one dimensional space [15] or two dimensions [11,22] if there exists intra-specific competition for v and w. From the above discussions, we know that the global boundedness and stabilization of solution for the system (1.1) were only established in two-dimensional spaces in the case of $\phi(u,v)=v$ [21] or $\phi(u,v)=uv$ with quadratic decay terms (i.e., intra-specific competition) for v and w [22]. Hence it is natural to ask whether the results are still valid in higher dimensions for the more general functional response functions $F_i(i=1,2)$ and the signal intensity function $\phi(u,v)$. To this end, we shall study the global dynamics for system (1.1) with the functional response functions $F_1(u,v)$ and $F_2(v,w)$, and the signal intensity function $\phi(u,v)$ satisfying the following assumptions:

(H1): $F_1(u,v)\in C^2([0,\infty)\times[0,\infty)), F_1(0,v)=0$, and $F_{1u}(u,v)>0, F_{1v}(u,v) \leq 0$ for all $u,v\geq 0$. Moreover, there exists a constant $K_1>0$ such that $F_1(u,v)\leq K_1(u+1)$.

(H2): $F_2(v,w) \in C^2([0,\infty)\times[0,\infty))$, $F_2(0,w)=0$, and $F_{2v}(v,w)>0$, $F_{2w}(v,w) \leq 0$ for all $v,w \geq 0$. Moreover, there exists a constant $K_2>0$ such that $F_2(v,w) \leq K_2(v+1)$.

(H3): $\phi(u,v) \in C^2([0,\infty)\times[0,\infty))$ and $\phi(u,v) \geq 0$ for all $u,v \geq 0$.

Then under the assumptions (H1)-(H3), the existence and uniqueness of local solution of (1.1) can be readily proved by Amann's theorem [2,3]. We omit the details of the proof for simplicity.

Lemma 1.1. (Local existence) *Let $\Omega \subset \mathbb{R}^n (n \geq 1)$ be a bounded domain with smooth boundary and suppose (H1)–(H3) hold. Assume $(u_0,v_0,w_0) \in [W^{2,\infty}(\Omega)]^3$ with $u_0,v_0,w_0 \gneqq 0$, then there exists a $T_{max} \in (0,\infty]$ such that the system (1.1) has a unique classical solution*

$$(u,v,w) \in [C^0(\bar{\Omega}\times[0,T_{max})) \cap C^{2,1}(\bar{\Omega}\times(0,T_{max}))]^3$$

satisfying $u,v,w>0$ for all $t>0$. Moreover, it holds that

$$\text{either } T_{max}=\infty \text{ or } \lim_{t\to T_{max}} (\|u(\cdot,t)\|_{W^{1,\infty}} + \|v(\cdot,t)\|_{W^{1,\infty}} + \|w(\cdot,t)\|_{L^\infty}) = \infty.$$

Remark 1.2. The standard method allows us to consider $(u_0,v_0,w_0) \in W^{1,p}(\Omega)$ with $p>\frac{n}{2}$, we do not pursue the sharpest result in terms of the class of initial data.

From Lemma 1.1, we know that the global boundedness of the classical solution exist if there exists a constant $C>0$ such that

$$\|u(\cdot,t)\|_{W^{1,\infty}} + \|v(\cdot,t)\|_{W^{1,\infty}} + \|w(\cdot,t)\|_{L^\infty} \leq C. \tag{1.5}$$

However, the condition (1.5) is hard to verify directly (i.e., see [21,22]). Hence we want to know whether or not there exists more explicit and concise boundedness criterion to ensure the existence of global classical solution in any spatial dimensions. To this end, we establish our first result on the boundedness criterion for the system (1.1) under the assumptions (H1)–(H3) as follows.

Proposition 1.3. (Boundedness criterion) *Let (u,v,w) be the solution of (1.1) obtained in Lemma 1.1. Suppose that there exist $p_0>\frac{n}{2}$ and a constant $M_0>0$ independent of t such that*

$$\sup_{t\in(0,T_{max})} \|v(\cdot,t)\|_{L^{p_0}} + \sup_{t\in(0,T_{max})} \|w(\cdot,t)\|_{L^{p_0}} \leq M_0,$$

then one can find a constant $C>0$ independent of t such that

$$\|u(\cdot,t)\|_{W^{1,\infty}} + \|v(\cdot,t)\|_{W^{1,\infty}} + \|w(\cdot,t)\|_{L^\infty} \leq C \quad \text{for all} \quad t \in (0,T_{max}).$$

Remark 1.4. The result in Proposition 1.3 implies that the global boundedness of the solution for the system (1.1) with assumptions (H1)–(H3) can be ensured if we can find a $p_0>\frac{n}{2}$ such that

$$\|v(\cdot,t)\|_{L^{p_0}} + \|w(\cdot,t)\|_{L^{p_0}} \leq C, \quad \text{for all} \quad t \in (0,T_{max}), \tag{1.6}$$

where $C>0$ is a constant independent of t. Indeed, the condition (1.6) is easier to verify than (1.5).

Remark 1.5. For the system (1.1) with the assumptions (H1)–(H3), we can easily check that there exists a constant $C > 0$ independent of t such that $\|v(\cdot,t)\|_{L^1} + \|w(\cdot,t)\|_{L^1} \leq C$, see Lemma 2.1. Hence using the boundedness criterion in Theorem (1.3), we can derive the global existence of solution for the system (1.1) with uniform-in-time bound in one-dimensional space directly. For the higher dimensions ($n \geq 2$), we need more regularity of u and v, which, however, is not easily obtained due to the complex coupled structure. To our knowledge, we can obtain (1.6) with $p_0 > 1$ in two-dimensional spaces (see [21,22]) only under some special functional response functions $F_i (i = 1, 2)$ and signal intensity function $\phi(u, v)$, which motivates us to study the global boundedness of solution for system (1.1) in higher dimensional spaces ($n \geq 2$) for general functional response functions $F_i (i = 1, 2)$ and signal intensity function $\phi(u, v)$.

With the above boundedness criterion, we shall establish the global boundedness of solution of the system (1.1) in higher-dimensional space ($n \geq 2$) as follows.

Theorem 1.6. (Global boundedness) *Let $\Omega \subset \mathbb{R}^n (n \geq 2)$ be a bounded domain with smooth boundary. Assume that $0 \lneqq (u_0, v_0, w_0) \in [W^{2,\infty}(\Omega)]^3$ and (H1)–(H3) hold. Suppose one of the following conditions holds:*

(1): *The parameters ξ and χ are small such that*

$$\xi \leq \frac{1}{(n+2)M_1} \quad \textit{where} \quad M_1 := \max\{1, \|u_0\|_{L^\infty}\} \tag{1.7}$$

and

$$\chi \|\phi_v\|_{L^\infty} \leq \frac{1}{(n+2)M_2} \quad \textit{where} \quad M_2 := C(\xi^\delta + 1). \tag{1.8}$$

Here $C > 0$ and $\delta > 1$ are two constants independent of ξ, χ and t.

(2): *There exist two positive constants μ_1 and μ_2 such that*

$$F_1(u, v)v \leq \mu_1 u \quad \text{and} \quad F_2(v, w)w \leq \mu_2 v. \tag{1.9}$$

Then the problem (1.1) has a unique global classical solution $(u, v, w) \in [C^0(\bar{\Omega} \times [0, \infty)) \cap C^{2,1}(\bar{\Omega} \times (0, \infty))]^3$ satisfying $u, v, w > 0$ for all $t > 0$. Moreover, there exists a constant $C > 0$ independent of t such that

$$\|u(\cdot,t)\|_{W^{1,\infty}} + \|v(\cdot,t)\|_{W^{1,\infty}} + \|w(\cdot,t)\|_{L^\infty} \leq C. \tag{1.10}$$

Remark 1.7. Below we give some remarks on the results obtained in Theorem 1.6.

- Applying the comparison principle to the first equation of (1.1), we have
$$\|u(\cdot,t)\|_{L^\infty} \leq M_1 := \max\{1, \|u_0\|_{L^\infty}\}.$$
Moreover, under the condition (1.7), we can derive that there exists a positive constant M_2 such that
$$\|v(\cdot,t)\|_{L^\infty} \leq M_2 := C(\xi^\delta + 1)$$
where $C > 0$ and $\delta > 1$ are constants independent of ξ and χ. Since $\phi(u, v) \in C^2([0, \infty) \times [0, \infty))$ and the boundedness of u, v, which are

independent of χ, we can find a constant $\eta_0 > 0$ independent of χ such that

$$\|\phi_v\|_{L^\infty} \leq \eta_0.$$

Then (1.8) can be satisfied if

$$\chi \leq \frac{1}{\eta_0(n+2)M_2}.$$

- The conditions (1.9) can be satisfied by various types of functional response functions $F_i (i = 1, 2)$ such as Beddington–DeAngelis type [4,9, 34,51], ratio-dependent type [18] and Harrison type [14] and so on.

At last, we give some applications of the results obtained in Proposition 1.3 and Theorem 1.6. The first example for the application of our results is the food chain model with alarm-taxis and Holling type I functional response function, that is the system (1.1) with (1.4). More precisely, we have the following results.

Proposition 1.8. *Let $\Omega \subset \mathbb{R}^n (n \geq 2)$ be a bounded domain with smooth boundary. Assume that $0 \lneqq (u_0, v_0, w_0) \in [W^{2,\infty}(\Omega)]^3$ and the parameters ξ and χ satisfy the following conditions*

$$\xi \leq \frac{1}{(n+2)M_1} \quad \text{and} \quad \chi \leq \frac{1}{(n+2)M_1 M_2},$$

with M_1 and M_2 defined as in Theorem 1.6. Then the system (1.1) with (1.4) has a unique global classical solution $(u, v, w) \in [C^0(\bar{\Omega} \times [0,\infty)) \cap C^{2,1}(\bar{\Omega} \times (0,\infty))]^3$ satisfying $u, v, w > 0$ for all $t > 0$ with uniform-in-time bound in the sense of (1.10).

Remark 1.9. Our results in proposition 1.8 imply that the boundedness of the solution for the system (1.1) with (1.4) can be established in any dimensional space with some smallness assumptions on the taxis coefficients ξ and χ. As far as we know, the global existence of classical solution for the system (1.1) with (1.4) for large ξ and χ is still open even in two-dimensional spaces.

The second example will be discussed is the system (1.1) with functional response functions $F_1(u, v)$ and $F_2(v, w)$, and the signal intensity function $\phi(u, v)$ taking the following form

$$F_1(u,v) = \frac{u}{m_1 + v}, \quad F_2(v,w) = \frac{v}{m_2 + w} \quad \text{and} \quad \phi(u,v) = uv, \tag{1.11}$$

where m_1 and m_2 are two positive constants. The system (1.1) with (1.11) can be rewritten as follows

$$\begin{cases} u_t = \Delta u + u(1-u) - \frac{b_1 uv}{m_1+v}, & x \in \Omega,\ t > 0, \\ v_t = \Delta v - \xi \nabla \cdot (v \nabla u) + \frac{uv}{m_1+v} - \frac{b_2 vw}{m_2+w} - \theta_1 v, & x \in \Omega,\ t > 0, \\ w_t = \Delta w - \chi \nabla \cdot [w \nabla (uv)] + \frac{vw}{m_2+w} - \theta_2 w, & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), v(x,0) = v_0(x), w(x,0) = w_0(x), & x \in \Omega. \end{cases} \tag{1.12}$$

For the system (1.12) without spatial movement, the conditions for existence and stability of extinction and coexistence equilibrium states are determined in [5]. When $\chi = \xi = 0$, the global dynamics of solution for the system (1.12) have been established in [27]. Recently, by using the semigroup estimates, the authors in [50] proved the global bounded classical solution of (1.12) with $\xi = 0$ in all dimensions. In fact, we notice that the functional response functions $F_1(u,v)$ and $F_2(v,w)$ satisfy

$$F_1(u,v)v = \frac{uv}{m_1+v} \le u \text{ and } F_1(v,w)w = \frac{vw}{m_2+w} \le v.$$

Then the conditions (1.9) are satisfied and hence we can directly obtain the global boundedness of the solution for the system (1.12) as follows.

