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Global Dynamics of a Three-Species Lotka–Volterra Food Chain Model with Intraguild Predation and Taxis Mechanisms

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Abstract

In this paper, we study a three-species food chain model with intraguild predation and taxis mechanisms (prey-taxis and alarm-taxis) in an open interval $\Omega \subset \mathbb{R}$ with smooth boundary. Based on energy estimates, we first establish the existence of global classical solutions with a uniform-in-time bound. Moreover, we build the global stability of the spatially homogeneous prey-only steady states, semi-coexistence and coexistence steady states under certain conditions on parameters by using the Lyapunov functionals and LaSalle's invariant principle. With numerical simulations, we further demonstrate that the combination of taxis mechanisms and intraguild predation can produce stationary spatially inhomogeneous patterns, chaotic spatiotemporal patterns and spatial-periodic patterns for the parameters outside the stability regime. We also find from numerical simulations that prey-taxis could destabilize a positive equilibrium in a three-species Lotka–Volterra model with intraguild predation, which is in contrast to the well-known results that the attractive prey-taxis serves to enhance the stability of the spatially homogeneous steady state in two-species predator system or three-species food chain model without intraguild predation.

Keywords Boundedness · Global stability · Prey-taxis · Alarm-taxis · Pattern formation

Mathematics Subject Classification Primary 35A01 · 35B40 · 35B44 · 35K57 · 35Q92 · 92C17

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1 Introduction and main results

To understand the complex ecological interactions, various ordinary differential equation (ODE)-type food chain models have been proposed, and some interesting and impressive results have been established on the dynamics of three-species food chain model (Vance 1978; Gilpin 1979; Krikorian 1979; Hasting and Powell 1991; Holt and Polis 1997; McCann and Hastings 1997; Klebanoff and Hastings 1994; Polis 1991; Tanabe and Namba 2005; McCann and Yodzis 1994). In particular, the chaos phenomenon can be found for the three-species food chain models with nonlinear functional responses (Hasting and Powell 1991; Klebanoff and Hastings 1994) or for the simple Lotka–Volterra-type functional responses with intraguild predation (i.e., a simple kind of omnivory in which a predator and a prey share a common resource) (Tanabe and Namba 2005). As we know, the spatial movement plays an indispensable role for the population species to survive and thrive. However, compared with the well-known results on the temporal three-species predator–prey systems (Vance 1978; Gilpin 1979; Krikorian 1979; Hasting and Powell 1991; Holt and Polis 1997; McCann and Hastings 1997, ?; Polis 1991; Tanabe and Namba 2005), few results are available for the food chain model with spatial movement. In this paper, we shall consider the three-species Lotka–Volterra food chain model with spatial movement:

$$\begin{cases} u_t = d_1\Delta u + u(1-u) - b_1uv - \gamma_1uw, & x\in\Omega, t>0,\\ v_t = d_2\Delta v - \xi\nabla\cdot(v\nabla u) + uv - b_2vw - \theta_1v, & x\in\Omega, t>0,\\ w_t = \Delta w - \chi\nabla\cdot[w\nabla\phi(u,v)] + vw + \gamma_2uw - \theta_2w, & x\in\Omega, t>0, \end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^n$ is a bounded domain, and $(u, v, w) := (u, v, w)(x, t)$ denotes the densities of the prey species, primary and top predators, respectively. The parameters $d_i > 0$ $(i = 1, 2)$ are diffusion coefficients, the term $-\xi\nabla\cdot(v\nabla u)$ describes the directional movement of primary predators toward their prey density gradient (called prey-taxis mechanism (Kareiva and Odell 1987)). Similarly, the term $-\chi\nabla\cdot[w\nabla\phi(u, v)]$ describes the top predators move toward to high gradient of the signal produced as a result of the interaction between the prey and primary predator. For $i = 1, 2$, the parameters $b_i > 0$ and $\gamma_i \geq 0$ describe the interaction of interspecies, and $\theta_i > 0$ represent the mortality rates of the primary and top predators, respectively.

Before stating our main results, we first recall some related results for the system (1.1). If $w\equiv 0$, the system (1.1) becomes the two-species predator–prey system with prey-taxis (called the prey-taxis system), which was first proposed by Kareiva and Odell to interpret the heterogeneous aggregative patterns due to the area-restricted search strategy (Kareiva and Odell 1987). In recent years, the solution behaviors for two-species prey-taxis system have been extensively studied, including the global boundedness and large time behavior as well as pattern formations (cf. (Jin and Wang 2017; Kareiva and Odell 1987; Wu et al. 2016; Jin and Wang 2021; Winkler 2017; Cai et al. 2022) and references therein). Moreover, one can find more related results on the two-species predator–prey system with other types of taxis mechanisms such as the indirect prey-taxis mechanism (Ahn and Yoon 2020; Wang and Wang 2020; Tello and Wrzosek 2016), predator-taxis mechanism (Wu et al. 2018), dual-taxis mechanism

(Tao and Winkler 2022; Fuest 2020), and signal-dependent prey-taxis mechanism (Jin and Wang 2021). However, compared with the substantial results on the two-species predator–prey systems with various taxis mechanisms, few results are known for the three-species spatial food chain model (1.1) (i.e., $w \neq 0$). Recently, the second author and his collaborators (Jin et al. 2022) studied the global dynamics of system (1.1) in a two-dimensional bounded domain with homogeneous Neumann boundary conditions and under the following assumptions:

$$\gamma_1 = \gamma_2 = 0 \ \text{ and } \ \phi(u, v) = v. \tag{1.2}$$

The ideas/methods used in Jin et al. (2022) depend on that the system (1.1) with (1.2) has a nice entropy estimate, which was first developed in Tao and Winkler (2012) for the classical chemotaxis system with consumption of chemoattractant and later was used to study the prey-taxis system (Jin and Wang 2017).

If $\gamma_1, \gamma_2 > 0$, the corresponding ODE version of (1.1) (i.e., ignored the spatial movement) was called intraguild predation model, which exhibits very complex dynamics and has been studied for a long time (see Holt and Polis (1997); McCann and Hastings (1997); Polis (1991); Tanabe and Namba (2005) and references therein). Particularly, it has been proved in Tanabe and Namba (2005) that the intraguild predation sometimes destabilizes food webs and induces chaos, even if the functional responses are linear (Lotka–Volterra type). However, to our knowledge, for the spatial food chain model (1.1) with intraguild predation (i.e., $\gamma_1, \gamma_2 > 0$), there is no such a result. On the other hand, if the signal intensity function $\phi(u, v) = uv$, the system (1.1) was proposed in Haskell and Bell (2021) to test the "burglar alarm" hypothesis (cf. (Burkenroad 1943)): a prey species renders itself dangerous to a primary predator by generating an alarm call to attract a second predator at higher trophic levels in the food chain that preys on the primary predator. Hence, the system (1.1) with $\phi(u, v) = uv$, also called alarm-taxis system, has been studied for the global boundedness and stability of solutions: in one-dimensional space (Haskell and Bell 2021) and in two dimensions (Jin et al. 2023) in the presence of intraspecific competition for v and w.

Our goal in this paper is to study the global dynamics for system (1.1) with $\gamma_1, \gamma_2 > 0$ and more general signal functional $\phi(u, v)$. However, if $\gamma_1, \gamma_2 > 0$ or $\phi(u, v) \neq v$, the ideas used in Jin et al. (2022) are not available anymore. Moreover, due to the lack of quadratic decay terms (i.e., intraspecific competition) for v and w, the methods developed in Jin et al. (2023) are also inapplicable, which motivates us to develop new ideas to study this problem. To explore the combined effects of the intraguild predation and taxis mechanisms more clearly, we focus on studying the global dynamics of the system (1.1) in an open interval $\Omega \subset \mathbb{R}$:

$$\begin{cases} u_t = d_1 u_{xx} + u(1-u) - b_1 uv - \gamma_1 uw, & x \in \Omega, t > 0, \\ v_t = d_2 v_{xx} - \xi(v u_x)_x + uv - b_2 vw - \theta_1 v, & x \in \Omega, t > 0, \\ w_t = w_{xx} - \chi(w\phi(u, v)_x)_x + vw + \gamma_2 uw - \theta_2 w, & x \in \Omega, t > 0, \\ u_x = v_x = w_x = 0, & x \in \partial\Omega, t > 0, \\ (u, v, w)(x, 0) = (u_0, v_0, w_0)(x), & x \in \Omega. \end{cases} \tag{1.3}$$

For more generally, we assume that the signal intensity function $\phi(u, v)$ satisfies the following conditions:

(H0) The function $\phi(y, z) : (0, \infty) \times (0, \infty) \to \mathbb{R}$ is positive and $\phi(y, z)$ belongs to $C^2([0, \infty) \times [0, \infty))$.

Then we first show the global existence of classical solution as follows.

Theorem 1.1 (Global boundedness) *Let $\Omega \subset \mathbb{R}$ be a bounded open interval with smooth boundary. Suppose that the initial data $0 \lneqq (u_0, v_0, w_0) \in [W^{1,\infty}(\Omega)]^3$ and the assumptions in (H0) hold. Then the system (1.3) admits a unique global classical solution (u, v, w) fulfilling $u, v, w > 0$. Moreover, there exists a constant $M > 0$ independent of t such that*

$$\|u(\cdot, t)\|_{W^{1,2}} + \|v(\cdot, t)\|_{W^{1,2}} + \|w(\cdot, t)\|_{L^\infty} \le M.$$

Remark 1.2 The upper bounds of $\|u(\cdot, t)\|_{L^\infty}$ and $\|v(\cdot, t)\|_{L^\infty}$ play an important role in studying the large time behavior of solutions. In fact, we can show that

$$\|u(\cdot, t)\|_{L^\infty} \le M_0 := \max\{1, \|u_0\|_{L^\infty}\}, \tag{1.4}$$

and

$$\|v(\cdot, t)\|_{L^\infty} \le K_0 := C[1 + \xi(\xi^6 + 1)^{\frac{1}{2}}], \tag{1.5}$$

where the constant $C > 0$ depends on the parameters $u_0, v_0, \gamma_i, \theta_i, b_i, d_i$ $(i = 1, 2)$ and $|\Omega|$ but it is independent of ξ and χ.

A central question in population dynamics is whether the interacting species population will arrive at the coexistence, exclusion or extinction eventually.

If $\gamma_1 = \gamma_2 = 0$ and $\phi(u, v) = v$, it has been proved in Jin et al. (2022) that the globally bounded solution will converge to the constant steady state as $t \to \infty$ and no pattern formation occurs. Hence, there exist some interesting questions:

(1) How about the global dynamics of solution for the system (1.3) with $\gamma_1, \gamma_2 > 0$? Whether or not pattern formation occurs?
(2) If $\gamma_1 = \gamma_2 = 0$, whether or not pattern formation occurs for other kinds of $\phi(u, v)$ instead of $\phi(u, v) = v$?

To answer the above questions, we first classify the constant steady state (u_c, v_c, w_c) of the system (1.3) with $\gamma_1, \gamma_2 > 0$, which satisfies

$$\begin{cases} 0 = u_c(1 - u_c - b_1 v_c - \gamma_1 w_c), \\ 0 = v_c(u_c - b_2 w_c - \theta_1), \\ 0 = w_c(v_c + \gamma_2 u_c - \theta_2). \end{cases} \tag{1.6}$$

A direct calculation implies that the constant steady state (u_c, v_c, w_c) takes the following five cases:

- Trivial steady states: $E_0 := (0, 0, 0)$ and $E_1 := (1, 0, 0)$;
- Semi-trivial steady states: $E_{12} := \left(\theta_1, \frac{1-\theta_1}{b_1}, 0\right)$ and $E_{13} := \left(\frac{\theta_2}{\gamma_2}, 0, \frac{\gamma_2-\theta_2}{\gamma_1\gamma_2}\right)$;
- Coexistence steady state: $E_* := (u_*, v_*, w_*)$, where

$$\begin{cases} u_* = \frac{b_2(1-b_1\theta_2)+\gamma_1\theta_1}{b_2+\gamma_1-b_1b_2\gamma_2} > 0, \\ v_* = \frac{\gamma_1(\theta_2-\gamma_2\theta_1)+b_2(\theta_2-\gamma_2)}{b_2+\gamma_1-b_1b_2\gamma_2} > 0, \\ w_* = \frac{b_1(\gamma_2\theta_1-\theta_2)+(1-\theta_1)}{b_2+\gamma_1-b_1b_2\gamma_2} > 0. \end{cases} \tag{1.7}$$

One can check that the coexistence steady state $E_* := (u_*, v_*, w_*)$ is linearly unstable if $b_2+\gamma_1-b_1b_2\gamma_2 < 0$. Therefore, for the case of coexistence steady state (u_*, v_*, w_*), we only focus on studying the dynamics in the following range of parameters

$$\begin{cases} b_2 + \gamma_1 - b_1b_2\gamma_2 > 0, \\ \gamma_1(\theta_2 - \gamma_2\theta_1) + b_2(\theta_2 - \gamma_2) > 0, \\ b_1(\gamma_2\theta_1 - \theta_2) + (1 - \theta_1) > 0, \end{cases} \iff \begin{cases} b_2 + \gamma_1 - b_1b_2\gamma_2 > 0, \\ \theta_2 > \frac{\gamma_1\gamma_2}{b_2+\gamma_1}\theta_1 + \frac{b_2\gamma_2}{b_2+\gamma_1}, \\ \theta_2 < \frac{b_1\gamma_2-1}{b_1}\theta_1 + \frac{1}{b_1}. \end{cases} \tag{1.8}$$

Then by constructing some appropriate energy functionals, we can derive the global stability of the constant steady states as follows.