Proposition 1.10. *Let $\Omega \subset \mathbb{R}^n (n \ge 2)$ be a bounded domain with smooth boundary. Assume that $0 \lneqq (u_0, v_0, w_0) \in [W^{2,\infty}(\Omega)]^3$. Then the system (1.1) with (1.12) has a unique non-negative global classical solution $(u,v,w) \in [C^0(\bar\Omega \times [0,\infty)) \cap C^{2,1}(\bar\Omega \times (0,\infty))]^3$ satisfying $u, v, w > 0$ for all $t > 0$ with uniform-in-time bound.*

Remark 1.11. The results in Proposition 1.10 not only cover the results the global boundedness results of the system (1.12) in reference [50], but also extend these results to the case of $\xi > 0$.

2. Preliminaries and basic lemmas

In the following, we shall abbreviate $\int_\Omega f dx$ as $\int_\Omega f$ for simplicity without confusion. Moreover, we will use c_i and $M_i (i = 1, 2, ...)$ to denote generic positive constants independent of t which may vary in this paper. In this section, we first give some basic estimates and lemmas that will be used later.

Lemma 2.1. *Let (u,v,w) be the solution obtained in Lemma 1.1. Then it holds that*

$$\|u(\cdot,t)\|_{L^\infty} \le M_1 := \max\{1, \|u_0\|_{L^\infty}\} \quad \text{for all} \quad t \in (0, T_{max}), \tag{2.1}$$

and

$$\|v(\cdot,t)\|_{L^1} + \|w(\cdot,t)\|_{L^1} \le M_3 \quad \text{for all} \quad t \in (0, T_{max}), \tag{2.2}$$

where $M_3 > 0$ is a positive constant independent of ξ, χ and t.

Proof. Applying the comparison principle to the first equation of (1.1), we can derive (2.1) directly.

From the equations of (1.1), one can derive that

$$\frac{d}{dt}\int_\Omega (u + b_1 v + b_1 b_2 w) = \int_\Omega u(1-u) - b_1\theta_1 \int_\Omega v - b_1 b_2 \theta_2 \int_\Omega w,$$

and hence

$$\begin{aligned}\frac{d}{dt}\int_\Omega (u + b_1 v + b_1 b_2 w) + \int_\Omega u + b_1\theta_1 \int_\Omega v + b_1 b_2 \theta_2 \int_\Omega w \\ = \int_\Omega 2u - \int_\Omega u^2 = -\int_\Omega (u-1)^2 + |\Omega| \le |\Omega|,\end{aligned}$$

which, combined with Grönwall's inequality, gives (2.2). □

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, and $z \in C^2(\bar{\Omega})$ satisfy $\frac{\partial z}{\partial \nu} = 0$ on $\partial\Omega$, where ν is the outward unit normal vector on $\partial\Omega$. Then we have*

$$\frac{\partial |\nabla z|^2}{\partial \nu} \leq 2\kappa_1 |\nabla z|^2, \tag{2.3}$$

and

$$\int_\Omega |\nabla z|^{2(p+1)} \leq \kappa_2 \|z\|_{L^\infty}^2 \int_\Omega |\nabla z|^{2(p-1)} |D^2 z|^2 \quad \text{for } p \in [1, \infty), \tag{2.4}$$

where $\kappa_1 = \kappa_1(\Omega)$ is an upper bound of the curvatures of $\partial\Omega$ and $\kappa_2 = 2(n + 4p^2)$.

Proof. The proof of (2.3) can be found in [33]. The estimate (2.4) has been proved in [26]. □

Lemma 2.3. [44] *Let $T > 0$ and $\tau = \min\{1, \frac{T}{2}\}$. Suppose that the non-negative functions $y \in C([0,T)) \cap C^1((0,T))$ and $f \in L^1_{loc}([0,T))$ satisfy*

$$y'(t) + ay(t) \leq f(t), \quad t \in (0,T),$$

and

$$\int_t^{t+\tau} f(s)ds \leq b, \quad t \in (0, T-\tau),$$

then it holds that

$$y(t) \leq \max\left\{y(0) + b, \frac{b}{a} + 2b\right\}, \quad t \in (0,T).$$

Lemma 2.4. [12,46] *Let $e^{t\Delta}(t \geq 0)$ be the Neumann heat semigroup in Ω, and denote $\lambda_1 > 0$ as the first non-zero eigenvalue of $-\Delta$ in Ω under Neumann boundary conditions. Then there exist positive constants γ_i $(i = 1, 2, 3, 4)$ depending only on Ω such that:*

(i) *If $2 \leq p < \infty$, then*

$$\|\nabla e^{t\Delta} z\|_{L^p} \leq \gamma_1 e^{-\lambda_1 t} \|\nabla z\|_{L^p} \quad \text{for all } t > 0 \tag{2.5}$$

holds for all $z \in W^{1,p}(\Omega)$.

(ii) *If $1 \leq q \leq p \leq \infty$, then*

$$\|\nabla e^{t\Delta} z\|_{L^p} \leq \gamma_2 \left(1 + t^{-\frac{1}{2} - \frac{n}{2}(\frac{1}{q} - \frac{1}{p})}\right) e^{-\lambda_1 t} \|z\|_{L^q} \quad \text{for all } t > 0 \tag{2.6}$$

holds for all $z \in L^q(\Omega)$.

(iii) *If $1 \leq q \leq p \leq \infty$, then*

$$\|e^{t\Delta} z\|_{L^p} \leq \gamma_3 \left(1 + t^{-\frac{n}{2}(\frac{1}{q} - \frac{1}{p})}\right) \|z\|_{L^q} \quad \text{for all } t > 0 \tag{2.7}$$

holds for all $z \in L^q(\Omega)$.

(iv) *If $1 < q \leq p \leq \infty$, then*

$$\|e^{t\Delta} \nabla \cdot z\|_{L^p} \leq \gamma_4 \left(1 + t^{-\frac{1}{2} - \frac{n}{2}(\frac{1}{q} - \frac{1}{p})}\right) e^{-\lambda_1 t} \|z\|_{L^q} \quad \text{for all } t > 0 \tag{2.8}$$

holds for all $z \in (C_0^\infty(\Omega))^n$.

We emphasize that the result in Lemma 2.4(iv) is also applicable to any $z \in L^q(\Omega)$ with $1 \le q < \infty$, because $C_0^\infty(\Omega)$ is dense in $L^q(\Omega)(1 \le q < \infty)$ (see also [46]).

3. Boundedness criterion: proof of Proposition 1.3

In this section, we are devoted to establishing the boundedness criterion of for the system (1.1) in any dimensional space. To this end, we first utilize the Neumann semigroup theory motivated by [19, Lemma 3.1] to establish the boundedness of $\|v(\cdot,t)\|_{L^\infty}$ provided the boundedness of $\|v(\cdot,t)\|_{L^{p_0}}$ for some $p_0 > \frac{n}{2}$. For the two species predator–prey system (1.2), the boundedness of $\|v(\cdot,t)\|_{L^\infty}$ is enough to ensure the existence of global classical solution. However, for the three-species food chain model, we need further estimates on the boundedness of $\|w(\cdot,t)\|_{L^\infty}$, which is not easily to obtain. In fact, since the appearance of cross-diffusion term $-\chi\nabla\cdot[w\nabla\phi(u,v)]$ in the third equation, to derive the L^∞-bound of w, we need some estimates on ∇v, whose estimates however depends on the estimates of w itself. To overcome this problem, we first establish coupling energy estimates of $\|w(\cdot,t)\|_{L^p} + \|\nabla v(\cdot,t)\|_{L^{2p}}$ provided the boundedness of $\|w(\cdot,t)\|_{L^{p_0}}$ with $p_0 > \frac{n}{2}$, and then apply semigroup estimates to derive boundedness of $\|w(\cdot,t)\|_{L^\infty}$. Then the boundedness criterion for system (1.1) follows.

3.1. Boundedness of $\|v(\cdot,t)\|_{L^\infty}$

In this subsection, we first show that the boundedness of $\|v(\cdot,t)\|_{L^\infty}$ can be established provided the boundedness of $\|v(\cdot,t)\|_{L^{p_0}}$ for some $p_0 > \frac{n}{2}$.

Lemma 3.1. *Let (u,v,w) be the solution of (1.1) obtained in Lemma 1.1. If there exist constants $p_0 > \frac{n}{2}$ and $C_0 > 0$ independent of ξ and t such that*

$$\sup_{t\in(0,T_{max})} \|v(\cdot,t)\|_{L^{p_0}} \le C_0, \tag{3.1}$$

then we can find two constants $C_1 > 0$ and $\delta > 1$ independent of ξ, χ and t such that

$$\|v(\cdot,t)\|_{L^\infty} \le M_2 := C_1(\xi^\delta + 1). \tag{3.2}$$

Moreover, it holds that

$$\|\nabla u(\cdot,t)\|_{L^\infty} \le M_4, \tag{3.3}$$

where $M_4 > 0$ is a constant depending on ξ but independent of χ and t.

Proof. If $\|v(\cdot,t)\|_{L^{p_0}} \le C_0$, we claim that there exists a constant $c_1 > 0$ such that

$$\|\nabla u(\cdot,t)\|_{L^r} \le c_1, \quad \text{for all } t \in (0,T_{max}), \tag{3.4}$$

with

$$r \in \begin{cases} [1, \frac{np_0}{n-p_0}), & \text{if } p_0 \le n, \\ [1,\infty], & \text{if } p_0 > n. \end{cases} \tag{3.5}$$

In fact, applying the variation-of-constants formula to the first equation of (1.1), we have

$$u(\cdot,t)=e^{(\Delta-1)t}u_0+\int_0^t e^{(\Delta-1)(t-s)}u(2-u)ds-b_1\int_0^t e^{(\Delta-1)(t-s)}F_1(u,v)vds,$$

and hence

$$\begin{aligned}\|\nabla u(\cdot,t)\|_{L^r}&\le\|\nabla e^{(\Delta-1)t}u_0\|_{L^r}+\int_0^t\|\nabla e^{(\Delta-1)(t-s)}u(2-u)\|_{L^r}ds\\&\quad+b_1\int_0^t\|\nabla e^{(\Delta-1)(t-s)}F_1(u,v)v\|_{L^r}ds\\&=I_1+I_2+I_3.\end{aligned}\tag{3.6}$$

Then using (2.5) and $u_0\in W^{2,\infty}(\Omega)$, we can derive that

$$I_1=\|\nabla e^{(\Delta-1)t}u_0\|_{L^r}\le\gamma_1e^{-(\lambda_1+1)t}\|\nabla u_0\|_{L^r}\le c_2.\tag{3.7}$$

Noting the fact $\|u(\cdot,t)\|_{L^\infty}\le M_1$ in Lemma 2.1 and using (2.6), we can estimate I_2 as follows:

$$\begin{aligned}I_2&=\int_0^t\|\nabla e^{(\Delta-1)(t-s)}u(2-u)\|_{L^r}ds\\&\le\gamma_2\int_0^t\left(1+(t-s)^{-\frac12+\frac{n}{2r}}\right)e^{-(\lambda_1+1)(t-s)}\|u(2-u)\|_{L^\infty}ds\\&\le\gamma_2M_1(2+M_1)\int_0^\infty\left(1+(t-s)^{-\frac12+\frac{n}{2r}}\right)e^{-(\lambda_1+1)(t-s)}ds\\&\le c_3.\end{aligned}\tag{3.8}$$

Using the properties of $F_1(u,v)$ in assumption (H1) and noting the fact $\|u(\cdot,t)\|_{L^\infty}\le M_1$ (see Lemma 2.1), we can find a positive constant $c_4:=K_1(M_1+1)$ such that

$$0<F_1(u,v)\le c_4.\tag{3.9}$$

Then using (3.1) and (3.9), we can apply the semigroup estimates in (2.6) to derive

$$\begin{aligned}I_3&=b_1\int_0^t\|\nabla e^{(\Delta-1)(t-s)}F_1(u,v)v\|_{L^r}ds\\&\le b_1\gamma_2\int_0^t\left(1+(t-s)^{-\frac12-\frac n2\left(\frac1{p_0}-\frac1r\right)}\right)e^{-(\lambda_1+1)(t-s)}\|F_1(u,v)\|_{L^\infty}\|v\|_{L^{p_0}}ds\\&\le b_1\gamma_2C_0c_4\int_0^\infty\left(1+(t-s)^{-\frac12-\frac n2\left(\frac1{p_0}-\frac1r\right)}\right)e^{-(\lambda_1+1)(t-s)}ds\\&\le\frac{b_1\gamma_2C_0c_4}{\lambda_1+1}\left(1+(\lambda_1+1)^{\frac12+\frac n2\left(\frac1{p_0}-\frac1r\right)}\Gamma\left(\frac12-\frac n2\left(\frac1{p_0}-\frac1r\right)\right)\right)\\&\le c_5,\end{aligned}\tag{3.10}$$

where Γ denotes the Gamma function defined by $\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}dt$.

Then substituting (3.7), (3.8) and (3.10) into (3.6), we obtain (3.4).

We are now in a position to show (3.2). Applying the variation-of-constants formula to the second equation of system (1.1), and noting $F_2(v,w)>0$ from the assumption (H2), it holds that

$$\begin{aligned} v(\cdot,t) &= e^{(\Delta-\theta_1)t}v_0 - \xi\int_0^t e^{(\Delta-\theta_1)(t-s)}\nabla\cdot(v\nabla u)ds + \int_0^t e^{(\Delta-\theta_1)(t-s)}F_1(u,v)vds \\ &\quad - b_2\int_0^t e^{(\Delta-\theta_1)(t-s)}F_2(v,w)wds \\ &\le e^{(\Delta-\theta_1)t}v_0 - \xi\int_0^t e^{(\Delta-\theta_1)(t-s)}\nabla\cdot(v\nabla u)ds + \int_0^t e^{(\Delta-\theta_1)(t-s)}F_1(u,v)vds, \end{aligned} \tag{3.11}$$

which gives

$$\begin{aligned} \|v(\cdot,t)\|_{L^\infty} &\le \|e^{(\Delta-\theta_1)t}v_0\|_{L^\infty} + \xi\int_0^t \|e^{(\Delta-\theta_1)(t-s)}\nabla\cdot(v\nabla u)\|_{L^\infty}ds \\ &\quad + \int_0^t \|e^{(\Delta-\theta_1)(t-s)}F_1(u,v)v\|_{L^\infty}ds. \end{aligned} \tag{3.12}$$

For each fixed $T\in(0,T_{max})$, if we let

$$\mathcal{M}(T) := \sup_{t\in(0,T)}\|v(\cdot,t)\|_{L^\infty}, \tag{3.13}$$

then $\mathcal{M}(T)$ is finite due to local existence of the solution. Next, we shall estimate $\mathcal{M}(T)$ from (3.12). Without loss of generality, we assume that $\frac{n}{2}<p_0\le n$ and hence $n<\frac{np_0}{n-p_0}$, which entails us to find a $n<r<\frac{np_0}{n-p_0}$ such that (3.4) holds. Moreover, using the fact $n<r<\frac{np_0}{n-p_0}$, we can fix $n<q<r$ such that

$$-\frac{1}{2}-\frac{n}{2q}>-1 \quad\text{and}\quad \delta_0 := 1-\frac{r-q}{rq}\in(0,1). \tag{3.14}$$

Then using the fact $\|v(\cdot,t)\|_{L^1}\le M_3$ in (2.2) and the estimate (3.4), we have

$$\|v\nabla u\|_{L^q} \le \|v\|_{L^{\frac{rq}{r-q}}}\|\nabla u\|_{L^r} \le \|v\|_{L^1}^{\frac{r-q}{rq}}\|v\|_{L^\infty}^{1-\frac{r-q}{rq}}\|\nabla u\|_{L^r} \le c_6M_3^{1-\delta_0}\mathcal{M}^{\delta_0}(T). \tag{3.15}$$

On the other hand, noting (3.1) and (3.9) one has

$$\|F_1(u,v)v\|_{L^{p_0}} \le \|F_1(u,v)\|_{L^\infty}\|v\|_{L^{p_0}} \le C_0c_4. \tag{3.16}$$

With the estimates (3.15) and (3.16) in hand, from (3.14) and (3.12), we can use the semigroup estimates (2.7) and (2.8) to derive

$$
\begin{aligned}
\|v(\cdot,t)\|_{L^\infty} &\le \gamma_3\|v_0\|_{L^\infty} + \gamma_4\xi\int_0^\infty \left(1+(t-s)^{-\frac{1}{2}-\frac{n}{2q}}\right)e^{-(\lambda_1+\theta_1)(t-s)}\|v\nabla u\|_{L^q}ds \\
&\quad + \gamma_3\int_0^\infty \left(1+(t-s)^{-\frac{n}{2p_0}}\right)e^{-\theta_1(t-s)}\|F_1(u,v)v\|_{L^{p_0}}ds \\
&\le \gamma_3\|v_0\|_{L^\infty} \\
&\quad + \frac{\gamma_4 c_6\xi M_3^{1-\delta_0}}{\lambda_1+\theta_1}\mathcal{M}^{\delta_0}(T)\left(1+(\lambda_1+\theta_1)^{\frac{1}{2}+\frac{n}{2q}}\Gamma\left(\frac{1}{2}-\frac{n}{2q}\right)\right) \\
&\quad + \frac{\gamma_3 C_0 c_4}{\theta_1}\left(1+\theta_1^{\frac{n}{2p_0}}\Gamma\left(1-\frac{n}{2p_0}\right)\right) \\
&\le c_7\xi\mathcal{M}^{\delta_0}(T)+c_8.
\end{aligned}
\tag{3.17}
$$

Then noting the definition of $\mathcal{M}(T)$ in (3.13), from (3.17), one can obtain

$$
\mathcal{M}(T)\le c_7\xi\mathcal{M}^{\delta_0}(T)+c_8,
$$

which gives

$$
\mathcal{M}(T)\le 2(1-\delta_0)(2\delta_0)^{\frac{\delta_0}{1-\delta_0}}(c_7\xi)^{\frac{1}{1-\delta_0}}+2c_8\le c_9(1+\xi^{\frac{1}{1-\delta_0}})
$$

and then (3.2) follows. Moreover, using (3.2), we can obtain (3.3) directly from (3.4) and (3.5). Then we complete the proof of this lemma. □

3.2. Coupled energy estimates: $\|w(\cdot,t)\|_{L^p}+\|\nabla v(\cdot,t)\|_{L^{2p}}$

Next, we shall improve the regularity of v to establish the boundedness of $\|w(\cdot,t)\|_{L^\infty}$. To this end, we first establish the following coupled energy estimates.

Lemma 3.2. *Assume the conditions in Lemma 3.1 hold. Then it holds that*

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega\left(w^p+|\nabla v|^{2p}\right)+M_5\int_\Omega(w^p+|\nabla v|^{2p})+\frac{2(p-1)}{p}\int_\Omega|\nabla w^{\frac{p}{2}}|^2 \\
&\quad\le M_6\left(\int_\Omega|\Delta u|^{p+1}+\int_\Omega w^{p+1}+1\right),
\end{aligned}
\tag{3.18}
$$

where $M_5=p\min\{\theta_1,\theta_2\}$ and $M_6>0$ is a constant depending on ξ and χ but independent of t.

Proof. Using the second equation of (1.1) and the integration by parts, we can derive that

$$
\begin{aligned}
\frac{1}{2p}\frac{d}{dt}\int_\Omega|\nabla v|^{2p} &= \int_\Omega|\nabla v|^{2p-2}\nabla v\cdot\nabla v_t \\
&= \int_\Omega|\nabla v|^{2p-2}\nabla v\cdot\nabla[\Delta v-\nabla\cdot(\xi v\nabla u) \\
&\quad + F_1(u,v)v-b_2F_2(v,w)w-\theta_1 v] \\
&= \int_\Omega|\nabla v|^{2p-2}\nabla v\cdot\nabla\Delta v+\xi\int_\Omega\nabla\cdot(|\nabla v|^{2p-2}\nabla v)\nabla\cdot(v\nabla u) \\
&\quad + \int_\Omega|\nabla v|^{2p-2}\nabla v\cdot\nabla[F_1(u,v)v] \\
&\quad - b_2\int_\Omega|\nabla v|^{2p-2}\nabla v\cdot\nabla[F_2(v,w)w]-\theta_1\int_\Omega|\nabla v|^{2p},
\end{aligned}
$$

which, together with the fact $\nabla v\cdot\nabla\Delta v=\frac{1}{2}\Delta|\nabla v|^2-|D^2v|^2$, gives

$$\begin{aligned}&\frac{1}{2p}\frac{d}{dt}\int_\Omega|\nabla v|^{2p}+\frac{p-1}{2}\int_\Omega|\nabla v|^{2(p-2)}|\nabla|\nabla v|^2|^2\\&\quad+\int_\Omega|\nabla v|^{2(p-1)}|D^2v|^2+\theta_1\int_\Omega|\nabla v|^{2p}\\&=\frac{1}{2}\int_{\partial\Omega}|\nabla v|^{2(p-1)}\frac{\partial|\nabla v|^2}{\partial\nu}+\xi\int_\Omega\nabla\cdot(|\nabla v|^{2p-2}\nabla v)\nabla\cdot(v\nabla u)\\&\quad+\int_\Omega|\nabla v|^{2p-2}\nabla v\cdot\nabla[F_1(u,v)v]-b_2\int_\Omega|\nabla v|^{2p-2}\nabla v\cdot\nabla[F_2(v,w)w]\\&=J_1+J_2+J_3+J_4.\end{aligned}\tag{3.19}$$

Using (2.3) and the following trace inequality (see [37, Remark 52.9])

$$\|z\|_{L^2(\partial\Omega)}\le\varepsilon\|\nabla z\|_{L^2(\Omega)}+C_\varepsilon\|z\|_{L^2(\Omega)}\quad\text{for any}\quad\varepsilon>0.$$

We first estimate the term J_1 as follows:

$$\begin{aligned}J_1=\frac{1}{2}\int_{\partial\Omega}|\nabla v|^{2(p-1)}\frac{\partial|\nabla v|^2}{\partial\nu}&\le\kappa_1\int_{\partial\Omega}|\nabla v|^{2p}\\&\le\frac{p-1}{16}\int_\Omega|\nabla v|^{2(p-2)}|\nabla|\nabla v|^2|^2+c_1\int_\Omega|\nabla v|^{2p}.\end{aligned}\tag{3.20}$$

Using the facts $\|v(\cdot,t)\|_{L^\infty}\le M_2$ and $\|\nabla u(\cdot,t)\|_{L^\infty}\le M_4$, and noting $|\Delta v|\le\sqrt{n}|D^2v|$, we can estimate the term J_2 as follows:

$$\begin{aligned}J_2&=\xi\int_\Omega\nabla\cdot(|\nabla v|^{2p-2}\nabla v)\nabla\cdot(v\nabla u)\\&=\xi(p-1)\int_\Omega|\nabla v|^{2(p-2)}\nabla|\nabla v|^2\cdot\nabla v(\nabla v\cdot\nabla u+v\Delta u)\\&\quad+\xi\int_\Omega|\nabla v|^{2p-2}\Delta v(\nabla v\cdot\nabla u+v\Delta u)\\&\le\xi(p-1)M_4\int_\Omega|\nabla v|^{2(p-1)}|\nabla|\nabla v|^2|\\&\quad+\xi(p-1)M_2\int_\Omega|\nabla v|^{2p-3}|\nabla|\nabla v|^2||\Delta u|\\&\quad+\xi M_4\int_\Omega|\nabla v|^{2p-1}|\Delta v|+\xi M_2\int_\Omega|\nabla v|^{2p-2}|\Delta u||\Delta v|\\&\le\frac{p-1}{8}\int_\Omega|\nabla v|^{2(p-2)}|\nabla|\nabla v|^2|^2+c_2\int_\Omega|\nabla v|^{2p}\\&\quad+\frac{1}{8}\int_\Omega|\nabla v|^{2(p-1)}|D^2v|^2+c_3\int_\Omega|\nabla v|^{2(p-1)}|\Delta u|^2,\end{aligned}\tag{3.21}$$

where $c_2:=4(n+p-1)\xi^2M_4^2$ and $c_3:=4(n+p-1)\xi^2M_2^2$.