Theorem 1.3 (Global stability) *Assume M_0 and K_0 are defined in* (1.4) *and* (1.5)*, respectively. Then the solution (u, v, w) of* (1.3) *obtained in* Theorem 1.1 *has the following convergence properties:*

- *If $\theta_1 > 1$ and $\theta_2 > \gamma_2$, then it holds that*

$$\lim_{t\to\infty} (\|u - 1\|_{L^\infty} + \|v\|_{L^\infty} + \|w\|_{L^\infty}) = 0.$$

- *If $0 < \theta_1 < 1$ and $\theta_2 > \ell_1$ with*

$$\ell_1 := \frac{\gamma_1}{b_1b_2}\theta_1 - \frac{\theta_1}{b_1} + \frac{1}{b_1} + \frac{\max\{b_1b_2\gamma_2 - \gamma_1, 0\}}{b_1b_2},$$

then there exists $\xi_0 > 0$ such that whenever $\xi \in (0, \xi_0)$, it holds that

$$\lim_{t\to\infty} \left(\|u - \theta_1\|_{L^\infty} + \|v - \frac{1-\theta_1}{b_1}\|_{L^\infty} + \|w\|_{L^\infty}\right) = 0.$$

- *If $\theta_1 > 1$, $\theta_2 < \min\{\gamma_2, \ell_2\}$ with*

$$\ell_2 := \frac{\gamma_1\gamma_2}{b_1b_2\gamma_2 + b_2}\theta_1 + \frac{b_2\gamma_2}{b_1b_2\gamma_2 + b_2} + \frac{\gamma_2 \min\{b_1b_2\gamma_2 - \gamma_1, 0\}}{b_1b_2\gamma_2 + b_2},$$

then there exist $\xi_1 > 0$ *and* $\chi_1 > 0$ *such that whenever* $\xi \in (0, \xi_1)$ *and* $\chi \in (0, \chi_1)$, *it holds that*

$$\lim_{t\to\infty}\left(\|u-\frac{\theta_2}{\gamma_2}\|_{L^\infty}+\|v\|_{L^\infty}+\|w-\frac{\gamma_2-\theta_2}{\gamma_1\gamma_2}\|_{L^\infty}\right)=0.$$

- *If* (1.8) *and* $\gamma_1 = b_1b_2\gamma_2$ *hold, then there exist* $\xi_2 > 0$ *and* $\chi_2 > 0$ *such that whenever* $\xi \in (0, \xi_2)$ *and* $\chi \in (0, \chi_2)$, *it holds that*

$$\lim_{t\to\infty}(\|u-u_*\|_{L^\infty}+\|v-v_*\|_{L^\infty}+\|w-w_*\|_{L^\infty})=0,$$

where the coexistence steady state (u_*, v_*, w_*) *is defined in* (1.7).

In view of the results obtained in Theorem 1.3, there exists an interesting question: whether or not pattern formations (non-constant steady states) are possible when parameters outside the stability regimes found in Theorem 1.3. To answer this question, we first do some linearly stable analysis (see Proposition 5.1), which together with the global stability results for the corresponding ODE system obtained in Hsu et al. (2015), implies that the pattern (if any) can only arise from the homogeneous coexistence steady state (u_*, v_*, w_*). In Section 5, we shall use linear stability analysis to find the conditions on parameters for the instability of coexistence steady state and then perform numerical simulations to illustrate that spatially inhomogeneous patterns indeed can be found under certain conditions in Section 6. By comparing with the results obtained for the food chain model without intraguild predation (i.e., $\gamma_1 = \gamma_2 = 0$), we also demonstrate that the intraguild predation plays an important role in generating the pattern formation.

2 Local Existence and Preliminaries

In the following context, the $\int_\Omega f\,\mathrm{d}x$ and $\|f\|_{L^p(\Omega)}$ will be abbreviated as $\int_\Omega f$ and $\|f\|_{L^p}$, respectively. Moreover, the constants k_i and M_i $(i = 1, 2, 3\cdots)$ represent generic positive constants independent of t and will vary line-by-line. The local existence of solutions can be proved by using the Amann's theorem (Amann 1990, Theorem 7.3), we omit the proof details for brevity.

Lemma 2.1 (Local existence) *Let* $\Omega \subset \mathbb{R}$ *be a bounded open interval with smooth boundary. Suppose that* $0 \lneqq (u_0, v_0, w_0) \in [W^{1,\infty}(\Omega)]^3$ *and the assumption (H0) holds. Then there admits* $T_{\max} \in (0,\infty]$ *such that the system* (1.3) *has a unique classical solution*

$$(u, v, w) \in \left[C^0([0, T_{\max}); W^{1,2}(\Omega)] \cap C^{2,1}(\bar{\Omega}\times(0, T_{\max}))\right]^3,$$

satisfying $u, v, w > 0$ *for all* $t > 0$. *Moreover, it holds that if* $T_{\max} < \infty$, *then for all* $p > 1$,

$$\limsup_{t\nearrow T_{\max}}\left(\|u(\cdot,t)\|_{W^{1,p}}+\|v(\cdot,t)\|_{W^{1,p}}+\|w(\cdot,t)\|_{L^\infty}\right)=\infty.$$

Using similar arguments as in (Jin and Wang 2017, Lemma 2.2), we obtain the boundedness of u immediately as follows.

Lemma 2.2 *Suppose the assumptions in* Lemma 2.1 *hold. Then it holds that*

$$0 < u(x,t) \le M_0 := \max\{1, \|u_0\|_{L^\infty}\} \quad \textit{for all } (x,t) \in \Omega \times (0, T_{\max}); \tag{2.1}$$

Moreover, one has

$$\limsup_{t\to\infty} u(x,t) \le 1 \quad \textit{for all } \ x \in \bar{\Omega}. \tag{2.2}$$

Lemma 2.3 *Let* (u, v, w) *be a solution to the system* (1.3) *obtained in* Lemma 2.1. *Then there exist two constants* $M_1 > 0$ *and* $M_2 > 0$ *independent of* ξ *and* χ *such that for all* $t \in (0, T_{\max})$

$$\|v(\cdot,t)\|_{L^1} \le M_1 := \frac{\theta_1\|u_0\|_{L^1} + \theta_1 b_1\|v_0\|_{L^1} + (1+\theta_1)M_0|\Omega|}{\theta_1 b_1}, \tag{2.3}$$

and

$$\|w(\cdot,t)\|_{L^1} \le M_2 := \begin{cases} \frac{\gamma_0(\|u_0\|_{L^1}+b_1\|v_0\|_{L^1}+b_1b_2\|w_0\|_{L^1})+2M_0|\Omega|}{b_1b_2\gamma_0}, & \textit{if } \gamma_1=\gamma_2=0,\\ \frac{\gamma_0(b_2\gamma_2\|u_0\|_{L^1}+b_2\gamma_1\|w_0\|_{L^1}+\gamma_1\|v_0\|_{L^1})+2b_2\gamma_2M_0|\Omega|+M_0M_1\gamma_1}{\gamma_0b_2\gamma_1}, & \textit{if } \gamma_1,\gamma_2>0. \end{cases} \tag{2.4}$$

Proof Using the first and second equations of (1.3) and applying the homogeneous Neumann boundary conditions, we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u+b_1v) + \int_\Omega u^2 &= \int_\Omega u - b_1\theta_1\int_\Omega v - \gamma_1\int_\Omega uw - b_1b_2\int_\Omega vw,\\ &\le \int_\Omega u - b_1\theta_1\int_\Omega v, \end{aligned}$$

which along with $\theta_1 > 0$ and (2.1) can be updated as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u+b_1v) + \theta_1\int_\Omega (u+b_1v) + \int_\Omega u^2 \le (1+\theta_1)\int_\Omega u \le (1+\theta_1)M_0|\Omega|,$$

and hence, applying Grönwall's inequality, one has

$$\|v(\cdot,t)\|_{L^1} \le \frac{\|u_0\|_{L^1}}{b_1} + \|v_0\|_{L^1} + \frac{(1+\theta_1)M_0|\Omega|}{\theta_1 b_1} =: M_1, \tag{2.5}$$

which gives (2.3).

Next, we shall show the boundedness of $\|w(\cdot,t)\|_{L^1}$. To this end, we divide our proof into two cases: $\gamma_1=\gamma_2=0$ and $\gamma_1,\gamma_2>0$.

Case 1: $\gamma_1=\gamma_2=0$. In this case, we deduce from the equations of (1.3) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u+b_1v+b_1b_2w)+\int_\Omega u^2+b_1\theta_1\int_\Omega v+b_1b_2\theta_2\int_\Omega w=\int_\Omega u. \quad (2.6)$$

Denoting $\gamma_0:=\min\{1,\theta_1,\theta_2\}$ and using the fact $0<u\le M_0$ (see (2.1)), it follows from (2.6) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u+b_1v+b_1b_2w)+\gamma_0\int_\Omega (u+b_1v+b_1b_2w)\le 2M_0|\Omega|,$$

which, together with Grönwall's inequality, gives

$$\|w(\cdot,t)\|_{L^1}\le \frac{\gamma_0(\|u_0\|_{L^1}+b_1\|v_0\|_{L^1}+b_1b_2\|w_0\|_{L^1})+2M_0|\Omega|}{b_1b_2\gamma_0}. \quad (2.7)$$

Case 2: $\gamma_1,\gamma_2>0$. Using the equations of (1.3), one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left(\gamma_2u+\gamma_1w+\frac{\gamma_1}{b_2}v\right)+\gamma_2\int_\Omega u^2+\theta_2\gamma_1\int_\Omega w+\frac{\theta_1\gamma_1}{b_2}\int_\Omega v\le\gamma_2\int_\Omega u+\frac{\gamma_1}{b_2}\int_\Omega uv,$$

which together with (2.1) and (2.5) derives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left(\gamma_2u+\gamma_1w+\frac{\gamma_1}{b_2}v\right)+\gamma_0\int_\Omega\left(\gamma_2u+\gamma_1w+\frac{\gamma_1}{b_2}v\right)&\le 2\gamma_2\int_\Omega u+\frac{\gamma_1}{b_2}\int_\Omega uv\\&\le 2\gamma_2M_0|\Omega|+\frac{\gamma_1M_0M_1}{b_2},\end{aligned}$$

and hence, using Grönwall's inequality, we have

$$\|w(\cdot,t)\|_{L^1}\le\frac{\gamma_0(b_2\gamma_2\|u_0\|_{L^1}+b_2\gamma_1\|w_0\|_{L^1}+\gamma_1\|v_0\|_{L^1})+2b_2\gamma_2M_0|\Omega|+M_0M_1\gamma_1}{\gamma_0b_2\gamma_1},$$

which combined with (2.7) indicates (2.4). Then, the proof of Lemma 2.3 is completed. □

With the boundedness of $\|u(\cdot,t)\|_{L^\infty}$, $\|v(\cdot,t)\|_{L^1}$ and $\|w(\cdot,t)\|_{L^1}$ in hand, next we can use the semigroup estimates to obtain the boundedness of $\|u_x(\cdot,t)\|_{L^q}$ for any $q>1$ in one-dimensional space. More precisely, we have the following results.

Lemma 2.4 *Let (u,v,w) be the solution to the system (1.3) obtained in* Lemma 2.1. *Then for any $q>1$, it holds that*

$$\|u_x(\cdot,t)\|_{L^q}\le M_3:=M_3(q),\ \text{ for all }\ t\in(0,T_{\max}), \quad (2.8)$$

where the constant $M_3(q)>0$ is defined in (2.12), and is independent of ξ and χ.

Proof The first equation of (1.3) can be rewritten as

$$u_t - d_1(u_{xx} - u) = f(x,t), \tag{2.9}$$

where $f(x,t) = (d_1 + 1 - u - b_1 v - \gamma_1 w)u$. Using Hölder inequality, the facts $0 < u \le M_0$ in (2.1), $\|v(\cdot,t)\|_{L^1} \le M_1$ in (2.3) and $\|w(\cdot,t)\|_{L^1} \le M_2$ in (2.4), one has

$$\begin{aligned} \|f(\cdot,t)\|_{L^1} &= \|(d_1 + 1 - u - b_1 v - \gamma_1 w)u\|_{L^1} \\ &\le M_0\left(|\Omega|(d_1 + 1 + M_0) + M_1 b_1 + M_2\gamma_1\right) = \ell_3. \end{aligned} \tag{2.10}$$

Applying the variation-of-constants formula to (2.9) and using the well-known semigroup estimates (see Winkler 2010, Lemma 1.3) and (2.10) guarantee that there exist two constants $\sigma_1 > 0$ and $\sigma_2 > 0$ depending only on Ω such that

$$\begin{aligned} \|u_x(\cdot,t)\|_{L^q} &\le \|\partial_x e^{td_1(\Delta-1)}u_0\|_{L^q} + \int_0^t \|\partial_x e^{(t-s)d_1(\Delta-1)} f(\cdot,s)\|_{L^q} ds \\ &\le \sigma_1\|\partial_x u_0\|_{L^q} + \sigma_2 \int_0^t e^{-(\lambda_1+1)d_1(t-s)}\left(1 + (t-s)^{-1+\frac{1}{2q}}\right)\|f(\cdot,s)\|_{L^1} ds \\ &\le \sigma_1\|\partial_x u_0\|_{L^q} + \sigma_2\ell_3 \int_0^\infty e^{-(\lambda_1+1)d_1 z}\left(1 + z^{-1+\frac{1}{2q}}\right) ds \\ &\le \sigma_1\|\partial_x u_0\|_{L^q} + \frac{\sigma_2\ell_3}{(\lambda_1+1)d_1}\left(1 + \Gamma\left(\frac{1}{2q}\right)((\lambda_1+1)d_1)^{1-\frac{1}{2q}}\right), \end{aligned} \tag{2.11}$$

where $\Gamma(\cdot)$ represents the Gamma function defined by $\Gamma(y) := \int_0^\infty t^{-1+y} e^{-t} dt$, and $\lambda_1 > 0$ denotes the first nonzero eigenvalue of $-\Delta$ under Neumann boundary conditions. Then (2.8) follows directly from (2.11) by choosing

$$\begin{aligned} M_3(q) := &\frac{\sigma_2 M_0\left(|\Omega|(d_1 + 1 + M_0) + M_1 b_1 + M_2\gamma_1\right)}{(\lambda_1+1)d_1}\left(1 + \Gamma\left(\frac{1}{2q}\right)((\lambda_1+1)d_1)^{1-\frac{1}{2q}}\right) \\ &+ \sigma_1\|\partial_x u_0\|_{L^q}, \end{aligned} \tag{2.12}$$

which is independent of t, ξ and χ. Then the proof of Lemma 2.4 is completed. □

The following is an auxiliary result that will be used later.

Lemma 2.5 (Stinner et al. 2014, Lemma 3.4) *Let $T > 0$ and $T_0 \in (0, T)$ and suppose $f(t) : [0, T) \to [0, \infty)$ is an absolutely continuous function and satisfies*

$$f'(t) + \alpha f(t) \le h(t) \text{ for all } t \in (0, T),$$

where constant $\alpha > 0$ and the nonnegative function $h \in L^1_{loc}([0, T))$ fulfilling

$$\int_t^{t+T_0} h(s)ds \le \beta \text{ for all } t \in [0, T - T_0).$$

Then

$$f(t) \le \max\{f(0) + \beta, \frac{\beta}{\alpha T_0} + 2\beta\} \text{ for all } t \in (0, T).$$

3 Global Boundedness: Proof of Theorem 1.1

In this section, we shall prove the boundedness of the global classical solution for the system (1.3) as stated in Theorem 1.1. To this end, we first establish the boundedness of $\|v(\cdot,t)\|_{L^\infty}$.

3.1 Boundedness of $\|v(\cdot,t)\|_{L^\infty}$

Since the upper bound of $\|v(\cdot,t)\|_{L^\infty}$ plays a vital role in studying the global stability of coexistence steady state, in the following, we shall give the explicit relation between the upper bound of $\|v(\cdot,t)\|_{L^\infty}$ and ξ.

Lemma 3.1 *Let (u, v, w) be the solution of the system (1.3) obtained in* Lemma 2.1. *Then*

$$\int_\Omega v^2(\cdot,t) \le M_4(\xi^6+\xi^2+1), \quad \text{for all } t \in (0, T_{\max}), \tag{3.1}$$

and

$$\int_t^{t+\tau}\int_\Omega v_x^2(\cdot,s)\mathrm{d}x\mathrm{d}s \le \frac{2M_4}{d_2}(\xi^6+\xi^2+1), \quad \text{for all } t \in (0, T_{\max}-\tau), \tag{3.2}$$

where $\tau=\min\{1, \frac{T_{\max}}{2}\}$ and $M_4>0$ defined in (3.8), is a constant independent of χ, ξ and t.

Proof Multiplying the second equation of (1.3) by v and using Young's inequality and the fact $0<u(\cdot,t)\le M_0$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega v^2+d_2\int_\Omega v_x^2+b_2\int_\Omega wv^2+\theta_1\int_\Omega v^2 &= \xi\int_\Omega vv_x\cdot u_x+\int_\Omega uv^2\\ &\le \xi\|v\|_{L^\infty}\|v_x\|_{L^2}\|u_x\|_{L^2}+M_0\|v\|_{L^2}^2.\end{aligned} \tag{3.3}$$

Taking $q=2$ in (2.8), it follows that

$$\begin{aligned}\|u_x(\cdot.t)\|_{L^2} &\le \frac{\sigma_2M_0\left(|\Omega|(d_1+1+M_0)+M_1b_1+M_2\gamma_1\right)}{(\lambda_1+1)d_1}\left(1+\Gamma\left(\frac{1}{4}\right)((\lambda_1+1)d_1)^{\frac{3}{4}}\right)\\ &\quad+\sigma_1\|\partial_xu_0\|_{L^2}\\ &=: \Gamma_1,\end{aligned} \tag{3.4}$$

and then applying Gagliardo–Nirenberg inequality, Young's inequality as well as $\|v(\cdot,t)\|_{L^1}\le M_1$ in (2.3), one derives

$$\begin{aligned}\xi\|v\|_{L^\infty}\|v_x\|_{L^2}\|u_x\|_{L^2} &\le k_1\xi(\|v_x\|_{L^2}^{\frac{2}{3}}\|v\|_{L^1}^{\frac{1}{3}}+\|v\|_{L^1})\|v_x\|_{L^2}\|u_x\|_{L^2}\\ &\le k_1\xi M_1^{\frac{1}{3}}\Gamma_1\|v_x\|_{L^2}^{\frac{5}{3}}+k_1\xi M_1\Gamma_1\|v_x\|_{L^2}\\ &\le \frac{d_2}{4}\|v_x\|_{L^2}^2+k_2(\xi^6+\xi^2),\end{aligned} \tag{3.5}$$

where $k_2 := \left\{\left(\frac{20}{3d_2}\right)^5 \frac{k_1^6}{6} + \frac{2k_1^2}{d_2}\right\} M_1^2\Gamma_1^2(1 + \Gamma_1^4)$. Similarly, using Gagliardo–Nirenberg inequality and the fact $\|v(\cdot,t)\|_{L^1} \le M_1$ again, we have

$$\begin{aligned}\left(\frac{1}{2}+M_0\right)\|v\|_{L^2}^2 &\le k_3\left(\frac{1}{2}+M_0\right)\left(\|v_x\|_{L^2}^{\frac{2}{3}}\|v\|_{L^1}^{\frac{4}{3}}+\|v\|_{L^1}^2\right)\\ &\le k_3\left(\frac{1}{2}+M_0\right)M_1^{\frac{4}{3}}\|v_x\|_{L^2}^{\frac{2}{3}}+k_3\left(\frac{1}{2}+M_0\right)M_1^2 \\ &\le \frac{d_2}{4}\|v_x\|_{L^2}^2+k_4,\end{aligned} \tag{3.6}$$

where $k_4 := k_3\left(\frac{1}{2}+M_0\right)M_1^2\left\{1+\left(\frac{k_3}{3d_2}\right)^{\frac{1}{2}}\frac{4}{3}\left(\frac{1}{2}+M_0\right)^{\frac{1}{2}}\right\}$ is independent of ξ and χ. Then substituting (3.5) and (3.6) into (3.3) ensures a constant $k_5 := 2(k_2+k_4)$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega v^2+\int_\Omega v^2+d_2\int_\Omega v_x^2 \le 2k_2(\xi^6+\xi^2)+2k_4 \le k_5(\xi^6+\xi^2+1), \tag{3.7}$$

which along with Grönwall's inequality gives

$$\|v(\cdot,t)\|_{L^2}^2 \le k_5(\xi^6+\xi^2+1)+\|v_0\|_{L^2}^2,$$

and hence, (3.1) follows by taking

$$\begin{aligned}M_4 &:= k_5+\|v_0\|_{L^2}\\ &= 2\left(\left(\frac{20}{3d_2}\right)^5\frac{k_1^6}{6}+\frac{2k_1^2}{d_2}\right)M_1^2\Gamma_1^2(1+\Gamma_1^4)+\|v_0\|_{L^2}\\ &\quad+2k_3\left(\frac{1}{2}+M_0\right)M_1^2\left(1+\left(\frac{k_3}{3d_2}\right)^{\frac{1}{2}}\frac{4}{3}\left(\frac{1}{2}+M_0\right)^{\frac{1}{2}}\right).\end{aligned} \tag{3.8}$$

Finally, we integrate (3.7) with respect to t to obtain that for all $t \in (0, T_{\max}-\tau)$,

$$\begin{aligned}d_2\int_t^{t+\tau}\int_\Omega v_x^2(\cdot,s)\mathrm{d}x\mathrm{d}s &\le k_5(\xi^6+\xi^2+1)+\int_\Omega v^2(\cdot,t)\\ &\le 2k_5(\xi^6+\xi^2+1)+\|v_0\|_{L^2}\\ &\le 2M_4(\xi^6+\xi^2+1),\end{aligned}$$

and hence, (3.2) follows directly. Then the proof of Lemma 3.1 is completed. □

Lemma 3.2 *Let (u, v, w) be the solution of the system* (1.3) *obtained in* Lemma 2.1. *Then there exists a positive constant M_5 defined in* (3.14)*, which is independent of ξ,*

χ, such that

$$\|v(\cdot,t)\|_{L^\infty} \le M_5[1+\xi(\xi^6+\xi^2+1)^{\frac{1}{2}}], \text{ for all } t \in (0,T_{\max}). \tag{3.9}$$

Proof We rewrite the second equation of (1.3) as

$$v_t = d_2 v_{xx} - d_2 v - (\xi v u_x)_x + (d_2+u)v - (b_2 w + \theta_1)v. \tag{3.10}$$

Applying the variation-of-constants formula to (3.10), one has

$$\begin{aligned} v(\cdot,t) =& e^{td_2(\Delta-1)}v_0 - \xi\int_0^t e^{(t-s)d_2(\Delta-1)}(vu_x)_x ds + \int_0^t e^{(t-s)d_2(\Delta-1)}(d_2+u)v ds \\ &- \int_0^t e^{(t-s)d_2(\Delta-1)}(b_2 w+\theta_1)v ds, \end{aligned}$$

which, combined with the facts $b_2, w, v > 0$ and the semigroup estimates (Winkler 2010, Lemma 1.3), entails us to find two constants $\sigma_3 > 0$ and $\sigma_4 > 0$ depending only on Ω such that

$$\begin{aligned} \|v(\cdot,t)\|_{L^\infty} \le& \|e^{td_2(\Delta-1)}v_0\|_{L^\infty} + \xi\int_0^t \|e^{(t-s)d_2(\Delta-1)}(vu_x)_x\|_{L^\infty} ds \\ &+ \int_0^t \|e^{(t-s)d_2(\Delta-1)}(u+d_1-\theta_1)v\|_{L^\infty} ds \\ \le& \sigma_3\|v_0\|_{L^\infty} + \xi\sigma_4\int_0^t e^{-(\lambda_1+1)d_2(t-s)}(1+(t-s)^{-\frac{5}{6}})\|vu_x\|_{L^{\frac{3}{2}}} ds \\ &+ \sigma_3\int_0^t e^{-(\lambda_1+1)d_2(t-s)}(1+(t-s)^{-\frac{1}{2}})\|(u+d_2)v\|_{L^1} ds \\ =:& \sigma_3\|v_0\|_{L^\infty} + J_1 + J_2. \end{aligned} \tag{3.11}$$

Choosing $q = 6$ in (2.12), we can find a constant $\Gamma_2 > 0$ independent of χ and ξ such that

$$\begin{aligned} \|u_x(\cdot.t)\|_{L^6} \le& \frac{\sigma_2 M_0\,(|\Omega|(d_1+1+M_0)+M_1 b_1+M_2\gamma_1)}{(\lambda_1+1)d_1}\left(1+\Gamma\left(\frac{1}{12}\right)((\lambda_1+1)d_1)^{\frac{11}{12}}\right) \\ &+ \sigma_1\|\partial_x u_0\|_{L^6} \\ =:& \Gamma_2, \end{aligned}$$

which, along with Hölder inequality, and (3.1), indicates

$$\|vu_x\|_{L^{\frac{3}{2}}} \le \|v\|_{L^2}\|u_x\|_{L^6} \le M_4^{\frac{1}{2}}(\xi^6+\xi^2+1)^{\frac{1}{2}}\Gamma_2,$$

and hence,

$$
\begin{aligned}
J_1 :&= \sigma_4 \xi \int_0^t e^{-d_2(\lambda_1+1)(t-s)}(1+(t-s)^{-\frac{5}{6}})\|vu_x\|_{L^{\frac{3}{2}}} ds \\
&\le \sigma_4 M_4^{\frac{1}{2}} \xi (\xi^6+\xi^2+1)^{\frac{1}{2}} \Gamma_2 \int_0^t e^{-d_2(\lambda_1+1)(t-s)}(1+(t-s)^{-\frac{5}{6}}) ds \\
&\le \sigma_4 M_4^{\frac{1}{2}} \xi (\xi^6+\xi^2+1)^{\frac{1}{2}} \Gamma_2 \int_0^\infty e^{-d_2(\lambda_1+1)z}\left(1+z^{-1+\frac{1}{6}}\right) ds \\
&\le k_1 \xi (\xi^6+\xi^2+1)^{\frac{1}{2}},
\end{aligned} \tag{3.12}
$$

where $k_1 := \frac{\sigma_4 M_4^{\frac{1}{2}} \Gamma_2}{d_2(\lambda_1+1)}\left(1+\Gamma\left(\frac{1}{6}\right) d_2^{\frac{5}{6}}(\lambda_1+1)^{\frac{5}{6}}\right)$ is independent of χ and ξ. Noting the facts $0 < u \le M_0$ and $\|v(\cdot,t)\|_{L^1} \le M_1$, one derives

$$
\begin{aligned}
J_2 :&= \sigma_3 \int_0^t e^{-d_2(\lambda_1+1)(t-s)}(1+(t-s)^{-\frac{1}{2}})\|(u+d_2)v\|_{L^1} ds \\
&\le \sigma_3 \int_0^t e^{-d_2(\lambda_1+1)(t-s)}(1+(t-s)^{-\frac{1}{2}})\|u+d_2\|_{L^\infty}\|v\|_{L^1} ds \\
&\le \sigma_3 (M_0+d_2) M_1 \int_0^t e^{-d_2(\lambda_1+1)(t-s)}(1+(t-s)^{-\frac{1}{2}}) ds \\
&\le k_2,
\end{aligned} \tag{3.13}
$$

with

$$
k_2 := \frac{\sigma_3(M_0+d_2)M_1}{\lambda_1 d_2 + d_2}\left(1+\Gamma\left(\frac{1}{2}\right)(\lambda_1 d_2+d_2)^{\frac{1}{2}}\right).
$$

Then substituting (3.12) and (3.13) into (3.11), we have

$$
\|v(\cdot,t)\|_{L^\infty} \le \sigma_3\|v_0\|_{L^\infty} + k_1 \xi (\xi^6+\xi^2+1)^{\frac{1}{2}} + k_2,
$$

which gives (3.9) by choosing

$$
\begin{aligned}
M_5 :&= \frac{\sigma_4 M_4^{\frac{1}{2}} \Gamma_2}{d_2(\lambda_1+1)}\left(1+\Gamma\left(\frac{1}{6}\right) d_2^{\frac{5}{6}}(\lambda_1+1)^{\frac{5}{6}}\right) \\
&\quad + \frac{\sigma_3(M_0+d_2)M_1}{\lambda_1 d_2+d_2}\left(1+\Gamma\left(\frac{1}{2}\right)(\lambda_1 d_2+d_2)^{\frac{1}{2}}\right) + \sigma_3\|v_0\|_{L^\infty}.
\end{aligned} \tag{3.14}
$$

Hence, the proof of Lemma 3.2 is finished. □

3.2 Boundedness of $\|w(\cdot,t)\|_{L^\infty}$

To establish the boundedness of $\|w(\cdot,t)\|_{L^\infty}$, we first prove the space-time bound for w based on some ideas in Tao and Winkler (2019).