Using (3.2) and noting the properties of $F_2(v,w)$ in assumption (H2), one has

$$0<F_2(v,w)\le K_2(M_2+1).\tag{3.22}$$

Then noting facts (3.2), (3.9) and (3.22) as well as $|\Delta v| \le \sqrt{n}|D^2 v|$, and using Young's inequality, we estimate the terms J_3 and J_4 as follows:

$$\begin{aligned} J_3 &= \int_\Omega |\nabla v|^{2p-2}\nabla v\cdot\nabla(F_1(u,v)v) \\ &= -\int_\Omega \nabla\cdot(|\nabla v|^{2p-2}\nabla v)F_1(u,v)v \\ &\le (p-1)\int_\Omega |\nabla v|^{2p-3}|\nabla|\nabla v|^2|F_1(u,v)v + \int_\Omega |\nabla v|^{2p-2}|\Delta v|F_1(u,v)v \\ &\le \frac{p-1}{16}\int_\Omega |\nabla v|^{2(p-2)}|\nabla|\nabla v|^2|^2 + \frac{1}{16}\int_\Omega |\nabla v|^{2(p-1)}|D^2 v|^2 + c_4\int_\Omega |\nabla v|^{2(p-1)} \end{aligned} \tag{3.23}$$

and

$$\begin{aligned} J_4 &= -b_2\int_\Omega |\nabla v|^{2p-2}\nabla v\cdot\nabla(F_2(v,w)w) \\ &= b_2\int_\Omega \nabla\cdot(|\nabla v|^{2p-2}\nabla v)F_2(v,w)w \\ &\le b_2(p-1)\int_\Omega |\nabla v|^{2p-3}|\nabla|\nabla v|^2|F_2(v,w)w + b_2\int_\Omega |\nabla v|^{2p-2}|\Delta v|F_2(v,w)w \\ &\le \frac{p-1}{16}\int_\Omega |\nabla v|^{2(p-2)}|\nabla|\nabla v|^2|^2 + \frac{1}{16}\int_\Omega |\nabla v|^{2(p-1)}|D^2 v|^2 + c_5\int_\Omega |\nabla v|^{2(p-1)}w^2. \end{aligned} \tag{3.24}$$

Substituting (3.20), (3.21), (3.23) and (3.24) into (3.19), and using Hölder inequality and Young's inequality, we can derive that

$$\begin{aligned} &\frac{1}{2p}\frac{d}{dt}\int_\Omega |\nabla v|^{2p} + \frac{3(p-1)}{16}\int_\Omega |\nabla v|^{2(p-2)}|\nabla|\nabla v|^2|^2 \\ &\quad + \frac{3}{4}\int_\Omega |\nabla v|^{2(p-1)}|D^2 v|^2 + \theta_1\int_\Omega |\nabla v|^{2p} \\ &\le (c_1+c_2)\int_\Omega |\nabla v|^{2p} + c_3\int_\Omega |\nabla v|^{2(p-1)}|\Delta u|^2 + c_4\int_\Omega |\nabla v|^{2(p-1)} \\ &\quad + c_5\int_\Omega |\nabla v|^{2(p-1)}w^2 \\ &\le (c_1+c_2)\int_\Omega |\nabla v|^{2p} + c_4\int_\Omega |\nabla v|^{2(p-1)} + \frac{1}{4\kappa_2 M_2^2}\int_\Omega |\nabla v|^{2(p+1)} \\ &\quad + c_6\int_\Omega |\Delta u|^{p+1} + c_7\int_\Omega w^{p+1}, \end{aligned} \tag{3.25}$$

where $\kappa_2 = 2(n+4p^2)$ is the constant defined in Lemma 2.2. Then applying Young's inequality and (2.4), and noting $\|v(\cdot,t)\|_{L^\infty} \le M_2$, we can obtain

$$\begin{aligned} &(c_1+c_2)\int_\Omega |\nabla v|^{2p} + c_4\int_\Omega |\nabla v|^{2(p-1)} + \frac{1}{4\kappa_2 M_2^2}\int_\Omega |\nabla v|^{2(p+1)} \\ &\quad \le \frac{1}{2\kappa_2 M_2^2}\int_\Omega |\nabla v|^{2(p+1)} + c_8 \\ &\quad \le \frac{1}{2}\int_\Omega |\nabla v|^{2(p-1)}|D^2 v|^2 + c_8, \end{aligned}$$

which substituted into (3.25) entails that

$$\begin{aligned} &\frac{d}{dt}\int_\Omega |\nabla v|^{2p} + \frac{3p(p-1)}{8}\int_\Omega |\nabla v|^{2(p-2)}|\nabla|\nabla v|^2|^2 \\ &\quad + \frac{p}{2}\int_\Omega |\nabla v|^{2(p-1)}|D^2 v|^2 + 2p\theta_1\int_\Omega |\nabla v|^{2p} \\ &\le 2c_6 p\int_\Omega |\Delta u|^{p+1} + 2c_7 p\int_\Omega w^{p+1} + 2c_8 p. \end{aligned} \tag{3.26}$$

On the other hand, from the third equation of (1.1), we can derive that

$$\begin{aligned} \frac{1}{p}\frac{d}{dt}\int_\Omega w^p &= \int_\Omega w^{p-1}w_t \\ &= \int_\Omega w^{p-1}[\Delta w - \nabla\cdot[\chi w\nabla\phi(u,v)] + F_2(v,w)w - \theta_2 w] \\ &= -(p-1)\int_\Omega w^{p-2}|\nabla w|^2 + \chi(p-1)\int_\Omega \phi_u w^{p-1}\nabla u\cdot\nabla w \\ &\quad + \chi(p-1)\int_\Omega \phi_v w^{p-1}\nabla v\cdot\nabla w + \int_\Omega F_2(v,w)w^p - \theta_2\int_\Omega w^p. \end{aligned} \tag{3.27}$$

Moreover, noting the boundedness of $\|u(\cdot,t)\|_{L^\infty}$ (see Lemma 2.1) and $\|v(\cdot,t)\|_{L^\infty}$(see (3.2)) and using the assumption (H3), we can find a constant $\eta>0$ independent of t such that

$$\|\phi_u\|_{L^\infty} + \|\phi_v\|_{L^\infty} \le \eta. \tag{3.28}$$

Then noting the facts (3.3), (3.22) and (3.28), and using Young's inequality, we can derive from (3.27) that

$$\begin{aligned} &\frac{1}{p}\frac{d}{dt}\int_\Omega w^p + (p-1)\int_\Omega w^{p-2}|\nabla w|^2 + \theta_2\int_\Omega w^p \\ &\quad \le \eta\chi(p-1)M_4\int_\Omega w^{p-1}|\nabla w| + \eta\chi(p-1)\int_\Omega w^{p-1}|\nabla v||\nabla w| + \int_\Omega F_2(v,w)w^p \\ &\quad \le \frac{p-1}{2}\int_\Omega w^{p-2}|\nabla w|^2 + c_9\int_\Omega w^p|\nabla v|^2 + c_{10}\int_\Omega w^p, \end{aligned}$$

which, together with the fact $\int_\Omega w^{p-2}|\nabla w|^2 = \frac{4}{p^2}\int_\Omega |\nabla w^{\frac{p}{2}}|^2$, gives

$$\frac{d}{dt}\int_\Omega w^p + \frac{2(p-1)}{p}\int_\Omega |\nabla w^{\frac{p}{2}}|^2 + \theta_2 p\int_\Omega w^p \le c_9 p\int_\Omega w^p|\nabla v|^2 + c_{10}p\int_\Omega w^p. \tag{3.29}$$

On the other hand, using Young's inequality noting $\|v(\cdot,t)\|_{L^\infty}\le M_2$, we can derive from (2.4) that

$$\begin{aligned} c_9 p\int_\Omega w^p|\nabla v|^2 + c_{10}p\int_\Omega w^p &\le \frac{p}{2\kappa_2 M_2^2}\int_\Omega |\nabla v|^{2(p+1)} + c_{11}\int_\Omega w^{p+1} + c_{12} \\ &\le \frac{p}{2}\int_\Omega |\nabla v|^{2(p-1)}|D^2 v|^2 + c_{11}\int_\Omega w^{p+1} + c_{12}. \end{aligned} \tag{3.30}$$

Then substituting (3.30) into (3.29) and combining (3.26), we can obtain (3.18). □

3.3. Boundedness of $\|w(\cdot,t)\|_{L^\infty}$

In this subsection, we are devoted to establishing the boundedness of $\|w(\cdot,t)\|_{L^\infty}$. To this end, we first improve the regularity of u as follows.

Lemma 3.3. *Let (u,v,w) be the solution obtained in Lemma 1.1, and suppose that (3.1) hold. Then there exists a constant $M_7>0$ depending on ξ but independent of χ and t such that for any $p>1$*

$$\int_t^{t+\tau}\int_\Omega |D^2u|^p \le M_7 \quad \text{for all } \ t\in(0,\widetilde{T}_{max}), \tag{3.31}$$

and

$$\int_\tau^t e^{-p(t-s)}\|\Delta u\|_{L^p}^p \le M_7 \quad \text{for all } \ t\in(\tau,T_{max}), \tag{3.32}$$

where

$$\tau:=\min\left\{1,\frac{T_{\max}}{2}\right\} \text{ and } \widetilde{T}_{\max}=\begin{cases}T_{\max}-\tau, & \text{if } T_{\max}<\infty,\\ \infty, & \text{if } T_{\max}=\infty.\end{cases}$$

Proof. Letting $\mathcal{G}(x,t):=u(2-u)-b_1F_1(u,v)v$, we can rewrite the first equation of (1.1) as follows:

$$u_t-\Delta u+u=\mathcal{G}(x,t).$$

The combination of Lemma 2.1, (3.2) and (3.9) gives

$$\|\mathcal{G}(\cdot,t)\|_{L^\infty}=\|u(2-u)-b_1F_1(u,v)v\|_{L^\infty}\le c_1. \tag{3.33}$$

With $u_0\in W^{2,\infty}(\Omega)$ and (3.33) in hand, we can use the similar arguments as in [28, Lemma 4.2] to derive (3.31) directly. Then the estimate (3.32) is a consequence of the maximal Sobolev regularity property (see [8, Lemma 2.5]). □

Lemma 3.4. *Suppose that (u,v,w) is the solution obtained in Lemma 1.1 and assume that there exist constants $p_0>\frac{n}{2}$ and $M_0>0$ such that*

$$\sup_{t\in(0,T_{max})}\|v(\cdot,t)\|_{L^{p_0}}+\sup_{t\in(0,T_{max})}\|w(\cdot,t)\|_{L^{p_0}}\le M_0,$$

then there exist two positive constants M_8 and M_9, which depend on ξ and χ but are independent of t such that

$$\|\nabla v(\cdot,t)\|_{L^\infty}\le M_8 \quad \text{for all } \ t\in(0,T_{max}), \tag{3.34}$$

and

$$\|w(\cdot,t)\|_{L^\infty}\le M_9 \quad \text{for all } \ t\in(0,T_{max}). \tag{3.35}$$