Lemma 3.3 *Let* (u, v, w) *be the solution of the system* (1.3) *obtained in* Lemma 2.1. *Then there exists a constant* $M_6 > 0$ *such that*

$$\int_t^{t+\tau}\int_\Omega w^2(\cdot,s)\mathrm{d}x\mathrm{d}s \le M_6, \textit{ for all } t \in (0, T_{\max}-\tau), \tag{3.15}$$

where $\tau = \min\{1, \frac{T_{\max}}{2}\}$.

Proof Applying Gagliardo–Nirenberg inequality, Cauchy–Schwarz inequality and the fact $\|\sqrt{w+1}\|_{L^2}^2 = \int_\Omega (w+1) \le M_2 + |\Omega|$, we obtain

$$\begin{aligned}\int_\Omega w^2 \le& \int_\Omega (w+1)^2\\ =&\|\sqrt{w+1}\|_{L^4}^4\\ \le& k_1\|\partial_x\sqrt{w+1}\|_{L^1}^2\|\sqrt{w+1}\|_{L^2}^2 + k_1\|\sqrt{w+1}\|_{L^2}^4\\ \le& \frac{k_1(M_2+|\Omega|)}{4}\left(\int_\Omega \frac{|w_x|}{\sqrt{w+1}}\right)^2 + k_1(M_2+|\Omega|)^2\\ \le& \frac{k_1(M_2+|\Omega|)^2}{4}\int_\Omega \frac{w_x^2}{(w+1)^2} + k_1(M_2+|\Omega|)^2.\end{aligned} \tag{3.16}$$

On the other hand, we use the third equation of (1.3), (2.3) and Young's inequality to derive that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \ln(w+1) =& \int_\Omega \frac{w_t}{w+1}\\ =& \int_\Omega \frac{w_x^2}{(w+1)^2} - \chi\int_\Omega \frac{w\phi(u,v)_x\cdot w_x}{(w+1)^2} + \int_\Omega \frac{(v+\gamma_2 u)w}{w+1} - \theta_2\int_\Omega \frac{w}{w+1}\\ \ge& \int_\Omega \frac{w_x^2}{(w+1)^2} - \chi\int_\Omega \frac{w\phi(u,v)_x\cdot w_x}{(w+1)^2} - \theta_2|\Omega|\\ \ge& \frac{1}{2}\int_\Omega \frac{w_x^2}{(w+1)^2} - \frac{\chi^2}{2}\int_\Omega \frac{w^2|\phi(u,v)_x|^2}{(w+1)^2} - \theta_2|\Omega|.\end{aligned} \tag{3.17}$$

Noting the facts $0 \le \ln(w+1) \le w$ and $\frac{w^2}{(w+1)^2} \le 1$ for all $w \ge 0$ and integrating (3.17) from t to $(t+\tau)$, one has

$$\begin{aligned}\int_t^{t+\tau}\int_\Omega \frac{w_x^2}{(w+1)^2} \le& 2\theta_2|\Omega| + \chi^2\int_t^{t+\tau}\int_\Omega \frac{w^2|\phi(u,v)_x|^2}{(w+1)^2} + 2\int_\Omega \ln(w+1)(\cdot,t+\tau)\\ \le& 2\theta_2|\Omega| + 2M_2 + \chi^2\int_t^{t+\tau}\int_\Omega |\phi_u u_x + \phi_v v_x|^2.\end{aligned} \tag{3.18}$$

Furthermore, by (H0) and the L^∞-boundedness of u, v (see (2.1) and (3.9)), there exists a constant $\gamma > 0$ independent of t such that

$$|\phi_u| + |\phi_v| \le \gamma \quad \text{for all } t \in (0, T_{\max}), \tag{3.19}$$

and then using (3.2) and (3.4), one derives

$$\begin{aligned}\chi^2 \int_t^{t+\tau} \int_\Omega |\phi_u u_x + \phi_v v_x|^2 &\le 2\chi^2\gamma^2 \int_t^{t+\tau} \int_\Omega (u_x^2 + v_x^2) \\ &\le 2\chi^2\gamma^2 \left(\Gamma_1^2 + \frac{2M_4}{d_2} \left(\xi^6 + \xi^2 + 1 \right) \right).\end{aligned} \tag{3.20}$$

We substitute (3.20) into (3.18) to obtain that for all $t \in (0, T_{\max} - \tau)$

$$\int_t^{t+\tau} \int_\Omega \frac{w_x^2}{(w+1)^2} \le 2\theta_2|\Omega| + 2M_2 + 2\chi^2\gamma^2 \left(\Gamma_1^2 + \frac{2M_4}{d_2} \left(\xi^6 + \xi^2 + 1 \right) \right). \tag{3.21}$$

Hence, integrating (3.16) from t to $(t + \tau)$ and applying (3.21), we get (3.15) directly. Then the proof of Lemma 3.3 is finished. □

Lemma 3.4 *Let* (u, v, w) *be the solution to the system* (1.3) *obtained in* Lemma 2.1. *Then there exists a positive constant* M_7 *such that*

$$\int_t^{t+\tau} \int_\Omega u_{xx}^2(\cdot, s)\mathrm{d}x\mathrm{d}s \le M_7, \ \textit{for all} \ \ t \in (0, T_{\max} - \tau), \tag{3.22}$$

where $\tau := \min\{1, \frac{1}{2}T_{\max}\}$.

Proof We multiply the first equation by $-u_{xx}$ and use Young's inequality and (3.4) to derive

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u_x^2 + d_1 \int_\Omega u_{xx}^2 + 2\int_\Omega u u_x^2 &= \int_\Omega u_x^2 + b_1 \int_\Omega u v u_{xx} + \gamma_1 \int_\Omega u w u_{xx} \\ &\le \int_\Omega u_x^2 + \frac{d_1}{2}\int_\Omega u_{xx}^2 + \frac{b_1^2}{d_1}\int_\Omega u^2 v^2 + \frac{\gamma_1^2}{d_1}\int_\Omega u^2 w^2 \\ &\le \frac{d_1}{2}\int_\Omega u_{xx}^2 + \frac{\gamma_1^2 M_0^2}{d_1}\int_\Omega w^2 \\ &\quad + \frac{b_1^2 M_0^2 M_4(\xi^6 + \xi^2 + 1)}{d_1} + \Gamma_1^2,\end{aligned}$$

which gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u_x^2 + d_1 \int_\Omega u_{xx}^2 \le \frac{2\gamma_1^2 M_0^2}{d_1}\int_\Omega w^2 + \frac{2b_1^2 M_0^2 M_4(\xi^6 + \xi^2 + 1)}{d_1} + 2\Gamma_1^2. \tag{3.23}$$

Then integrating (3.23) with respect to t and using (3.15) and (3.4) imply that for all $t \in (0, T_{\max} - \tau)$,

$$\begin{aligned}\int_t^{t+\tau}\int_\Omega u_{xx}^2(\cdot,s)\mathrm{d}x\mathrm{d}s &\le \frac{2\gamma_1^2M_0^2}{d_1^2}\int_t^{t+\tau}\int_\Omega w^2 + \frac{1}{d_1}\int_\Omega u_x^2(\cdot,t) + \frac{2b_1^2M_0^2M_4(\xi^6+1)}{d_1^2} + \frac{2\Gamma_1^2}{d_1}\\ &\le \frac{2\gamma_1^2M_0^2M_6}{d_1^2} + \frac{3\Gamma_1^2}{d_1} + \frac{2b_1^2M_0^2M_4(\xi^6+\xi^2+1)}{d_1^2}\\ &=: M_7,\end{aligned}$$

which entails (3.22) immediately. Then the proof of Lemma 3.4 is completed. □

Lemma 3.5 *Let* (u, v, w) *be the solution of the system* (1.3) *obtained in* Lemma 2.1. *Then there exists a positive constant* M_8 *such that for all* $t \in (0, T_{\max})$*,*

$$\int_\Omega v_x^2(\cdot,t) \le M_8, \textit{ for all } t \in (0, T_{\max} - \tau). \tag{3.24}$$

Proof Multiplying the second equation of (1.3) by $-v_{xx}$, integrating the result over Ω, and using Hölder inequality and $\|u(\cdot,t)\|_{L^\infty} + \|v(\cdot,t)\|_{L^2} + \|v(\cdot,t)\|_{L^\infty} \le k_1$, one obtains

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega v_x^2 + 2d_2\int_\Omega v_{xx}^2 =& 2\xi\int_\Omega vu_{xx}v_{xx}\\ &+2\xi\int_\Omega v_xu_xv_{xx} + 2\theta_1\int_\Omega vv_{xx}\\ &+2b_2\int_\Omega vwv_{xx} - 2\int_\Omega uvv_{xx}\\ &\le 2\xi k_1\|u_{xx}\|_{L^2}\|v_{xx}\|_{L^2} + 2\xi\|v_xu_x\|_{L^2}\|v_{xx}\|_{L^2}\\ &+2k_1(\theta_1+k_1)\|v_{xx}\|_{L^2}|\Omega|^{\frac12} + 2b_2k_1\|w\|_{L^2}\|v_{xx}\|_{L^2}\\ &\le d_2\|v_{xx}\|_{L^2}^2 + \frac{4\xi^2k_1^2}{d_2}\|u_{xx}\|_{L^2}^2 + \frac{4\xi^2}{d_2}\|v_xu_x\|_{L^2}^2 + \frac{4b_2^2k_1^2}{d_2}\|w\|_{L^2}^2\\ &\quad+\frac{4k_1^2(\theta_1+k_1)^2|\Omega|}{d_2},\end{aligned}$$

which yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega v_x^2 + d_2\int_\Omega v_{xx}^2 \le& \frac{4\xi^2k_1^2}{d_2}\|u_{xx}\|_{L^2}^2 + \frac{4\xi^2}{d_2}\|v_xu_x\|_{L^2}^2 + \frac{4b_2^2k_1^2}{d_2}\|w\|_{L^2}^2\\ &+\frac{4k_1^2(\theta_1+k_1)^2|\Omega|}{d_2}.\end{aligned} \tag{3.25}$$

Furthermore, choosing $q = 4$ in Lemma 2.4 and using Hölder inequality and Gagliardo–Nirenberg inequality, we derive

$$\frac{4\xi^2}{d_2}\|v_x u_x\|_{L^2}^2 \le \frac{4\xi^2}{d_2}\|u_x\|_{L^4}^2\|v_x\|_{L^4}^2 \le k_2\|v_{xx}\|_{L^2}\|v\|_{L^\infty} + k_2\|v\|_{L^\infty}^2 \le \frac{d_2}{2}\|v_{xx}\|_{L^2}^2 + k_3, \tag{3.26}$$

and

$$\int_\Omega v_x^2 = \|v_x\|_{L^2}^2 \le k_4\left(\|v_{xx}\|_{L^2}\|v\|_{L^2} + \|v\|_{L^2}^2\right) \le \frac{d_2}{2}\|v_{xx}\|_{L^2}^2 + k_5. \tag{3.27}$$

Substituting (3.26) and (3.27) into (3.25), one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega v_x^2 + \int_\Omega v_x^2 \le \frac{4\xi^2k_1^2}{d_2}\|u_{xx}\|_{L^2}^2 + \frac{4b_2^2k_1^2}{d_2}\|w\|_{L^2}^2 + k_6, \tag{3.28}$$

with $k_6 = k_3 + k_5 + \frac{4k_1^2(\theta_1+k_1)^2|\Omega|}{d_2}$. Letting

$$h(t) := \frac{4\xi^2k_1^2}{d_2}\|u_{xx}\|_{L^2}^2 + \frac{4b_2^2k_1^2}{d_2}\|w\|_{L^2}^2 + k_6$$

and then using Lemma 3.4 and Lemma 3.3, we have

$$\begin{aligned}\int_t^{t+\tau} h(s)ds = \frac{4\xi^2k_1^2}{d_2}\int_t^{t+\tau}\int_\Omega u_{xx}^2(\cdot,s)\mathrm{d}x\mathrm{d}s \\ + \frac{4b_2^2k_1^2}{d_2}\int_t^{t+\tau}\int_\Omega w^2(\cdot,s)\mathrm{d}x\mathrm{d}s + k_6\tau \le k_7.\end{aligned} \tag{3.29}$$

Applying Lemma 2.5 to (3.28) and using (3.29), one gets (3.24). Then, we complete the proof of Lemma 3.5. □

Lemma 3.6 *Let (u, v, w) be the solution of the system (1.3) obtained in* Lemma 2.1. *Then it holds that*

$$\|w(\cdot,t)\|_{L^4} \le M_9, \quad \textit{for all } t \in (0, T_{\max}), \tag{3.30}$$

where $M_9 > 0$ is a constant independent of t.

Proof We multiply the third equation of (1.3) by w^3, integrate the results over Ω and use Young's inequality with the boundedness of $\|u(\cdot,t)\|_{L^\infty}$ and $\|v(\cdot,t)\|_{L^\infty}$ to derive

$$\begin{aligned}\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega w^4 + 3\int_\Omega w^2w_x^2 &= 3\chi\int_\Omega w^3(\phi_u u_x\cdot w_x + \phi_v v_x\cdot w_x) + \int_\Omega w^4(v+\gamma_2 u - \theta_2) \\ &\le 3\chi\int_\Omega w^3(|\phi_u||u_x| + |\phi_v||v_x|)|w_x| + k_1\int_\Omega w^4 \\ &\le \frac{3}{2}\int_\Omega w^2w_x^2 + \frac{3\chi^2}{2}\int_\Omega w^4(|\phi_u||u_x| + |\phi_v||v_x|)^2 + k_1\int_\Omega w^4,\end{aligned}$$

which, together with the basic inequality $(y+z)^2 \le 2(y^2+z^2)$ and the fact $\frac{1}{4}\int_\Omega |(w^2)_x|^2 = \int_\Omega w^2 w_x^2$, gives

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega w^4 + \int_\Omega w^4 + \frac{3}{2}\int_\Omega |(w^2)_x|^2 \\
&\le 12\chi^2 \int_\Omega w^4(\phi_u^2 u_x^2 + \phi_v^2 v_x^2) + (4k_1+1)\int_\Omega w^4 \\
&\le \|w\|_{L^\infty}^4 \Big(12\chi^2 \|\phi_u\|_{L^\infty}^2 \|u_x\|_{L^2}^2 + \|\phi_v\|_{L^\infty}^2 \|v_x\|_{L^2}^2 + (4k_1+1)|\Omega|\Big) \\
&\le k_2 \|w\|_{L^\infty}^4,
\end{aligned} \tag{3.31}
$$

where we have used Hölder inequality, (3.19) and (3.24) as well as (3.4). By Gagliardo–Nirenberg inequality, Young's inequality and (2.3), one has

$$
\begin{aligned}
k_2\|w\|_{L^\infty}^4 = k_2\|w^2\|_{L^\infty}^2 &\le k_3 \|(w^2)_x\|_{L^2}^{\frac{8}{5}} \|w^2\|_{L^{\frac{1}{2}}}^{\frac{2}{5}} + k_3\|w^2\|_{L^{\frac{1}{2}}}^2 \\
&= k_3 \|(w^2)_x\|_{L^2}^{\frac{8}{5}} \|w\|_{L^1}^{\frac{4}{5}} + k_3 \|w\|_{L^1}^4 \\
&\le \frac{3}{2}\|(w^2)_x\|_{L^2}^2 + k_4,
\end{aligned}
$$

which, substituted into (3.31), gives

$$
\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega w^4 + \int_\Omega w^4 \le k_4,
$$

and then (3.30) follows by Grönwall's inequality. Hence, the proof of Lemma 3.6 is completed. □

Lemma 3.7 *Let* (u, v, w) *be the solution of the system* (1.3) *obtained in* Lemma 2.1. *Then there exists a constant* $M_{10} > 0$ *independent of* t *such that*

$$
\|w(\cdot,t)\|_{L^\infty} \le M_{10}, \quad \text{for all } t \in (0, T_{\max}). \tag{3.32}
$$

Proof Applying the variation-of-constants formula to the third equation of (1.3) and using the well-known semigroup estimates, we have

$$
\begin{aligned}
\|w(\cdot,t)\|_{L^\infty} \le k_1\|w_0\|_{L^\infty} &+ k_2\int_0^t e^{-(\lambda_1+1)(t-s)}(1+(t-s)^{-\frac{7}{8}})\|\phi(u,v)_x w\|_{L^{\frac{4}{3}}} ds \\
&+ k_3\int_0^t e^{-(\lambda_1+1)(t-s)}(1+(t-s)^{-\frac{1}{4}})\|(v+\gamma_2 u + 1 - \theta_2)w\|_{L^2} ds \\
\le & k_1\|w_0\|_{L^\infty} + I_1 + I_2.
\end{aligned} \tag{3.33}
$$

Noting the facts $\|w(\cdot,t)\|_{L^4} \le M_9$, (3.19), (3.24) and (3.4) and using Hölder inequality, one has

$$\begin{aligned}\|\phi(u,v)_x w\|_{L^{\frac{4}{3}}} =&\|(\phi_u u_x+\phi_v v_x)w\|_{L^{\frac{4}{3}}}\\ \le&\|\phi_u u_x+\phi_v v_x\|_{L^2}\|w\|_{L^4}\\ \le&\frac{M_9^2}{2}+\|\phi_u\|_{L^\infty}^2\|u_x\|_{L^2}^2+\|\phi_v\|_{L^\infty}^2\|v_x\|_{L^2}^2\\ \le&\frac{M_9^2}{2}+\gamma^2(M_8+\Gamma_1^2)=:k_4,\end{aligned}$$

and hence,

$$\begin{aligned}I_1 :=& k_2\int_0^t e^{-(\lambda_1+1)(t-s)}(1+(t-s)^{-\frac{7}{8}})\|\phi(u,v)_x w\|_{L^{\frac{4}{3}}}ds\\ \le& k_2k_4\int_0^t e^{-(\lambda_1+1)(t-s)}(1+(t-s)^{-\frac{7}{8}})ds\\ \le& k_5.\end{aligned} \tag{3.34}$$

On the other hand, using Hölder inequality and the boundedness of u, v and $\|w\|_{L^4}$, we can find a constant $k_6>0$ such that

$$\|(v+\gamma_2 u-\theta_2+1)w\|_{L^2} \le \|v+\gamma_2 u-\theta_2+1\|_{L^4}\|w\|_{L^4} \le k_6,$$

and hence,

$$\begin{aligned}I_2 :=&k_3\int_0^t e^{-(\lambda_1+1)(t-s)}(1+(t-s)^{-\frac{1}{4}})\|(v+\gamma_2 u-\theta_2-\alpha_2 w+1)w\|_{L^2}ds\\ \le&k_3k_6\int_0^t e^{-(\lambda_1+1)(t-s)}(1+(t-s)^{-\frac{1}{4}})ds\\ \le&k_7.\end{aligned} \tag{3.35}$$

Substituting (3.34) and (3.35) into (3.33) gives (3.32), and hence, the proof of Lemma 3.7 is completed. □

3.3 Proof of Theorem 1.1

Noting (2.1) and (3.4), we derive

$$\|u(\cdot,t)\|_{W^{1,2}} \le k_1, \quad \text{for all } t\in(0,T_{\max}). \tag{3.36}$$

And the combination of (3.9) and (3.24) gives

$$\|v(\cdot,t)\|_{W^{1,2}} \le k_2, \quad \text{for all } t\in(0,T_{\max}). \tag{3.37}$$

Then combining (3.36), (3.37) and (3.32) and using Lemma 2.1, we directly prove Theorem 1.1.

4 Global Stability: Proof of Theorem 1.3

In this section, we use Lyapunov functionals and LaSalle's invariant principle to establish global stability of constant steady states for the system (1.3).

4.1 Case of Prey-Only

In this subsection, we shall study the global stability of (1, 0, 0) (i.e., prey-only steady state) provided $\theta_1 > 1$ and $\theta_2 > \gamma_2$. To this end, we introduce the energy functional as below:

$$\mathcal{F}_1(t) := \mathcal{F}_1(u, v, w) = \alpha_1 \int_\Omega (u - 1 - \ln u) + b_1 \int_\Omega v + b_1 b_2 \int_\Omega w,$$

where

$$\alpha_1 := \begin{cases} 1, & \text{if } \gamma_1 = \gamma_2 = 0, \\ \min\left\{\frac{\theta_1 - 1}{4}, \frac{b_1 b_2(\theta_2 - \gamma_2)}{4\gamma_1}\right\}, & \text{if } \gamma_1, \gamma_2 > 0. \end{cases}$$

Lemma 4.1 *Let* (u, v, w) *be the solution of* (1.3) *obtained in* Theorem 1.1. *Then if* $\theta_1 > 1$ *and* $\theta_2 > \gamma_2$*, one has*

$$\lim_{t\to\infty} (\|u(\cdot, t) - 1\|_{L^\infty} + \|v(\cdot, t)\|_{L^\infty} + \|w(\cdot, t)\|_{L^\infty}) = 0.$$

Proof Letting $g(z) = z - z_* \ln z$ and noting $g'(z_*) = 0$, we use Taylor's expansion to obtain that for all $z, z_* > 0$

$$z - z_* - z_* \ln \frac{z}{z_*} = g(z) - g(z_*) = \frac{1}{2} g''(\tilde{z})(z - z_*)^2 = \frac{z_*}{2\tilde{z}^2}(z - z_*)^2 \geq 0, \quad (4.1)$$

where $\tilde{z}$ is between z and z_*. Choosing $z = u$ and $z_* = 1$, from (4.1) one has

$$u - 1 - \ln u = \frac{1}{2y_1^2}(u - 1)^2 \geq 0, \quad (4.2)$$

where y_1 is between u and 1. Hence, using (4.2) and the definition of $\mathcal{F}_1(t)$, we derive that $\mathcal{F}_1(t) \geq 0$ and $\mathcal{F}_1(t) = 0$ if and only if $(u, v, w) = (1, 0, 0)$. Moreover, some calculations give

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_1(t) &= \alpha_1 \int_\Omega \frac{u-1}{u} u_t + b_1 \int_\Omega v_t + b_1 b_2 \int_\Omega w_t \\ &= -\alpha_1 d_1 \int_\Omega \frac{u_x^2}{u^2} - \alpha_1 \int_\Omega (u-1)^2 - \alpha_1 b_1 \int_\Omega uv - \alpha_1 \gamma_1 \int_\Omega uw \\ &\quad + b_1 \int_\Omega (u - \theta_1 + \alpha_1) v + \int_\Omega (b_1 b_2 \gamma_2 u - b_1 b_2 \theta_2 + \alpha_1 \gamma_1) w. \end{aligned} \tag{4.3}$$

Case 1: $\gamma_1 = \gamma_2 = 0$. In this case, substituting $\alpha_1 = 1$ and $\gamma_1 = \gamma_2 = 0$ into (4.3), one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_1(t) = -d_1 \int_\Omega \frac{u_x^2}{u^2} - \int_\Omega (u-1)^2 - b_1(\theta_1 - 1) \int_\Omega v - b_1 b_2 \theta_2 \int_\Omega w,$$

which, along with $\theta_1 > 1$, gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_1(t) \le 0.$$

Case 2: $\gamma_1, \gamma_2 > 0$. Noting the facts $\limsup_{t\to\infty} u(x,t) \le 1$ in (2.2) and $\theta_1 > 1$ as well as $\theta_2 > \gamma_2$, for $\varepsilon_1 := \min\left\{\frac{\theta_1 - 1}{2}, \frac{\theta_2 - \gamma_2}{2\gamma_2}\right\}$, we can find a $t_1 > 0$ such that

$$u(x,t) \le 1 + \varepsilon_1 \text{ for any } x \in \bar{\Omega} \text{ and } t > t_1,$$

which, together with $\alpha_1 := \min\left\{\frac{\theta_1 - 1}{4}, \frac{b_1 b_2(\theta_2 - \gamma_2)}{4\gamma_1}\right\}$, entails

$$\begin{aligned} u - \theta_1 + \alpha_1 &\le 1 + \varepsilon_1 + \alpha_1 - \theta_1 \\ &\le 1 + \frac{\theta_1 - 1}{2} - \theta_1 + \frac{\theta_1 - 1}{4} \\ &= -\frac{\theta_1 - 1}{4} < 0 \text{ for all } t > t_1, \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} b_1 b_2 \gamma_2 u - b_1 b_2 \theta_2 + \alpha_1 \gamma_1 &\le b_1 b_2 \gamma_2 (1 + \varepsilon_1) - b_1 b_2 \theta_2 + \alpha_1 \gamma_1 \\ &\le b_1 b_2 (\gamma_2 - \theta_2) + b_1 b_2 \gamma_2 \frac{\theta_2 - \gamma_2}{2\gamma_2} + \frac{b_1 b_2 (\theta_2 - \gamma_2)}{4\gamma_1} \gamma_1 \\ &= -\frac{b_1 b_2(\theta_2 - \gamma_2)}{4} < 0 \text{ for all } t > t_1. \end{aligned} \tag{4.5}$$

The combination of (4.3), (4.4) and (4.5) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_1(t) \le -\alpha_1 d_1 \int_\Omega \frac{u_x^2}{u^2} - \alpha_1 \int_\Omega (u-1)^2 - \alpha_1 b_1 \int_\Omega uv - \alpha_1\gamma_1 \int_\Omega uw \\ - \frac{b_1(\theta_1 - 1)}{4}\int_\Omega v - \frac{b_1 b_2(\theta_2-\gamma_2)}{4}\int_\Omega w,$$

and thus, $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_1(t) \le 0$ for all $t > t_1$.

Moreover, all the above cases indicate that $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_1(t) = 0$ iff $(u, v, w) = (1, 0, 0)$. Hence, by LaSalle's invariance principle (e.g., see (Shankar 1999, pp.198–199, Theorem 5.24)), we know that (u, v, w) converges to $(1, 0, 0)$ in L^∞ as $t \to \infty$. □

4.2 Case of Semi-coexistence

In this subsection, we first study the global stability of semi-coexistence $E_{12} := \left(\theta_1, \frac{1-\theta_1}{b_1}, 0\right)$ based on the following energy functional:

$$\mathcal{F}_2(t) := \mathcal{F}_2(u, v, w) = \int_\Omega \left(u - \theta_1 - \theta_1 \ln\frac{u}{\theta_1}\right) + b_1 \int_\Omega \left(v - V - V\ln\frac{v}{V}\right) \\ + b_1 b_2 \int_\Omega w,$$

where $V := \frac{1-\theta_1}{b_1}$.