Proof. Since $\|w(\cdot,t)\|_{L^{p_0}} \le M_0$ with $p_0 > \frac{n}{2}$, then we can use Gagliardo-Nirenberg inequality to find a constant $c_1 > 0$ such that

$$\begin{aligned} M_6\int_\Omega w^{p+1} &= M_6\|w^{\frac{p}{2}}\|^{\frac{2(p+1)}{p}}_{L^{\frac{2(p+1)}{p}}} \\ &\le c_1M_6\Big(\|\nabla w^{\frac{p}{2}}\|^{\frac{2\theta(p+1)}{p}}_{L^2}\|w^{\frac{p}{2}}\|^{\frac{2(1-\theta)(p+1)}{p}}_{L^{\frac{2p_0}{p}}} + \|w^{\frac{p}{2}}\|^{\frac{2(p+1)}{p}}_{L^{\frac{2p_0}{p}}}\Big), \\ &\le \frac{2(p-1)}{p}\int_\Omega|\nabla w^{\frac{p}{2}}|^2 + c_2, \end{aligned} \tag{3.36}$$

where $\theta = \frac{\frac{p}{2p_0} - \frac{p}{2(p+1)}}{\frac{p}{2p_0} - (\frac{1}{2} - \frac{1}{n})} \in (0,1)$ and $\frac{2\theta(p+1)}{p} < 2$. Substituting (3.36) into (3.18), we have

$$\frac{d}{dt}\int_\Omega\Big(w^p + |\nabla v|^{2p}\Big) + M_5\int_\Omega(w^p + |\nabla v|^{2p}) \le M_6\int_\Omega|\Delta u|^{p+1} + c_3. \tag{3.37}$$

Then noting (3.31), we can apply Lemma 2.3 to (3.37) to find a constant $c_4 > 0$ independent of t such for all $p > p_0 > \frac{n}{2}$ that

$$\|w(\cdot,t)\|_{L^p} + \|\nabla v(\cdot,t)\|_{L^{2p}} \le c_4 \quad \text{for all } t \in (0,T_{max}). \tag{3.38}$$

From Lemma (1.1), we know that (3.34) holds for all $t \in (0,\tau]$ with τ is defined in Lemma 3.3. Hence we only need to show that (3.34) holds for all $t \in (\tau, T_{max})$. Applying the variation-of-constants formula to second equation of (1.1), we have

$$\begin{aligned} \nabla v(\cdot,t) &= \nabla e^{(\Delta-1)(t-\tau)}v(\cdot,\tau) - \xi\int_\tau^t\nabla e^{(\Delta-1)(t-s)}\nabla\cdot(v\nabla u)ds \\ &\quad + \int_\tau^t\nabla e^{(\Delta-1)(t-s)}(F_1(u,v) \\ &\quad + 1-\theta_1)vds - b_2\int_\tau^t\nabla e^{(\Delta-1)(t-s)}F_2(v,w)wds. \end{aligned} \tag{3.39}$$

Then noting the L^∞-bound of $v, \nabla u, F_1(u,v)$ and $F_2(v,w)$, from (3.39), we can derive that

$$\begin{aligned} \|\nabla v(\cdot,t)\|_{L^\infty} &\le \|\nabla e^{(\Delta-1)(t-\tau)}v(\cdot,\tau)\|_{L^\infty} + \xi\int_\tau^t\|\nabla e^{(\Delta-1)(t-s)}\nabla\cdot(v\nabla u)\|_{L^\infty}ds \\ &\quad + \int_\tau^t\|\nabla e^{(\Delta-1)(t-s)}(F_1(u,v)+1-\theta_1)v\|_{L^\infty}ds \\ &\quad + b_2\int_\tau^t\|\nabla e^{(\Delta-\theta_1)(t-s)}F_2(v,w)w\|_{L^\infty}ds. \end{aligned} \tag{3.40}$$

Then using the semigroup estimates in Lemma 2.4 to (3.40), it holds that

$$
\begin{aligned}
\|\nabla v(\cdot,t)\|_{L^\infty} &\leq \gamma_2\|v(\cdot,\tau)\|_{L^\infty} \\
&+ \gamma_2\xi\int_\tau^t \left(1+(t-s)^{-\frac{1}{2}-\frac{n}{4p}}\right)e^{-(\lambda_1+1)(t-s)}\|\nabla\cdot(v\nabla u)\|_{L^{2p}}ds \\
&+ \gamma_2 c_5\int_\tau^t \left(1+(t-s)^{-\frac{1}{2}}\right)e^{-(\lambda_1+1)(t-s)}ds \\
&+ \gamma_2 b_2 c_6\int_\tau^t \left(1+(t-s)^{-\frac{1}{2}-\frac{n}{2p}}\right)e^{-(\lambda_1+1)(t-s)}\|w\|_{L^p}ds.
\end{aligned} \tag{3.41}
$$

Choosing $p>n$ in (3.38), and using the boundedness of $\|v(\cdot,t)\|_{L^\infty}$ and $\|\nabla u(\cdot,t)\|_{L^\infty}$, we have

$$
\begin{aligned}
\|\nabla\cdot(v\nabla u)\|_{L^{2p}} = \|\nabla v\cdot\nabla u+v\Delta u\|_{L^{2p}} &\leq c_7\|\nabla v\|_{L^{2p}} \\
&+c_8\|\Delta u\|_{L^{2p}} \leq c_4c_7+c_8\|\Delta u\|_{L^{2p}},
\end{aligned}
$$

and

$$
\int_\tau^t \left(1+(t-s)^{-\frac{1}{2}-\frac{n}{4p}}\right)e^{-(\lambda_1+1)(t-s)} + \int_\tau^t \left(1+(t-s)^{-\frac{1}{2}-\frac{n}{2p}}\right)e^{-(\lambda_1+1)(t-s)} \leq c_9,
$$

which substituted into (3.41) gives

$$
\begin{aligned}
\|\nabla v(\cdot,\tau)\|_{L^\infty} &\leq c_8\gamma_2\xi\int_\tau^t \left(1+(t-s)^{-\frac{1}{2}-\frac{n}{4p}}\right)e^{-(\lambda_1+1)(t-s)}\|\Delta u\|_{L^{2p}}ds \\
&+ \gamma_2\|v(\cdot,t)\|_{L^\infty}+\gamma_2c_5c_9+\gamma_2b_2c_6c_9.
\end{aligned} \tag{3.42}
$$

On the other hand, using Hölder inequality and choosing $p>n$ in (3.32), we can derive that

$$
\begin{aligned}
&\int_\tau^t \left(1+(t-s)^{-\frac{1}{2}-\frac{n}{4p}}\right)e^{-(\lambda_1+1)(t-s)}\|\Delta u\|_{L^{2p}}ds \\
&\leq \left(\int_\tau^t \left(1+(t-s)^{-\frac{1}{2}-\frac{n}{4p}}\right)^{\frac{2p}{2p-1}} e^{-\frac{2p\lambda_1}{2p-1}(t-s)}\right)^{\frac{2p-1}{2p}} \\
&\quad\cdot\left(\int_\tau^t e^{-2p(t-s)}\|\Delta u\|_{L^{2p}}^{2p}ds\right)^{\frac{1}{2p}} \\
&\leq c_{10}.
\end{aligned} \tag{3.43}
$$

Substituting (3.43) into (3.42), we obtain (3.34) directly.

Similarly, we can apply the variation-of-constants formula to the third equation of (1.1) and use the semigroup estimates in Lemma 2.4 to obtain

$$\begin{aligned}\|w(\cdot,t)\|_{L^\infty} &\le \|e^{(\Delta-\theta_2)t}w_0\|_{L^\infty} + \chi\int_0^t \|e^{(\Delta-\theta_2)(t-s)}\nabla\cdot[w\nabla\phi(u,v)]\|_{L^\infty}ds \\ &\quad + \int_0^t \|e^{(\Delta-\theta_2)(t-s)}F_2(v,w)w\|_{L^\infty}ds \\ &\le \gamma_3\|w_0\|_{L^\infty} \\ &\quad + c_{11}\int_0^t (1+(t-s)^{-\frac{1}{2}-\frac{n}{2p}})e^{-(\lambda+\theta_2)(t-s)}\|w\nabla\phi(u,v)\|_{L^p}ds \\ &\quad + c_{12}\int_0^t \left(1+(t-s)^{-\frac{n}{2p}}\right)e^{-\theta_2(t-s)}\|w\|_{L^p}ds,\end{aligned}$$

where $p > n$. Using the facts (3.28), $\|\nabla u(\cdot,t)\|_{L^\infty} \le M_4$ in (3.3) and $\|\nabla v(\cdot,t)\|_{L^\infty} \le M_8$ in (3.34), for $p > n$, we have

$$\begin{aligned}\|w\|_{L^p} + \|w\nabla\phi(u,v)\|_{L^p} &\le (1+\|\nabla\phi(u,v)\|_{L^\infty})\|w\|_{L^p} \\ &\le [1+\eta(M_4+M_8)]\|w\|_{L^p} \le c_{13}.\end{aligned} \tag{3.44}$$

Substituting (3.44) into (3.45), and using the fact $p > n$, we have

$$\begin{aligned}\|w(\cdot,t)\|_{L^\infty} &\le \gamma_3\|w_0\|_{L^\infty} + c_{11}c_{13}\int_0^\infty (1+(t-s)^{-\frac{1}{2}-\frac{n}{2p}})e^{-(\lambda+\theta_2)(t-s)}ds \\ &\quad + c_{12}c_{13}\int_0^\infty \left(1+(t-s)^{-\frac{n}{2p}}\right)e^{-\theta_2(t-s)}ds \\ &\le c_{14},\end{aligned} \tag{3.45}$$

which gives (3.35). □

Proof of Proposition 1.3. Proposition 1.3 is a consequence of the combination of Lemma 3.1–3.4. □

4. Global boundedness: Proof of Theorem 1.6

In this section, we will prove global boundedness of the classical solution as stated in Theorem 1.6. From the boundedness criterion established in Proposition 1.3, to prove the boundedness results in Theorem 1.6, we only need to show the boundedness of $\|v(\cdot,t)\|_{L^p} + \|w(\cdot,t)\|_{L^p}$ for $p > \frac{n}{2}$ based on the weighted energy estimates (see [39,45]) and the semigroup estimates.

4.1. Case I: general functional response functions

In this subsection, we first study the (1.1) with general functional response function satisfying the assumptions (H1) and (H2). Then based on the weighted energy estimates, we shall show the boundedness of $\|v(\cdot,t)\|_{L^p} + \|w(\cdot,t)\|_{L^p}$ with $p > \frac{n}{2}$ under the smallness assumptions on ξ and χ.

Lemma 4.1. *Let the (u, v, w) be the solution obtained in Lemma 1.1 and suppose assumptions (H1)–(H3) hold. If ξ satisfies that*

$$\xi \le \frac{1}{(n+2)M_1} \quad \text{with} \quad M_1 := \max\{1, \|u_0\|_{L^\infty}\}, \tag{4.1}$$

then it holds that

$$\|v(\cdot,t)\|_{L^\infty} \le M_2 \quad \text{for all } \ t\in(0,T_{max}), \tag{4.2}$$

where M_2 is the constant defined in Lemma 3.1.