Lemma 4.2 *Let (u, v, w) be the solution of (1.3) obtained in* Theorem 1.1. *If $0 < \theta_1 < 1$ and*

$$\theta_2 > \frac{\gamma_1}{b_1 b_2}\theta_1 - \frac{\theta_1}{b_1} + \frac{1}{b_1} + \frac{\max\{b_1 b_2\gamma_2 - \gamma_1, 0\}}{b_1 b_2} =: \ell_1, \tag{4.6}$$

then there exists $\xi_0 > 0$ such that for all $\xi \in (0, \xi_0)$, it holds that

$$\lim_{t\to\infty}\left(\|u(\cdot,t) - \theta_1\|_{L^\infty} + \left\|v(\cdot,t) - \frac{1-\theta_1}{b_1}\right\|_{L^\infty} + \|w(\cdot,t)\|_{L^\infty}\right) = 0. \tag{4.7}$$

Proof Using (4.1), we can check that $\mathcal{F}_2(t) \ge 0$ and $\mathcal{F}_2(t) = 0$ iff $(u, v, w) = \left(\theta_1, \frac{1-\theta_1}{b_1}, 0\right)$. Applying the equations of (1.3) and using the fact $1 = \theta_1 + b_1 V$, one has

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_2(t) &= \int_\Omega \frac{u-\theta_1}{u}u_t + b_1\int_\Omega \frac{v-V}{v}v_t + b_1b_2\int_\Omega w_t \\ &= -\theta_1 d_1\int_\Omega \frac{u_x^2}{u^2} - b_1d_2V\int_\Omega \frac{v_x^2}{v^2} + b_1\xi V\int_\Omega \frac{u_x\cdot v_x}{v} \\ &\quad + \int_\Omega (u-\theta_1)(1-u-b_1v-\gamma_1 w) \\ &\quad + b_1\int_\Omega (v-V)(u-b_2w-\theta_1) + b_1b_2\int_\Omega w(v+\gamma_2 u-\theta_2) \\ &= -\int_\Omega Y_1^T B_1 Y_1 + \int_\Omega h_1(x,t)w - \int_\Omega (u-\theta_1)^2,\end{aligned} \tag{4.8}$$

where

$$Y_1 = \begin{pmatrix} \frac{u_x}{v} \\ \frac{v_x}{v}\end{pmatrix}, \quad B_1 := \begin{pmatrix} \theta_1 d_1 & -\frac{b_1V\xi u}{2} \\ -\frac{b_1V\xi u}{2} & b_1d_2V\end{pmatrix}$$

and

$$h_1(x,t) := (b_1b_2\gamma_2-\gamma_1)u + b_1b_2V + \gamma_1\theta_1 - b_1b_2\theta_2.$$

After some calculations, one can check that B_1 is a positive definite matrix provided that

$$\xi^2(1-\theta_1)\|u\|_{L^\infty}^2 < 4\theta_1 d_1 d_2. \tag{4.9}$$

Since $0<\theta_1<1$ and $\|u\|_{L^\infty}$ is independent of ξ, we can find an appropriate constant $\xi_0>0$ such that if $0<\xi<\xi_0$, then (4.9) holds, which entails us to find a constant $k_1>0$ such that

$$-\int_\Omega Y_1^T B_1 Y_1 \le -k_1\int_\Omega \left(\frac{u_x^2}{u^2}+\frac{v_x^2}{v^2}\right). \tag{4.10}$$

Next, we shall show that under condition (4.6), there exists a constant $k_2>0$ such that

$$\int_\Omega h_1(x,t)w \le -\frac{b_1b_2k_2}{2}\int_\Omega w. \tag{4.11}$$

We divide our proof into two cases: $b_1b_2\gamma_2\le\gamma_1$ and $b_1b_2\gamma_2>\gamma_1$.
Case 1: $b_1b_2\gamma_2\le\gamma_1$. In this case, from (4.6), one has

$$\theta_2 > \frac{\gamma_1}{b_1b_2}\theta_1 - \frac{\theta_1}{b_1} + \frac{1}{b_1},$$

which indicates

$$\begin{aligned} h_1(x,t) &= (b_1b_2\gamma_2 - \gamma_1)u + b_1b_2V + \gamma_1\theta_1 - b_1b_2\theta_2 \\ &\leq b_1b_2\frac{1-\theta_1}{b_1} + \gamma_1\theta_1 - b_1b_2\theta_2 \\ &= -b_1b_2\left(\theta_2 - \frac{\gamma_1}{b_1b_2}\theta_1 + \frac{\theta_1}{b_1} - \frac{1}{b_1}\right) < 0. \end{aligned} \tag{4.12}$$

Case 2: $b_1b_2\gamma_2 > \gamma_1$. For this case,

(4.6) and the fact $\limsup_{t\to\infty} u(x,t) \leq 1$ in (2.2) can guarantee that for the positive constant $\varepsilon_2 := \frac{b_1b_2}{2(b_1b_2\gamma_2-\gamma_1)}(\theta_2 - \ell_1)$, there exists a constant $t_2 > 0$ such that

$$u(x,t) \leq 1 + \varepsilon_2 \text{ for any } x \in \bar{\Omega} \text{ and } t > t_2,$$

and hence,

$$\begin{aligned} h_1(x,t) &= (b_1b_2\gamma_2 - \gamma_1)u + b_1b_2V + \gamma_1\theta_1 - b_1b_2\theta_2 \\ &\leq (b_1b_2\gamma_2 - \gamma_1) + \frac{b_1b_2}{2}(\theta_2 - \ell_1) + b_1b_2\frac{1-\theta_1}{b_1} + \gamma_1\theta_1 - b_1b_2\theta_2 \\ &= -\frac{b_1b_2}{2}(\theta_2 - \ell_1) < 0. \end{aligned} \tag{4.13}$$

Combining (4.12) with (4.13) and letting

$$k_2 = \theta_2 - \frac{\gamma_1}{b_1b_2}\theta_1 + \frac{\theta_1}{b_1} - \frac{1}{b_1} - \frac{\max\{b_1b_2\gamma_2 - \gamma_1, 0\}}{b_1b_2},$$

we directly obtain (4.11). Then substituting (4.10) and (4.11) into (4.8), one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_2(t) \leq -k_1\int_\Omega\left(\frac{u_x^2}{u^2} + \frac{v_x^2}{v^2}\right) - \int_\Omega (u-\theta_1)^2 - \frac{b_1b_2k_2}{2}\int_\Omega w \leq 0,$$

and "=" holds iff $(u, v_x, w) = (\theta_1, 0, 0)$. Furthermore, the fact $v_x = 0$ entails $v = \tilde{v}$, where $\tilde{v}$ is a positive constant. Hence, $(u, v, w) = (\theta_1, \tilde{v}, 0)$ satisfies

$$0 = \theta_1(1 - \theta_1 - b_1\tilde{v}),$$

which yields $\tilde{v} = \frac{1-\theta_1}{b_1} = V$. Then $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_2(t) = 0$ implies $(u, v, w) = \left(\theta_1, \frac{1-\theta_1}{b_1}, 0\right)$.

Applying LaSalle's invariance principle, one obtains that the semi-coexistence $\left(\theta_1, \frac{1-\theta_1}{b_1}, 0\right)$ is globally asymptotically stable, which gives (4.7). □

Next, we shall study the global stability of the semi-coexistence steady state $\left(\frac{\theta_2}{\gamma_2}, 0, \frac{\gamma_2-\theta_2}{\gamma_1\gamma_2}\right)$ based on the following energy functional:

$$\begin{aligned}\mathcal{F}_3(t) := \mathcal{F}_3(u,v,w) =& \frac{b_1b_2\gamma_2}{\gamma_1}\int_\Omega \left(u - \frac{\theta_2}{\gamma_2} - \frac{\theta_2}{\gamma_2}\ln\frac{u\gamma_2}{\theta_2}\right) + b_1b_2\int_\Omega \left(w - W - W\ln\frac{w}{W}\right)\\ &+ b_1\int_\Omega v + \int_\Omega v^2,\end{aligned} \tag{4.14}$$

where $W := \frac{\gamma_2-\theta_2}{\gamma_1\gamma_2}$.

Lemma 4.3 *Let* (u, v, w) *be the solution of* (1.3) *obtained in* Theorem 1.1. *Then if* $\theta_1 > 1$, $\theta_2 < \gamma_2$ *and*

$$\theta_2 < \frac{\gamma_1\gamma_2}{b_1b_2\gamma_2 + b_2}\theta_1 + \frac{b_2\gamma_2}{b_1b_2\gamma_2 + b_2} + \frac{\gamma_2\min\{b_1b_2\gamma_2 - \gamma_1, 0\}}{b_1b_2\gamma_2 + b_2} =: \ell_2, \tag{4.15}$$

there exist $\xi_1 > 0$ *and* $\chi_1 > 0$ *such that if* $\xi \in (0, \xi_1)$ *and* $\chi \in (0, \chi_1)$, *the following holds:*

$$\lim_{t\to\infty}\left(\left\|u(\cdot,t) - \frac{\theta_2}{\gamma_2}\right\|_{L^\infty} + \|v(\cdot,t)\|_{L^\infty} + \left\|w(\cdot,t) - \frac{\gamma_2-\theta_2}{\gamma_1\gamma_2}\right\|_{L^\infty}\right) = 0.$$

Proof Applying (4.1), we can verify that $\mathcal{F}_3(t) \geq 0$ and "=" holds iff $(u, v, w) = \left(\frac{\theta_2}{\gamma_2}, 0, \frac{\gamma_2-\theta_2}{\gamma_1\gamma_2}\right)$. Moreover, by the definition of $\mathcal{F}_3(t)$ in (4.14), we utilize the equations of (1.3) and the fact $1 = \frac{\theta_2}{\gamma_2} + \gamma_1 W$ to derive

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_3(t) =& \frac{b_1b_2\gamma_2}{\gamma_1}\int_\Omega \frac{u - \frac{\theta_2}{\gamma_2}}{u}u_t + b_1b_2\int_\Omega \frac{w-W}{w}w_t + b_1\int_\Omega v_t + 2\int_\Omega vv_t\\ =& -\frac{b_1b_2d_1\theta_2}{\gamma_1}\int_\Omega \frac{u_x^2}{u^2} - b_1b_2W\int_\Omega \frac{w_x^2}{w^2} + b_1b_2W\chi\int_\Omega \frac{\phi_u u_x\cdot w_x + \phi_v v_x\cdot w_x}{w}\\ &+ \frac{b_1b_2\gamma_2}{\gamma_1}\int_\Omega\left(u - \frac{\theta_2}{\gamma_2}\right)(1 - u - b_1v - \gamma_1 w) + b_1b_2\int_\Omega (w-W)(v + \gamma_2 u - \theta_2)\\ &+ b_1\int_\Omega v(u - b_2w - \theta_1) - 2d_2\int_\Omega v_x^2 + 2\xi\int_\Omega vu_x\cdot v_x + 2\int_\Omega v^2(u - b_2w - \theta_1)\\ =& -\int_\Omega Y_2^T B_2 Y_2 - \frac{b_1b_2\gamma_2}{\gamma_1}\int_\Omega\left(u - \frac{\theta_2}{\gamma_2}\right)^2 + b_1\int_\Omega vh_2(x,t) + 2\int_\Omega v^2h_3(x,t),\end{aligned} \tag{4.16}$$

where

$$Y_2 = \begin{pmatrix} \frac{u_x}{u}\\ \frac{v_x}{v}\\ \frac{w_x}{w}\end{pmatrix}, \quad B_2 := \begin{pmatrix} \frac{b_1b_2d_1\theta_2}{\gamma_1} & -\xi uv^2 & -\frac{b_1b_2\chi Wu\phi_u}{2}\\ -\xi uv^2 & 2d_2v^2 & -\frac{b_1b_2\chi Wv\phi_v}{2}\\ -\frac{b_1b_2\chi Wu\phi_u}{2} & -\frac{b_1b_2\chi Wv\phi_v}{2} & b_1b_2W, \end{pmatrix}$$

and

$$h_2(x,t) := (1 - \frac{b_1 b_2 \gamma_2}{\gamma_1})u - \theta_1 - b_2 W + \frac{b_1 b_2 \theta_2}{\gamma_1} \quad \text{and} \quad h_3(x,t) := u - b_2 w - \theta_1. \tag{4.17}$$

After some calculations, we can check that B_2 is positive definite if

$$\begin{vmatrix} \frac{b_1 b_2 d_1 \theta_2}{\gamma_1} & -\xi u v^2 \\ -\xi u v^2 & 2 d_2 v^2 \end{vmatrix} = \left(\frac{2 b_1 b_2 d_1 d_2 \theta_2}{\gamma_1} - \xi^2 u^2 v^2 \right) v^2 > 0, \tag{4.18}$$

and

$$\begin{aligned} |B_2| = & b_1 b_2 W v^2 \left(\frac{2 b_1 b_2 d_1 d_2 \theta_2}{\gamma_1} - \xi^2 u^2 v^2 \right) \\ & - \frac{b_1^2 b_2^2 \chi^2 W^2 v^2}{4} \left(2 u v \phi_u \phi_v \xi u + 2 u^2 \phi_u^2 d_2 + \phi_v^2 \frac{d_1 \theta_2 b_1 b_2}{\gamma_1} \right) > 0. \end{aligned} \tag{4.19}$$

Indeed, it can be verified that (4.18) and (4.19) hold if

$$2 b_1 b_2 d_1 d_2 \theta_2 > \xi^2 \gamma_1 M_0^2 K_0^2 + \chi^2 M_*^c, \tag{4.20}$$

where M_0 and K_0 are defined in (1.4) and (1.5), respectively, and

$$M_*^c := \frac{b_1 b_2 (\gamma_2 - \theta_2)}{4 \gamma_2} \left(2 \xi M_0^2 K_0 \|\phi_v\|_{L^\infty} \|\phi_u\|_{L^\infty} + 2 d_2 M_0^2 \|\phi_u\|_{L^\infty}^2 + \frac{d_1 \theta_2 b_1 b_2}{\gamma_1} \|\phi_v\|_{L^\infty}^2 \right).$$

Since $M_0 \geq \|u\|_{L^\infty}$ is independent of ξ, χ and $K_0 \geq \|v\|_{L^\infty}$ is independent of χ, for any given $\phi(u,v) \in C^2([0,\infty))$, we can obtain the upper bounds of $\|\phi_u\|_{L^\infty}$ and $\|\phi_v\|_{L^\infty}$ are independent of χ. Then there exist $\xi_1 > 0$ and $\chi_1 > 0$ such that (4.20) holds if $\xi \in (0, \xi_1)$ and $\chi \in (0, \chi_1)$. Hence, we can find a constant $k_1 > 0$ such that

$$-\int_\Omega Y_2^T B_2 Y_2 \leq -k_1 \int_\Omega \left(\frac{u_x^2}{u^2} + \frac{v_x^2}{v^2} + \frac{w_x^2}{w^2} \right). \tag{4.21}$$

Next, we shall show $h_3(x,t) < 0$ and $h_2(x,t) < 0$, respectively. Noting $\theta_1 > 1$ and (4.15), we can take

$$\varepsilon_3 := \begin{cases} \frac{\theta_1 - 1}{2}, & \text{if } \gamma_1 \leq b_1 b_2 \gamma_2, \\ \min\left\{ \frac{\theta_1 - 1}{2}, \frac{(\ell_2 - \theta_2)(b_1 b_2 \gamma_2 + b_2)}{2(\gamma_1 - b_1 b_2 \gamma_2)\gamma_2} \right\}, & \text{if } \gamma_1 > b_1 b_2 \gamma_2. \end{cases}$$

From (2.2), we can find a constant $t_3 > 0$ such that

$$u(x,t) \leq 1 + \varepsilon_3 \text{ for all } x \in \bar{\Omega} \text{ and } t > t_3, \tag{4.22}$$

and hence,

$$h_3(x,t) := u - b_2 w - \theta_1 \le 1 + \varepsilon_3 - \theta_1 \le \frac{\theta_1 - 1}{2} + 1 - \theta_1 = -\frac{\theta_1 - 1}{2} < 0. \tag{4.23}$$

As for h_2, we need to distinguish in two cases:

Case 1: $\gamma_1 \le b_1 b_2 \gamma_2$. This case means $1 - \frac{b_1 b_2 \gamma_2}{\gamma_1} \le 0$; thus, it follows from (4.15) and (4.17) that

$$\begin{aligned} h_2(x,t) &\le -\theta_1 - \frac{b_2\gamma_2 - b_2\theta_2}{\gamma_1\gamma_2} + \frac{b_1 b_2 \theta_2}{\gamma_1} \\ &= -\frac{b_1 b_2 \gamma_2 + b_2}{\gamma_1 \gamma_2}(\ell_2 - \theta_2) < 0. \end{aligned} \tag{4.24}$$

Case 2: $\gamma_1 > b_1 b_2 \gamma_2$. In this case, we have $1 - \frac{b_1 b_2 \gamma_2}{\gamma_1} > 0$, which along with (4.17), (4.22) and (4.15) gives

$$\begin{aligned} h_2(x,t) &\le (1 - \frac{b_1 b_2 \gamma_2}{\gamma_1}) + (1 - \frac{b_1 b_2 \gamma_2}{\gamma_1}) \frac{(\ell_2 - \theta_2)(b_1 b_2 \gamma_2 + b_2)}{2(\gamma_1 - b_1 b_2 \gamma_2)\gamma_2} - \theta_1 - b_2 W + \frac{b_1 b_2 \theta_2}{\gamma_1} \\ &= \frac{(b_1 b_2 \gamma_2 + b_2)(\ell_2 - \theta_2)}{2\gamma_1\gamma_2} + \frac{\gamma_1 - b_1 b_2 \gamma_2}{\gamma_1} - \theta_1 - \frac{b_2\gamma_2 - b_2\theta_2}{\gamma_1\gamma_2} + \frac{b_1 b_2 \theta_2}{\gamma_1} \\ &= -\frac{b_1 b_2 \gamma_2 + b_2}{2\gamma_1\gamma_2}(\ell_2 - \theta_2) < 0. \end{aligned} \tag{4.25}$$

Then combining (4.23), (4.24) and (4.25), we derive that

$$b_1 \int_\Omega v h_2(x,t) + 2\int_\Omega v^2 h_3(x,t) \le -\frac{b_1 b_2 (b_1\gamma_2 + 1)(\ell_2 - \theta_2)}{2\gamma_1\gamma_2} \int_\Omega v,$$

which, along with (4.21) and (4.16), gives

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_3(t) \le& -k_1 \int_\Omega \left(\frac{u_x^2}{u^2} + \frac{v_x^2}{v^2} + \frac{w_x^2}{w^2} \right) \\ & - \frac{b_1 b_2 \gamma_2}{\gamma_1} \int_\Omega \left(u - \frac{\theta_2}{\gamma_2} \right)^2 - \frac{b_1(b_1 b_2 \gamma_2 + b_2)(\ell_2 - \theta_2)}{2\gamma_1\gamma_2} \int_\Omega v \\ \le& \ 0. \end{aligned}$$

Thus, $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_3(t) = 0$ iff $(u, v, w_x) = \left(\frac{\theta_2}{\gamma_2}, 0, 0\right)$. This indicates $w = \tilde{w}$, where $\tilde{w} > 0$ is a constant. Since $\left(\frac{\theta_2}{\gamma_2}, 0, \tilde{w}\right)$ is a solution of (1.6), then one has

$$\frac{\theta_2}{\gamma_2}\left(1 - \frac{\theta_2}{\gamma_2} - \gamma_1 \tilde{w}\right) = 0,$$

which implies $\tilde{w} = \frac{\gamma_2-\theta_2}{\gamma_1\gamma_2}$. Hence, $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_3(t) = 0$ iff $(u, v, w) = \left(\frac{\theta_2}{\gamma_2}, 0, \frac{\gamma_2-\theta_2}{\gamma_1\gamma_2}\right)$. Then, one obtains that $\left(\frac{\theta_2}{\gamma_2}, 0, \frac{\gamma_2-\theta_2}{\gamma_1\gamma_2}\right)$ is globally asymptotically stable by applying LaSalle's invariance principle. □

4.3 Case of Coexistence

In this subsection, we shall study the global stability of coexistence steady state (u_*, v_*, w_*) defined in (1.7) under the condition (1.8). To this end, we introduce the energy function as follows

$$\mathcal{F}_4(t) := \mathcal{F}_4(u, v, w) = \mathcal{F}_u(t) + b_1\mathcal{F}_v(t) + b_1b_2\mathcal{F}_w(t),$$

where

$$\mathcal{F}_y(t) = \int_\Omega \left(y - y_* - y_* \ln \frac{y}{y_*}\right), \ y = u, v, w.$$

Lemma 4.4 *Let (u, v, w) be the solution of* (1.3) *obtained in* Theorem 1.1. *If* (1.8) *holds and $\gamma_1 = b_1b_2\gamma_2$, then there exist $\xi_2 > 0$ and $\chi_2 > 0$ such that for all $\xi \in (0, \xi_2)$ and $\chi \in (0, \chi_2)$, it holds that*

$$\lim_{t\to\infty} (\|u(\cdot, t) - u_*\|_{L^\infty} + \|v(\cdot, t) - v_*\|_{L^\infty} + \|w(\cdot, t) - w_*\|_{L^\infty}) = 0.$$

Proof Using (4.1), we can check that $\mathcal{F}(t) \geq 0$ and $\mathcal{F}(t) = 0$ iff $(u, v, w) = (u_*, v_*, w_*)$.

Next, we shall show $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_4(t) \leq 0$ under certain conditions for the parameters. In fact, using the first equation of (1.3) and $u_* + b_1v_* + \gamma_1w_* = 1$, we derive

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_u(t) &= \int_\Omega \frac{u-u_*}{u}u_t \\
&= -u_*d_1\int_\Omega \frac{u_x^2}{u^2} + \int_\Omega (u-u_*)(1-u-b_1v-\gamma_1w) \\
&= -u_*d_1\int_\Omega \frac{u_x^2}{u^2} - \int_\Omega (u-u_*)^2 - b_1\int_\Omega (u-u_*)(v-v_*) \\
&\quad - \gamma_1\int_\Omega (u-u_*)(w-w_*).
\end{aligned} \tag{4.26}$$

Applying $u_* - b_2w_* = \theta_1$ and the second equation of (1.3), one has

$$\begin{aligned}
b_1\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_v(t) &= b_1\int_\Omega \frac{v-v_*}{v}v_t \\
&= -b_1v_*d_2\int_\Omega \frac{v_x^2}{v^2} + b_1\xi v_*\int_\Omega \frac{u_x\cdot v_x}{v} + b_1\int_\Omega (v-v_*)(u-b_2w-\theta_1)
\end{aligned}$$

$$= -b_1 v_* d_2 \int_\Omega \frac{v_x^2}{v^2} + b_1 \xi v_* \int_\Omega \frac{u_x \cdot v_x}{v} + b_1 \int_\Omega (v - v_*)(u - u_*) \\ - b_1 b_2 \int_\Omega (v - v_*)(w - w_*). \tag{4.27}$$

Similarly, noting $v_* + \gamma_2 u_* = \theta_2$ and applying the third equation of (1.3), we derive that

$$\begin{aligned} b_1 b_2 \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{F}_w(t) &= b_1 b_2 \int_\Omega \frac{w - w_*}{w} w_t \\ &= -b_1 b_2 w_* \int_\Omega \frac{w_x^2}{w^2} + b_1 b_2 w_* \chi \int_\Omega \frac{\phi_u u_x \cdot w_x + \phi_v v_x \cdot w_x}{w} \\ &\quad + b_1 b_2 \int_\Omega (w - w_*)(v + \gamma_2 u - \theta_2) \\ &= -b_1 b_2 w_* \int_\Omega \frac{w_x^2}{w^2} + b_1 b_2 w_* \chi \int_\Omega \frac{\phi_u u_x \cdot w_x + \phi_v v_x \cdot w_x}{w} \\ &\quad + b_1 b_2 \int_\Omega (w - w_*)(v - v_*) + b_1 b_2 \gamma_2 \int_\Omega (w - w_*)(u - u_*). \end{aligned} \tag{4.28}$$

We combine (4.26), (4.27) and (4.28) and use $b_1 b_2 \gamma_2 - \gamma_1 = 0$ to obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{F}_4(t) &= -u_* d_1 \int_\Omega \frac{u_x^2}{u^2} - b_1 v_* d_2 \int_\Omega \frac{v_x^2}{v^2} - b_1 b_2 w_* \int_\Omega \frac{w_x^2}{w^2} \\ &\quad + b_1 \xi v_* \int_\Omega \frac{u_x \cdot v_x}{v} + b_1 b_2 w_* \chi \int_\Omega \frac{\phi_u u_x \cdot w_x + \phi_v v_x \cdot w_x}{w} - \int_\Omega (u - u_*)^2 \\ &= - \int_\Omega Y_3^T B_3 Y_3 - \int_\Omega (u - u_*)^2, \end{aligned} \tag{4.29}$$

where

$$Y_3 = \begin{pmatrix} \frac{u_x}{u} \\ \frac{v_x}{v} \\ \frac{w_x}{w} \end{pmatrix} \text{ and } B_3 = \begin{pmatrix} u_* d_1 & -\frac{b_1 \xi v_* u}{2} & -\frac{\chi b_1 b_2 w_* \phi_u u}{2} \\ -\frac{b_1 \xi v_* u}{2} & b_1 v_* d_2 & -\frac{\chi b_1 b_2 w_* \phi_v v}{2} \\ -\frac{\chi b_1 b_2 w_* \phi_u u}{2} & -\frac{\chi b_1 b_2 w_* \phi_v v}{2} & b_1 b_2 w_* \end{pmatrix}.$$

After some calculations, one can verify that the matrix B_3 is positive definite if and only if

$$\begin{vmatrix} u_* d_1 & -\frac{b_1 \xi v_* u}{2} \\ -\frac{b_1 \xi v_* u}{2} & b_1 v_* d_2 \end{vmatrix} = \frac{v_* b_1 (4 u_* d_1 d_2 - b_1 v_* \xi^2 u^2)}{4} > 0, \tag{4.30}$$

and

$$
\begin{aligned}
|B_3| = & \frac{b_1^2 b_2 w_*}{4}(4d_1 d_2 u_* v_* - b_1 \xi^2 v_*^2 u^2) \\
& - \frac{b_1^2 b_2 w_* \chi^2}{4}(u_* d_1 b_2 w_* \phi_v^2 v^2 + \xi v_* u w_* \phi_v v \cdot b_1 b_2 \phi_u u + b_1 b_2 v_* d_2 w_* \phi_u^2 u^2) \\
> & \, 0.
\end{aligned} \tag{4.31}
$$

Since $M_0 \geq \|u\|_{L^\infty}$ is independent of ξ, χ and $K_0 \geq \|v\|_{L^\infty}$ is independent of χ (see Remark 1.2), we can find appropriate numbers $\xi_2 > 0$ and $\chi_2 > 0$ such that if $\xi \in (0, \xi_2)$ and $\chi \in (0, \chi_2)$, then

$$
4d_1 d_2 u_* v_* > b_1 v_*^2 M_0^2 \xi^2 + \chi^2 M_*(\xi, u, v),
$$

where

$$
\begin{aligned}
M_*(\xi, u, v) := & \, u_* w_* b_2 d_1 \|\phi_v\|_{L^\infty}^2 K_0^2 + \xi v_* w_* b_1 b_2 \|\phi_v\|_{L^\infty} \|\phi_u\|_{L^\infty} M_0^2 K_0 \\
& + b_1 b_2 v_* w_* d_2 \|\phi_u\|_{L^\infty}^2 K_0^2,
\end{aligned}
$$

which gives (4.30) and (4.31). Hence, there exists a constant $k_1 > 0$ such that (4.29) can be updated as

$$
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_4(t) \leq -k_1 \int_\Omega \left(\frac{u_x^2}{u^2} + \frac{v_x^2}{v^2} + \frac{w_x^2}{w^2} \right) - \int_\Omega (u - u_*)^2 \leq 0. \tag{4.32}
$$

Then (4.32) implies $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_4(t) \leq 0$ and "=" holds iff $(u, v_x, w_x) = (u_*, 0, 0)$, this indicates $v = \tilde{v}_*$ and $w = \tilde{w}_*$, where $\tilde{v}_*$ and $\tilde{w}_*$ are positive constants satisfying

$$
\begin{cases}
0 = u_*(1 - u_* - b_1 \tilde{v}_* - \gamma_1 \tilde{w}_*), \\
0 = \tilde{v}_*(u_* - b_2 \tilde{w}_* - \theta_1), \\
0 = \tilde{w}_*(\tilde{v}_* + \gamma_2 u_* - \theta_2).
\end{cases}
$$

This together with the definition of u_* in (1.7) gives

$$
\tilde{v}_* = \frac{\gamma_1(\theta_2 - \gamma_2 \theta_1) + b_2(\theta_2 - \gamma_2)}{b_2} = v_*,
$$

and

$$
\tilde{w}_* = \frac{b_1(\gamma_2 \theta_1 + \theta_2) + (1 - \theta_1)}{b_2} = w_*.
$$

Therefore, we conclude that $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_4(t) \leq 0$ and $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_4(t) = 0$ iff $(u, v, w) = (u_*, v_*, w_*)$. Then, LaSalle's invariance principle yields that (u_*, v_*, w_*) is globally asymptotically stable. □

4.4 Proof of Theorem 1.3

The combination of Lemmas 4.1–4.4 immediately implies Theorem 1.3.