Proof. Noting that $\|u(\cdot,t)\|_{L^\infty} \le M_1 := \max\{1, \|u_0\|_{L^\infty}\}$ from (2.1), based on some ideas in [39,45], we introduce a weight function as follows

$$\Phi(u) := e^{(\beta u)^2} \quad \text{with} \quad \beta^2 = \frac{p-1}{4pM_1^2}, \tag{4.3}$$

where $p = n+2$ is chosen to ensure the following two relations (will be used later) hold:

$$\xi^2 p^2 < 8\beta^2 \tag{4.4}$$

and

$$\frac{(p-1)\beta^2}{p} > \frac{4\beta^4 u^2}{p} \quad \text{and} \quad \frac{(p-1)\beta^2}{p} \ge \frac{\xi^2(p-1)^2}{4}. \tag{4.5}$$

We can easily check that $\Phi(u)$ satisfies

$$1 \le \Phi(u) \le c_1, \tag{4.6}$$

and

$$0 < \Phi'(u) = 2\beta^2 u\Phi(u) \quad \text{and} \quad 0 < \Phi''(u) = (2\beta^2 + 4\beta^4u^2)\Phi(u). \tag{4.7}$$

From equations of (1.1) and using formula of integration by parts, we can derive that

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\int_\Omega \Phi(u)v^p &= \frac{1}{p}\int_\Omega \Phi'(u)v^p u_t + \int_\Omega \Phi(u)v^{p-1}v_t \\
&= -\frac{1}{p}\int_\Omega \Phi''(u)v^p|\nabla u|^2 - 2\int_\Omega \Phi'(u)v^{p-1}\nabla u\cdot\nabla v \\
&\quad - (p-1)\int_\Omega \Phi(u)v^{p-2}|\nabla v|^2 + \xi\int_\Omega \Phi'(u)v^p|\nabla u|^2 \\
&\quad + \xi(p-1)\int_\Omega \Phi(u)v^{p-1}\nabla u\cdot\nabla v \\
&\quad + \frac{1}{p}\int_\Omega \Phi'(u)uv^p + \int_\Omega \Phi(u)v^pF_1(u,v) \\
&\quad - \frac{1}{p}\int_\Omega \Phi'(u)u^2v^p - \frac{b_1}{p}\int_\Omega \Phi'(u)v^{p+1}F_1(u,v) \\
&\quad - b_2\int_\Omega \Phi(u)v^{p-1}F_2(v,w)w - \theta_1\int_\Omega \Phi(u)v^p,
\end{aligned}$$

which, together with the facts $F_2(v,w)>0$ in assumption (H2), $\Phi(u)>0$ and $\Phi'(u)>0$ in (4.7) as well as $u,v,w>0$, gives

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int_\Omega \Phi(u)v^p \le &-\int_\Omega \left[\frac{1}{p}\Phi''(u)-\xi\Phi'(u)\right]v^p|\nabla u|^2-(p-1)\int_\Omega \Phi(u)v^{p-2}|\nabla v|^2\\ &-\int_\Omega \left[2\Phi'(u)-\xi(p-1)\Phi(u)\right]v^{p-1}\nabla u\cdot\nabla v\\ &+\frac{1}{p}\int_\Omega \Phi'(u)uv^p+\int_\Omega \Phi(u)v^pF_1(u,v)\\ =&-\int_\Omega XAX^T+\frac{1}{p}\int_\Omega \Phi'(u)uv^p+\int_\Omega \Phi(u)v^pF_1(u,v),\end{aligned}\tag{4.8}$$

where $X=(v^{\frac{p}{2}}\nabla u,\ \ v^{\frac{p-2}{2}}\nabla v)$ and

$$A=\begin{pmatrix}\frac{1}{p}\Phi''(u)-\xi\Phi'(u) & \frac{2\Phi'(u)-\xi(p-1)\Phi(u)}{2}\\ \frac{2\Phi'(u)-\xi(p-1)\Phi(u)}{2} & (p-1)\Phi(u)\end{pmatrix}.$$

Next, we shall prove the positive definite of the matrix A. To this end, we can use (4.7) and the fact (4.4) to obtain

$$\begin{aligned}\frac{1}{p}\Phi''(u)-\xi\Phi'(u)&=\left[\frac{1}{p}(2\beta^2+4\beta^4u^2)-2\xi\beta^2u\right]\Phi(u)\\ &=\frac{2\beta^2}{p}\left(2\beta^2u^2-\xi pu+1\right)\Phi(u)>0,\end{aligned}\tag{4.9}$$

where we have used the fact $2\beta^2u^2-\xi pu+1>0$ due to $\xi^2p^2<8\beta^2$ in (4.4). Using (4.6) and (4.7), and noting the fact (4.5), we can verify that

$$\begin{aligned}|A|=&\frac{p-1}{p}\left[\Phi''(u)-p\xi\Phi'(u)\right]\Phi(u)-\frac{[2\Phi'(u)-\xi(p-1)\Phi(u)]^2}{4}\\ =&\frac{p-1}{p}\Phi''(u)\Phi(u)-|\Phi'(u)|^2-\frac{\xi^2(p-1)^2\Phi^2(u)}{4}\\ =&\left[\frac{p-1}{p}(2\beta^2+4\beta^4u^2)-4\beta^4u^2-\frac{\xi^2(p-1)^2}{4}\right]\Phi^2(u)\\ =&\left[\frac{2(p-1)\beta^2}{p}-\frac{4\beta^4u^2}{p}-\frac{\xi^2(p-1)^2}{4}\right]\Phi^2(u)>0.\end{aligned}\tag{4.10}$$

Then the combination of (4.9) and (4.10) implies that the matrix A is positive definite, and thence from (4.8), one can find a constant $c_2>0$ independent of ξ such that

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\int_\Omega \Phi(u)v^p+c_2\int_\Omega v^p|\nabla u|^2+c_2\int_\Omega v^{p-2}|\nabla v|^2\\ &\quad\le\frac{1}{p}\int_\Omega \Phi'(u)uv^p+\int_\Omega \Phi(u)v^pF_1(u,v).\end{aligned}\tag{4.11}$$

On the other hand, we can use (4.6), (4.7) and (3.9) as well as (2.1) to find a positive constant c_3 such that

$$\frac{1}{p}\int_\Omega \Phi'(u)uv^p+\int_\Omega \Phi(u)v^pF_1(u,v)\le c_3\int_\Omega v^p,$$

which substituted into (4.11) yields

$$\frac{1}{p}\frac{d}{dt}\int_\Omega \Phi(u)v^p + c_2\int_\Omega v^{p-2}|\nabla v|^2 \le c_3\int_\Omega v^p. \tag{4.12}$$

Using the fact $1 \le \Phi(u) \le c_1$ in (4.6) and $\|v(\cdot,t)\|_{L^1} \le M_3$ in (2.2), then we can use Gagliardo-Nirenberg inequality to find $c_5 > 0$ such that

$$\begin{aligned} c_3\int_\Omega v^p + \int_\Omega \Phi(u)v^p \le c_4\int_\Omega v^p = c_4\|v^{\frac{p}{2}}\|_{L^2}^2 \\ &\le c_5\left(\|\nabla v^{\frac{p}{2}}\|_{L^2}^{\frac{2n(p-1)}{n(p-1)+2}}\|v^{\frac{p}{2}}\|_{L^{\frac{2}{p}}}^{\frac{4}{n(p-1)+2}} + \|v^{\frac{p}{2}}\|_{L^{\frac{2}{p}}}^2\right) \\ &\le c_2\int_\Omega v^{p-2}|\nabla v|^2 + c_6, \end{aligned}$$

which substituted into (4.12) gives

$$\frac{1}{p}\frac{d}{dt}\int_\Omega \Phi(u)v^p + \int_\Omega \Phi(u)v^p \le c_6. \tag{4.13}$$

Then applying Grönwall's inequality to (4.13), we have $\int_\Omega \Phi(u)v^p \le c_7$, which together with the facts $\Phi(u) \ge 1$ and $p = n+2$ gives

$$\|v(\cdot,t)\|_{L^{n+2}} \le c_8, \tag{4.14}$$

where $c_8 > 0$ is a constant independent of ξ. Then using Lemma 3.1 and noting (4.14), we can obtain (4.2) directly. □

Lemma 4.2. *Let (u, v, w) be the solution obtained in Lemma 1.1 and assume (H1)–(H3) hold. Suppose ξ satisfies (4.1) and χ satisfies*

$$\chi\|\phi_v\|_{L^\infty} \le \frac{1}{(n+2)M_2}, \tag{4.15}$$

we have

$$\|w(\cdot,t)\|_{L^\infty} \le M_9, \quad \text{for all } t \in (0, T_{max}), \tag{4.16}$$

where M_2 is the constant defined in Lemma 3.1 and M_9 is chosen in (3.35).

Proof. Since ξ satisfies (4.1), then from Lemma 4.1, we have $\|v(\cdot,t)\|_{L^\infty} \le M_2$. Letting $p := n+2$, we introduce a weight function

$$\Psi(v) := e^{(\rho v)^2} \quad \text{with} \quad \rho^2 = \frac{p-1}{4pM_2^2}, \tag{4.17}$$

which satisfies

$$1 \le \Psi(v) \le c_1, \tag{4.18}$$

and

$$0 < \Psi'(v) = 2\rho^2 v\Psi(v) \quad \text{and} \quad 0 < \Psi''(v) = (2\rho^2 + 4\rho^4 v^2)\Psi(v). \tag{4.19}$$

Using the second and third equation of (1.1), and noting the fact $\Psi(v) > 0$, after some calculations, we can derive

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int_\Omega \Psi(v)w^p =& \frac{1}{p}\int_\Omega \Psi'(v)w^p v_t + \int_\Omega \Psi(v)w^{p-1}w_t \\ \le& -\int_\Omega YBY^T + \int_\Omega \left[\frac{\xi}{p}\Psi''(v)v + \chi\Psi'(v)\phi_u\right] w^p \nabla u\cdot\nabla v \\ &+ \int_\Omega \left(\xi\Psi'(v)v + \chi(p-1)\Psi(v)\phi_u\right) w^{p-1}\nabla u\cdot\nabla w \\ &+ \frac{1}{p}\int_\Omega \Psi'(v)vw^pF_1(u,v) + \int_\Omega \Psi(v)w^pF_2(v,w),\end{aligned} \tag{4.20}$$

where $Y = (w^{\frac{p}{2}}\nabla v, \quad w^{\frac{p-2}{2}}\nabla w)$ and

$$B = \begin{pmatrix} \frac{1}{p}\Psi''(v) - \chi\phi_v\Psi'(v) & \frac{2\Psi'(u)-\chi(p-1)\phi_v\Psi(u)}{2} \\ \frac{2\Psi'(u)-\chi(p-1)\phi_v\Psi(u)}{2} & (p-1)\Psi(v) \end{pmatrix}. \tag{4.21}$$

Next, we shall prove that the matrix B is positive definite. In fact, noting (4.15) and the definition of ρ in (4.17), we can check that

$$\chi^2\phi_v^2 < \frac{8\rho^2}{p^2}, \tag{4.22}$$

and

$$\frac{(p-1)\rho^2}{p} > \frac{4\rho^4v^2}{p} \quad\text{and}\quad \frac{(p-1)\rho^2}{p} \ge \frac{\chi^2(p-1)^2\phi_v^2}{4}. \tag{4.23}$$

Then using (4.18) and (4.19), and noting (4.22), we can check that

$$\frac{1}{p}\Psi''(v) - \chi\phi_v\Psi'(v) = \frac{2\rho^2}{p}\left(2\rho^2v^2 - p\chi\phi_v v + 1\right)\Psi(v) > 0. \tag{4.24}$$

On the other hand, after some calculations, using (4.19) and (4.23), we can derive from (4.21) that

$$\begin{aligned}|B| =& \frac{p-1}{p}\left[\Psi''(v) - p\chi\phi_v\Psi'(v)\right]\Psi(v) - \frac{[2\Psi'(v) - \chi(p-1)\phi_v\Psi(v)]^2}{4} \\ =& \left[\frac{p-1}{p}(2\rho^2 + 4\rho^4v^2) - 4\rho^4v^2 - \frac{\chi^2(p-1)^2\phi_v^2}{4}\right]\Psi^2(v) \\ =& \left[\frac{2(p-1)\rho^2}{p} - \frac{4\rho^4v^2}{p} - \frac{\chi^2(p-1)^2\phi_v^2}{4}\right]\Psi^2(v) > 0.\end{aligned} \tag{4.25}$$

Then the combination of (4.24) and (4.25) gives the positive definite of the matrix B and hence there exists a constant $c_2 > 0$ such that

$$\int_\Omega YBY^T \ge c_2\int_\Omega w^p|\nabla v|^2 + c_2\int_\Omega w^{p-2}|\nabla w|^2. \tag{4.26}$$

On the other hand, noting the boundedness of $\|v(\cdot,t)\|_{L^\infty}$ and $\|\nabla u(\cdot,t)\|_{L^\infty}$, and using (4.18) and (4.19) as well as the assumptions (H1)–(H3) on $F_i(u,v)(i=1,2)$ and $\phi(u,v)$, we can derive that

$$\int_\Omega \left(\frac{\xi}{p}\Psi''(v)v+\chi\Psi'(v)\phi_u\right)w^p\nabla u\cdot\nabla v\le c_3\int_\Omega w^p|\nabla v| \tag{4.27}$$

and

$$\int_\Omega (\xi\Psi'(v)v+\chi(p-1)\Psi(v)\phi_u)\,w^{p-1}\nabla u\cdot\nabla w\le c_4\int_\Omega w^{p-1}|\nabla w| \tag{4.28}$$

as well as

$$\frac{1}{p}\int_\Omega \Psi'(v)vw^pF_1(u,v)+\int_\Omega \Psi(v)w^pF_2(v,w)\le c_5\int_\Omega w^p. \tag{4.29}$$

Then substituting (4.26)–(4.29) into (4.20), and using Young's inequality, we obtain that

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_\Omega \Psi(v)w^p\\
&\qquad+c_2\int_\Omega w^p|\nabla v|^2+c_2\int_\Omega w^{p-2}|\nabla w|^2\\
&\quad\le c_3\int_\Omega w^p|\nabla v|+c_4\int_\Omega w^{p-1}|\nabla w|+c_5\int_\Omega w^p\\
&\quad\le \frac{c_2}{2}\int_\Omega w^p|\nabla v|^2+\frac{c_2}{2}\int_\Omega w^{p-2}|\nabla w|^2+\left(\frac{c_3^2}{2c_2}+\frac{c_4^2}{2c_2}+c_5\right)\int_\Omega w^p,
\end{aligned}$$

which gives

$$\frac{1}{p}\frac{d}{dt}\int_\Omega \Psi(v)w^p+\frac{c_2}{2}\int_\Omega w^p|\nabla v|^2+\frac{c_2}{2}\int_\Omega w^{p-2}|\nabla w|^2\le c_6\int_\Omega w^p, \tag{4.30}$$

with $c_6:=\frac{c_3^2}{2c_2}+\frac{c_4^2}{2c_2}+c_5$.

At last, using the facts $1\le\Psi(v)\le c_1$ and $\|w(\cdot,t)\|_{L^1}\le M_3$. We can use the Gagliardo-Nirenberg inequality and Young's inequality to derive that

$$\begin{aligned}
c_6\int_\Omega w^p+\int_\Omega \Psi(v)w^p&\le c_7\int_\Omega w^p=c_7\|w^{\frac{p}{2}}\|_{L^2}^2\\
&\le c_8(\|\nabla w^{\frac{p}{2}}\|_{L^2}^{\frac{2n(p-1)}{n(p-1)+2}}\|w^{\frac{p}{2}}\|_{L^{\frac{2}{p}}}^{\frac{4}{n(p-1)+2}}+\|w^{\frac{p}{2}}\|_{L^{\frac{2}{p}}}^2)\\
&\le \frac{c_2}{2}\int_\Omega w^{p-2}|\nabla w|^2+c_9.
\end{aligned}\tag{4.31}$$

The combination of (4.30) and (4.31) gives

$$\frac{1}{p}\frac{d}{dt}\int_\Omega \Psi(v)w^p+\int_\Omega \Psi(v)w^p\le c_9. \tag{4.32}$$

Applying Grönwall's inequality to (4.32), we obtain

$$\int_\Omega w^p\le\int_\Omega \Psi(v)w^p\le c_{10}.$$

Since $p=n+2$, (4.16) follows Lemma 3.4. Then we complete the proof. □

4.2. Case II: special functional response functions

In this subsection, suppose $F_1(u,v)$ and $F_2(v,w)$ satisfy assumptions (H1)–(H3) and (1.9), then we shall establish the boundedness of system (1.1) by using semigroup estimates and the boundedness criterion obtained in Proposition 1.3.

Lemma 4.3. *Let (u,v,w) be the solution obtained in Lemma 1.1, and suppose the assumptions (H1)–(H3) and (1.9) hold. Then we have*

$$\|\nabla u(\cdot,t)\|_{L^\infty} \leq M_{10} \quad \textit{for all} \quad t \in (0,T_{max}), \tag{4.33}$$

and

$$\|v(\cdot,t)\|_{L^\infty} \leq M_{11} \quad \textit{for all} \quad t \in (0,T_{max}), \tag{4.34}$$

where the constant M_{10} is independent of ξ,χ and t, and M_{11} depends on ξ but is independent of χ and t.

Proof. Noting $\|u(\cdot,t)\|_{L^\infty} \leq M_1$ in Lemma 2.1, and using the assumption (H1) and (1.9), we have

$$0 < F_1(u,v)v \leq \mu_1 u \leq \mu_1 M_1 \quad \text{for all} \quad t \in (0,T_{max}). \tag{4.35}$$

Applying the variation-of-constants formula to the first equation of system (1.1), one can derive

$$\begin{aligned}\nabla u(\cdot,t) = \nabla e^{(\Delta-1)t}u_0 + \int_0^t \nabla e^{(\Delta-1)(t-s)}u(2-u)ds \\ -b_1\int_0^t \nabla e^{(\Delta-1)(t-s)}F_1(u,v)vds.\end{aligned} \tag{4.36}$$

Then using the semigroup estimates stated in Lemma 2.4 and (4.35), we can derive from (4.36) that

$$\begin{aligned}\|\nabla u(\cdot,t)\|_{L^\infty} &\leq \|\nabla e^{(\Delta-1)t}u_0\|_{L^\infty} + \int_0^t \|\nabla e^{(\Delta-1)(t-s)}u(2-u)\|_{L^\infty}ds \\ &\quad + b_1\int_0^t \|\nabla e^{(\Delta-1)(t-s)}F_1(u,v)v\|_{L^\infty}ds \\ &\leq \gamma_2\|u_0\|_{L^\infty} + \gamma_2\int_0^t (1+(t-s)^{-\frac{1}{2}})e^{-(\lambda_1+1)(t-s)}\|u(2-u)\|_{L^\infty}ds \\ &\quad + b_1\gamma_2\int_0^t (1+(t-s)^{-\frac{1}{2}})e^{-(\lambda_1+1)(t-s)}\|F_1(u,v)v\|_{L^\infty}ds \\ &\leq \gamma_2\|u_0\|_{L^\infty} \\ &\quad + \gamma_2 M_1(2+M_1+b_1\mu_1)\int_0^\infty (1+(t-s)^{-\frac{1}{2}})e^{-(\lambda_1+1)(t-s)}ds \\ &\leq \gamma_2\|u_0\|_{L^\infty} + \frac{\gamma_2 M_1(2+M_1+b_1\mu_1)}{\lambda_1+1}\left(1+(\lambda_1+1)^{\frac{1}{2}}\Gamma\left(\frac{1}{2}\right)\right),\end{aligned}$$

which gives (4.33).

Next, we shall prove the boundedness of $\|v(\cdot,t)\|_{L^\infty}$ in (4.34). By using the boundedness criterion established in Lemma 3.1, we only need to show $\|v(\cdot,t)\|_{L^p} \le c_1$ with $p > \frac{n}{2}$. To this end, for $T \in (0, T_{max})$ we first define

$$\mathcal{N}(T) := \sup_{t\in(0,T)} \|v(\cdot,t)\|_{L^p}. \tag{4.37}$$

For $p > \frac{n}{2}$, we choose $\frac{n}{3} < q < p$ such that

$$-\frac{1}{2} - \frac{n}{2}\left(\frac{1}{q} - \frac{1}{p}\right) > -1 \quad \text{and} \quad \delta_1 := \frac{1-\frac{1}{q}}{1-\frac{1}{p}} \in (0,1). \tag{4.38}$$

Then using (4.33), Hölder inequality and (4.37), we have

$$\|v\nabla u\|_{L^q} \le \|v\|_{L^q}\|\nabla u\|_{L^\infty} \le \|v\|_{L^1}^{1-\delta_1}\|v\|_{L^p}^{\delta_1}\|\nabla u\|_{L^\infty} \le M_3^{1-\delta_1} M_{10}\mathcal{N}^{\delta_1}(T). \tag{4.39}$$

Then applying the semigroup estimates in Lemma 2.4 and using (4.39), we can derive from (3.11) that

$$\begin{aligned}
\|v(\cdot,t)\|_{L^p} &\le \|e^{(\Delta-\theta_1)t}v_0\|_{L^p} + \xi\int_0^t \|e^{(\Delta-\theta_1)(t-s)}\nabla\cdot(v\nabla u)\|_{L^p}ds \\
&\quad + \int_0^t \|e^{(\Delta-\theta_1)(t-s)}F_1(u,v)v\|_{L^p}ds \\
&\le \gamma_3\|v_0\|_{L^\infty} + \gamma_3\int_0^t (1+(t-s)^{\frac{n}{2p}})e^{-\theta_1(t-s)}\|F_1(u,v)v\|_{L^\infty}ds \\
&\quad + \gamma_4\xi\int_0^t \left(1+(t-s)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\right)e^{-(\lambda_1+\theta_1)(t-s)}\|v\nabla u\|_{L^q}ds \\
&\le \gamma_3\|v_0\|_{L^\infty} + \gamma_3\mu_1 M_1\int_0^\infty (1+(t-s)^{\frac{n}{2p}})e^{-\theta_1(t-s)}ds \\
&\quad + \gamma_4\xi M_3^{1-\delta_1}M_{10}\mathcal{N}^{\delta_1}(T)\int_0^\infty \left(1+(t-s)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\right)e^{-(\lambda_1+\theta_1)(t-s)}ds \\
&\le c_1\xi\mathcal{N}^{\delta_1}(T) + c_2,
\end{aligned}$$

which combined (4.37) gives

$$\mathcal{N}(T) \le c_1\xi\mathcal{N}^{\delta_1}(T) + c_2$$

and hence $\mathcal{N}(T) \le c_3(1+\xi^{\frac{1}{1-\delta_1}}) := c_4$ by noting $\delta_1 \in (0,1)$ in (4.38). Then using the definition of $\mathcal{N}(T)$, we have for all $p > \frac{n}{2}$ that

$$\|v(\cdot,t)\|_{L^p} \le c_4 \quad \text{for all } t \in (0, T_{max}). \tag{4.40}$$

At last, applying Lemma 3.1 to (4.40) with $p > \frac{n}{2}$, we obtain (4.34). □

Lemma 4.4. *Let (u,v,w) be the solution obtained in Lemma 1.1 and the assumptions (H1)–(H3) as well as (1.9) hold. Then it holds that*

$$\|\nabla v(\cdot,t)\|_{L^\infty} \le M_{12} \quad \textit{for all} \quad t \in (0, T_{max}), \tag{4.41}$$

and

$$\|w(\cdot,t)\|_{L^\infty} \le M_{13} \quad \textit{for all} \quad t \in (0, T_{max}), \tag{4.42}$$

where M_{12} and M_{13} are positive constants depending on ξ and χ but independent of t.

Proof. Using the similar arguments in Lemma 3.2, we can derive

$$\begin{aligned}
&\frac{1}{2p}\frac{d}{dt}\int_\Omega |\nabla v|^{2p} + \frac{p-1}{2}\int_\Omega |\nabla v|^{2(p-2)}|\nabla|\nabla v|^2|^2 \\
&\quad + \int_\Omega |\nabla v|^{2(p-1)}|D^2 v|^2 + \theta_1 \int_\Omega |\nabla v|^{2p} \\
&= \frac{1}{2}\int_{\partial\Omega} |\nabla v|^{2(p-1)}\frac{\partial|\nabla v|^2}{\partial\nu} + \xi\int_\Omega \nabla\cdot(|\nabla v|^{2p-2}\nabla v)\nabla\cdot(v\nabla u) \\
&\quad + \int_\Omega |\nabla v|^{2p-2}\nabla v\cdot\nabla(F_1(u,v)v) - b_2\int_\Omega |\nabla v|^{2p-2}\nabla v\cdot\nabla(F_2(v,w)w) \\
&= J_1 + J_2 + J_3 + J_4.
\end{aligned} \tag{4.43}$$

The terms J_1, J_2 and J_3 can be estimated by using the same way in Lemma 3.2, and then we only need to estimate the term J_4. In fact, noting $\|v(\cdot,t)\|_{L^\infty} \le M_{11}$ in Lemma 4.3 and using the condition (1.9), we have

$$0 < F_2(v,w)w \le \mu_2 v \le \mu_2 M_{11} \quad \text{for all} \quad t\in(0,T_{max}). \tag{4.44}$$

Then using the integration by parts and Young's inequality as well as (4.44), we estimate J_4 as follows:

$$\begin{aligned}
J_4 &= -b_2\int_\Omega |\nabla v|^{2p-2}\nabla v\cdot\nabla(F_2(v,w)w) \\
&\le b_2(p-1)\int_\Omega |\nabla v|^{2p-3}|\nabla|\nabla v|^2|F_2(v,w)w + b_2\int_\Omega |\nabla v|^{2p-2}\Delta v F_2(v,w)w \\
&\le b_2(p-1)\mu_2 M_{11}\int_\Omega |\nabla v|^{2p-3}|\nabla|\nabla v|^2| + b_2\mu_2 M_{11}\sqrt{n}\int_\Omega |\nabla v|^{2p-2}|D^2 v| \\
&\le \frac{p-1}{16}\int_\Omega |\nabla v|^{2(p-2)}|\nabla|\nabla v|^2|^2 + \frac{1}{16}\int_\Omega |\nabla v|^{2p-2}|D^2 v|^2 + c_1\int_\Omega |\nabla v|^{2(p-1)}.
\end{aligned} \tag{4.45}$$

Then substituting J_1 in (3.20), J_2 in (3.21) and J_3 in (3.23) as well as J_4 in (4.45) into (4.43), we obtain

$$\begin{aligned}
&\frac{1}{2p}\frac{d}{dt}\int_\Omega |\nabla v|^{2p} + \frac{3(p-1)}{16}\int_\Omega |\nabla v|^{2(p-2)}|\nabla|\nabla v|^2|^2 \\
&\quad + \frac{3}{4}\int_\Omega |\nabla v|^{2p-2}|D^2 v|^2 + \theta_1\int_\Omega |\nabla v|^{2p} \\
&\le c_2\int_\Omega |\nabla v|^{2p} + c_3\int_\Omega |\nabla v|^{2(p-1)} + c_4\int_\Omega |\nabla v|^{2(p-1)}|\Delta u|^2.
\end{aligned} \tag{4.46}$$

Using Young's inequality and (2.4), we can derive that

$$\begin{aligned}
&c_2\int_\Omega |\nabla v|^{2p} + c_3\int_\Omega |\nabla v|^{2(p-1)} + c_4\int_\Omega |\nabla v|^{2(p-1)}|\Delta u|^2 \\
&\quad \le \frac{3}{4k_2 M_2^2}\int_\Omega |\nabla v|^{2(p+1)} + c_5\int_\Omega |D^2 u|^{p+1} + c_6 \\
&\quad \le \frac{3}{4}\int_\Omega |\nabla v|^{2p-2}|D^2 v|^2 + c_5\int_\Omega |D^2 u|^{p+1} + c_6,
\end{aligned}$$

which substituted into (4.46) gives

$$\frac{1}{2p}\frac{d}{dt}\int_\Omega |\nabla v|^{2p} + \theta_1 \int_\Omega |\nabla v|^{2p} \le c_5 \int_\Omega |D^2 u|^{p+1} + c_6. \tag{4.47}$$

Noting the boundedness of $\|v(\cdot,t)\|_{L^\infty}$, from Lemma 3.3, we have

$$\int_t^{t+\tau}\int_\Omega |D^2 u|^{p+1} \le M_7 \ \text{ for all } \ t \in (0, \widetilde{T}_{max}). \tag{4.48}$$

Then using Lemma 2.3 and noting (4.48), from (4.47), we obtain

$$\|\nabla v(\cdot,t)\|_{L^{2p}} \le c_7. \tag{4.49}$$

On the other hand, we rewrite the second equation of system (1.1) in the following:

$$v_t - \Delta v + v = -\nabla\cdot(\xi v \nabla u) + \mathcal{H}(u,v,w), \tag{4.50}$$

where $\mathcal{H}(u,v,w) := F_1(u,v)v - b_2 F_2(v,w)w + (1-\theta_1)v$ satisfying

$$\begin{aligned}\|\mathcal{H}(u,v,w)\|_{L^\infty} &\le \|F_1(u,v)v\|_{L^\infty} + b_2\|F_2(v,w)w\|_{L^\infty} + (1+\theta_1)\|v\|_{L^\infty}\\ &\le \mu_1 M_1 + b_2\mu_2 M_{11} + (1+\theta_1)M_{11} := c_8.\end{aligned} \tag{4.51}$$

Then applying the variation-of-constants formula to (4.50), we obtain

$$\begin{aligned}\nabla v(\cdot,t) =& \nabla e^{(\Delta-1)(t-\tau)}v(\cdot,\tau) - \xi\int_\tau^t \nabla e^{(\Delta-1)(t-s)}\nabla\cdot(v\nabla u)ds\\ &+ \int_\tau^t \nabla e^{(\Delta-1)(t-s)}\mathcal{H}(u,v,w)ds.\end{aligned} \tag{4.52}$$

Using the boundedness of $\|\nabla u(\cdot,t)\|_{L^\infty}$ (see (4.33)) and $\|\nabla v(\cdot,t)\|_{L^{2p}}$ (see (4.49)), for $p > \frac{n}{2}$, we can derive that

$$\|\nabla\cdot(v\nabla u)\|_{L^{2p}} \le c_9 + c_{10}\|\Delta u\|_{L^{2p}}. \tag{4.53}$$

Next, we apply the semigroup estimates in Lemma 2.4, and use (4.51)–(4.53) and (3.43) to obtain that

$$\begin{aligned}\|\nabla v(\cdot,t)\|_{L^\infty} \le& \|\nabla e^{(\Delta-1)(t-\tau)}v(\cdot,\tau)\|_{L^\infty} + \xi\int_\tau^t \|\nabla e^{(\Delta-1)(t-s)}\nabla\cdot(v\nabla u)\|_{L^\infty}ds\\ &+ \int_\tau^t \|\nabla e^{(\Delta-1)(t-s)}\mathcal{H}(u,v,w)\|_{L^\infty}ds\\ \le& c_{11} + \gamma_2\xi\int_\tau^t \left(1+(t-s)^{-\frac{1}{2}-\frac{n}{4p}}\right)e^{-(\lambda_1+1)(t-s)}\|\nabla\cdot(v\nabla u)\|_{L^{2p}}ds\\ &+ \gamma_2 c_8\int_\tau^t \left(1+(t-s)^{-\frac{1}{2}}\right)e^{-(\lambda_1+1)(t-s)}ds\\ \le& c_{11} + \gamma_2\xi c_9\int_\tau^t \left(1+(t-s)^{-\frac{1}{2}-\frac{n}{4p}}\right)e^{-(\lambda_1+1)(t-s)}ds\\ &+ \gamma_2\xi c_{10}\int_\tau^t \left(1+(t-s)^{-\frac{1}{2}-\frac{n}{4p}}\right)e^{-(\lambda_1+1)(t-s)}\|\Delta u\|_{L^{2p}}ds\\ &+ \gamma_2 c_8\int_\tau^t \left(1+(t-s)^{-\frac{1}{2}}\right)e^{-(\lambda_1+1)(t-s)}ds\\ \le& c_{12},\end{aligned}$$

which gives (4.41).

Next, we shall prove (4.42). In fact, using the third equation of system (1.1), we have

$$\begin{aligned} w(\cdot,t) = e^{(\Delta-\theta_2)t}w_0 - \chi\int_0^t e^{(\Delta-\theta_2)(t-s)}\nabla\cdot[w\nabla\phi(u,v)] \\ + \int_0^t e^{(\Delta-\theta_2)(t-s)}F_2(v,w)wds. \end{aligned} \tag{4.54}$$

For $p>\frac{n}{2}$ and $T\in(0,T_{max})$, we define

$$\mathcal{K}(T) := \sup_{t\in(0,T)}\|w(\cdot,t)\|_{L^p}. \tag{4.55}$$

Then noting $p>\frac{n}{2}$, we can find constant $\frac{n}{3}<q<p$ such that

$$-\frac{1}{2}-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right) > -1 \quad\text{and}\quad \delta_2 := \frac{\frac{1}{q}-1}{\frac{1}{p}-1}\in(0,1). \tag{4.56}$$

On the other hand, due the assumption (H3) and the boundedness of $\|u(\cdot,t)\|_{L^\infty}$ (see Lemma 2.1) and $\|v(\cdot,t)\|_{L^\infty}$ (see Lemma 4.3), we can find a positive constant c_{13} independent of t such that

$$\|\phi_u\|_{L^\infty}+\|\phi_v\|_{L^\infty}\le c_{13},$$

which, together with the facts (4.33) and (4.41), gives

$$\begin{aligned} \|w\nabla\phi(u,v)\|_{L^q} = \|w(\phi_u\nabla u+\phi_v\nabla v)\|_{L^q} &\le c_{13}\|w\|_{L^q}(\|\nabla u\|_{L^\infty}+\|\nabla v\|_{L^\infty}) \\ &\le c_{14}\|w\|_{L^1}^{1-\delta_2}\|w\|_{L^p}^{\delta_2} \\ &\le c_{14}M_3^{1-\delta_2}\mathcal{K}^{\delta_2}(T). \end{aligned} \tag{4.57}$$

Then using Lemma 2.4 and noting the facts (4.56) and (4.57), from (4.54), we have

$$\begin{aligned} \|w(\cdot,t)\|_{L^p} &\le \gamma_3\|w_0\|_{L^\infty} \\ &\quad + \chi\gamma_4\int_0^t\left(1+(t-s)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\right)e^{-(\lambda_1+\theta_2)(t-s)}\|w\nabla\phi(u,v)\|_{L^q}ds \\ &\quad + \gamma_3\mu_2M_{11}\int_0^t(1+(t-s)^{\frac{n}{2p}}e^{-\theta_2(t-s)}ds \\ &\le c_{15}+c_{16}\chi\mathcal{K}^{\delta_2}(T), \end{aligned}$$

which combined (4.55) gives

$$\mathcal{K}(T)\le c_{15}\chi\mathcal{K}^{\delta_2}(T)+c_{16}. \tag{4.58}$$

Then we apply Young's inequality to from (4.58) to obtain

$$\mathcal{K}(T)\le c_{17}(1+\chi^{\frac{1}{1-\delta_2}}) := c_{18},$$

and hence for $p>\frac{n}{2}$ it holds

$$\|w(\cdot,t)\|_{L^p}\le c_{18} \quad\text{for all}\quad t\in(0,T_{max}). \tag{4.59}$$

Then using the boundedness criterion in Lemma 3.4, from (4.59), we derive (4.42) and then it completes the proof of Lemma 4.4. □

Proof. Theorem 1.6 is a consequence of Lemmas 4.1–4.4. □

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