5 Linear Stability/Instability Analysis

In this section, we shall study the possible pattern formation for the system (1.3). In fact, for the space-absent ordinary differential equation (ODE) system of (1.3)

$$\begin{cases} u_t = u(1-u) - b_1uv - \gamma_1uw, \\ v_t = uv - b_2vw - \theta_1v, \\ w_t = vw + \gamma_2uw - \theta_2w, \end{cases}$$

it has been proved in Hsu et al. (2015) that:

(1) The trivial steady state $E_0 := (0, 0, 0)$ is always linearly unstable.
(2) The prey-only steady state $E_1 := (1, 0, 0)$ is linearly stable if $\theta_1 > 1$ and $\theta_2 > \gamma_2$.
(3) The semi-coexistence steady state $E_{12} := \left(\theta_1, \frac{1-\theta_1}{b_1}, 0\right)$ exists if $\theta_1 < 1$ and it is linearly stable provided

$$\theta_2 > \frac{b_1\gamma_2 - 1}{b_1}\theta_1 + \frac{1}{b_1}. \tag{5.1}$$

(4) The semi-coexistence steady state $E_{13} := \left(\frac{\theta_2}{\gamma_2}, 0, \frac{\gamma_2-\theta_2}{\gamma_1\gamma_2}\right)$ exists if $\theta_2 < \gamma_2$ and it is linearly stable provided

$$\theta_2 < \frac{\gamma_1\gamma_2}{b_2+\gamma_1}\theta_1 + \frac{b_2\gamma_2}{b_2+\gamma_1}. \tag{5.2}$$

For the system (1.3) with spatial movement, by the linear analysis, we can show that the steady states E_1, E_{12} and E_{13} are still linearly stable and hence no pattern formation occurs. More precisely, we have the following results:

Proposition 5.1 *Assume (H0) and $\phi_u \geq 0, \phi_v \geq 0$ hold. Then for the system (1.3), it holds that*

(a) If $\theta_1 > 1$ and $\theta_2 > \gamma_2$, the prey-only steady state E_1 is linearly stable.
(b) If $\theta_1 < 1$ and (5.1) hold, the semi-coexistence steady state E_{12} is linearly stable.
(c) If $\theta_2 < \gamma_2$ and (5.2) hold, the semi-coexistence steady state E_{13} is linearly stable.

Proof The proof can be found in the Appendix, see Sect. 7. □

And it has been shown in Hsu et al. (2015) that if the coexistence steady state (u_*, v_*, w_*) exists for the corresponding ODE system of (1.3), then it is linearly stable if and only if

$$\begin{cases} b_2 + \gamma_1 - b_1 b_2 \gamma_2 > 0, \\ \gamma_1 \gamma_2 u_* w_* + b_1 u_* v_* > (\gamma_1 - \gamma_2 b_1 b_2) v_* w_*. \end{cases} \tag{5.3}$$

Hence, in the following, we focus only on whether pattern formation emerges from the coexistence steady state (u_*, v_*, w_*) under the conditions (5.3) and (1.8).

As discussed in Appendix, the linear stability/instability of the constant steady state (u_*, v_*, w_*) is determined by the eigenvalue of the following characteristic equation

$$\mu^3 + P_1(\chi, \lambda_k)\mu^2 + P_2(\chi, \lambda_k)\mu + P_3(\chi, \lambda_k) = 0,$$

where $\{\lambda_k\}_{k=0}^{\infty} : 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \ldots$ denote the sequence of eigenvalues of $-\Delta$ under Neumann boundary conditions and $P_i(\chi, \lambda_k)$ $(i = 1, 2, 3)$ are given by the following equalities

$$\begin{aligned}
P_1(\chi, \lambda_k) :=& \lambda_k(d_1 + d_2 + 1) + u_* > 0, \\
P_2(\chi, \lambda_k) :=& \lambda_k^2(d_1 d_2 + d_1 + d_2) + \lambda_k[(d_2 + 1)u_* \\
&+ \chi \phi_u^* \gamma_1 u_* w_* + \chi \phi_v^* b_2 v_* w_* + \xi b_1 u_* v_*] \\
&+ \gamma_1 \gamma_2 u_* w_* + b_2 v_* w_* + b_1 u_* v_*, \\
P_3(\chi, \lambda_k) :=& \lambda_k^3 d_1 d_2 \\
&+ \lambda_k^2(d_2 u_* + \chi \phi_u^* d_2 \gamma_1 u_* w_* + \chi \phi_v^* d_1 b_2 v_* w_* \\
&+ \xi b_1 u_* v_* + \chi \phi_v^* \xi \gamma_1 u_* v_* w_*) \\
&+ \lambda_k \left[\gamma_1 \xi u_* v_* w_* + \chi(b_2 \phi_v^* + \gamma_1 \phi_v^* - \phi_u^* b_1 b_2) u_* v_* w_*\right] \\
&+ \lambda_k(\gamma_1 \gamma_2 d_2 u_* w_* + b_2 d_1 v_* w_* + b_1 u_* v_*) \\
&+ (b_2 + \gamma_1 - \gamma_2 b_1 b_2) u_* v_* w_*,
\end{aligned} \tag{5.4}$$

with $\phi_u^* = \phi_u(u_*, v_*)$ and $\phi_v^* = \phi_v(u_*, v_*)$. From Routh–Hurwitz criterion (e.g., Appendix B.1 in Murray (2002)), the coexistence steady state (u_*, v_*, w_*) is linearly stable if and only if for each $k \in \mathbb{N}$, it holds that

$$P_1(\chi, \lambda_k) > 0, \quad P_3(\chi, \lambda_k) > 0, \quad P_1(\chi, \lambda_k)P_2(\chi, \lambda_k) - P_3(\chi, \lambda_k) > 0.$$

A direct calculation gives

$$H(\chi, \lambda_k) := P_1(\chi, \lambda_k)P_2(\chi, \lambda_k) - P_3(\chi, \lambda_k) = \lambda_k^3 K_1 + \lambda_k^2 K_2 + \lambda_k K_3 + K_4, \tag{5.5}$$

where

$$
\begin{aligned}
K_1 :=& (d_1d_2 + d_1 + d_2 + 1)(d_1 + d_2) > 0,\\
K_2 :=& (d_1d_2 + d_1)u_* + \xi(d_1 + d_2)b_1u_*v_* + (d_1 + d_2 + 1)(d_2 + 1)u_*\\
&+ (d_1 + 1)\chi\phi_u^*\gamma_1u_*w_* + (d_2 + 1)\chi\phi_v^*b_2v_*w_* - \chi\phi_v^*\xi\gamma_1u_*v_*w_*,\\
K_3 :=& (d_2 + 1)u_*^2 + (d_1 + 1)\gamma_1\gamma_2u_*w_* + (d_2 + 1)b_2v_*w_*\\
&+ (d_1 + d_2)b_1u_*v_* + b_1\xi u_*^2v_*\\
&+ \chi\phi_u^*\gamma_1u_*^2w_* + \chi\phi_u^*b_1b_2u_*v_*w_* - (\chi\phi_v^* + \xi)\gamma_1u_*v_*w_*,\\
K_4 :=& u_*[\gamma_1\gamma_2u_*w_* + b_1u_*v_* - (\gamma_1 - \gamma_2b_1b_2)v_*w_*].
\end{aligned}
\tag{5.6}
$$

When $\chi = \xi = 0$, one can easily check that $P_3(\chi, \lambda_k) > 0$ and $H(\chi, \lambda_k) > 0$ for all $k \in \mathbb{N}$, which indicates that the coexistence steady state (u_*, v_*, w_*) is linearly stable. Hence, in the following we will study whether or not the taxis mechanisms can induce the pattern formations. $H(\chi, \lambda_k)$ depends on the values of $\phi_u^* = \phi_u(u_*, v_*)$, $\phi_v^* = \phi_v(u_*, v_*)$, γ_1 and γ_2. For a better understanding of the difference between the effect of prey-taxis and alarm-taxis in the food chain model with intraguild predation, we shall focus on the linear stability/instability of coexistence steady state for two types of $\phi(u, v)$: $\phi(u, v) = v$ and $\phi(u, v) = uv$, both under the conditions $\gamma_1, \gamma_2 \geq 0$.

5.1 Linear Stability/Instability Analysis: $\gamma_1 = \gamma_2 = 0$

In this subsection, we shall study the linear stability/instability of coexistence steady state (u_*, v_*, w_*) to (1.3) with $\phi(u, v) = v$ or $\phi(u, v) = uv$ in the case of $\gamma_1 = \gamma_2 = 0$. In this case, (1.3) can be simplified as

$$
\begin{cases}
u_t = d_1u_{xx} + u(1 - u) - b_1uv,\\
v_t = d_2v_{xx} - \xi(vu_x)_x + uv - b_2vw - \theta_1v,\\
w_t = w_{xx} - \chi(w\phi(u, v)_x)_x + vw - \theta_2w,
\end{cases}
\tag{5.7}
$$

which is the classical Lotka–Volterra food chain model with taxis mechanisms (i.e., $\xi, \chi > 0$). The coexistence steady state $(u_*, v_*, w_*) = (1-b_1\theta_2, \theta_2, \frac{1-\theta_1-b_1\theta_2}{b_2})$ exists provided

$$
\theta_1 + b_1\theta_2 < 1. \tag{5.8}
$$

It has been proved in Jin et al. (2022) that if $\phi(v) = v$, the coexistence steady state of the system (5.7) is globally stable if $\xi > 0$ and $\chi > 0$ are both small. Thus, it is natural to ask whether or not (u_*, v_*, w_*) is linearly unstable and pattern formation occurs for large ξ and χ. In fact, we have the following results.

Lemma 5.2 (Linear stability*: $\phi(u, v) = v$*) *Let $\phi(u, v) = v$ and assume (5.8) holds, then (u_*, v_*, w_*) of (5.7) is linearly stable for all $\chi, \xi \geq 0$.*

Proof Since $\phi(u, v) = v$, we have $\phi_u^* = 0$ and $\phi_v^* = 1$. Then noting $\gamma_1 = \gamma_2 = 0$, it follows from (5.4) that for each $k \in \mathbb{N}$

$$\begin{aligned} P_3(\chi, \lambda_k) = \lambda_k^3 d_1 d_2 + \lambda_k^2 \left(d_2 u_* + \chi d_1 b_2 v_* w_* + \xi b_1 u_* v_*\right) \\ + \lambda_k \left(b_2 d_1 v_* w_* + b_1 u_* v_* + \chi b_2 u_* v_* w_*\right) \\ + b_2 u_* v_* w_* > 0. \end{aligned}$$

On the other hand, by $K_i (i = 1, 2, 3, 4)$ in (5.6), one can check that

$$K_i > 0 \ \text{ for } \ i = 1, 2, 3, 4,$$

which implies for each $k \in \mathbb{N}$

$$H(\chi, \lambda_k) = P_1(\chi, \lambda_k) P_2(\chi, \lambda_k) - P_3(\chi, \lambda_k) = \lambda_k^3 K_1 + \lambda_k^2 K_2 + \lambda_k K_3 + K_4 > 0.$$

Then Routh–Hurwitz criterion implies that (u_*, v_*, w_*) is linearly stable. □

Remark 5.3 The results in Lemma 5.2 imply that no pattern formation occurs for the classical Lotka–Volterra food chain model with prey-taxis mechanisms for any $\xi, \chi \geq 0$.

In the following, we shall study the possibility of pattern formation for the Lotka–Volterra food chain model incorporating the alarm-taxis mechanism. The main results are as follows.

Lemma 5.4 (Linear stability/instability: $\phi(u, v) = uv$) *Let $\phi(u, v) = uv$ and assume (5.8) holds. It holds that*

(1) *If $2b_1\theta_2 \leq 1$, then (u_*, v_*, w_*) is linearly stable for all $\chi > 0$.*
(2) *If $2b_1\theta_2 > 1$, then (u_*, v_*, w_*) is linearly unstable provided $\chi > 0$ is large enough and there exists some $k \in \mathbb{N}^+$ such that*

$$0 < \lambda_k < \frac{2b_1\theta_2 - 1}{d_1}. \tag{5.9}$$

Proof For $\phi(u, v) = uv$, one has $\phi_u^* = \phi_u(u_*, v_*) = v_*$ and $\phi_v^* = \phi_v(u_*, v_*) = u_*$. Noting $\gamma_1 = \gamma_2 = 0$ and the definitions of $K_i (i = 1, 2, 3, 4)$ in (5.6), we have

$$K_i > 0 \ \text{ for all } \ i = 1, 2, 3, 4,$$

which implies that for each $k \in \mathbb{N}$

$$H(\chi, \lambda_k) = P_1(\chi, \lambda_k) P_2(\chi, \lambda_k) - P_3(\chi, \lambda_k) > 0.$$

Moreover, using $u_* - b_1 v_* = 1 - 2b_1\theta_2$ and the facts $\gamma_1 = \gamma_2 = 0$, $\phi_u^* = v_*$, $\phi_v^* = u_*$ again, we deduce from (5.4) that

$$P_3(\chi,\lambda_k)=\lambda_k^3 d_1 d_2+\lambda_k^2(d_2+\xi b_1 v_*)u_*+\lambda_k(d_1 b_2 w_*+b_1 u_*)v_*+b_2 u_* v_* w_* \\ +\lambda_k\chi b_2 u_* v_* w_*(\lambda_k d_1+1-2b_1\theta_2). \tag{5.10}$$

Then if $2b_1\theta_2\le 1$, one has $P_3(\chi,\lambda_k)>0$ for any $k\in\mathbb{N}$, and hence, (u_*, v_*, w_*) is linearly stable by Routh–Hurwitz criterion.

On the other hand, if $2b_1\theta_2>1$ and (5.9) holds, we get that $\lambda_k d_1+1-2b_1\theta_2<0$ for some $k\in\mathbb{N}^+$. Since λ_k, u_*, v_*, w_* are independent of χ, it follows that $P_3(\chi,\lambda_k)\le 0$ for sufficiently large $\chi>0$. Therefore, according to Routh–Hurwitz criterion, (u_*, v_*, w_*) is linearly unstable. □

Remark 5.5 For the Lotka–Volterra food chain model (5.7), our results imply that the taxis function $\phi(u,v)$ plays an important role on the pattern formation. If $\phi(u,v)=v$ (i.e., prey-taxis mechanism), no pattern formation occurs. If $\phi(u,v)=uv$ (i.e., alarm-taxis mechanism), the potential steady state bifurcations generated from the constant coexistence (u_*, v_*, w_*) may happen. Compared with the results obtained in Haskell and Bell (2021), our results confirm that the alarm-taxis mechanism can trigger the pattern formation by itself even without logistic growth source.

5.2 Linear Stability/Instability Analysis: $\gamma_1,\gamma_2>0$

In this subsection, we shall study the possibility of pattern formation for the system (1.3) with intraguild predation (i.e., $\gamma_1,\gamma_2>0$). To this end, we analyze the linear stability/instability of the coexistence steady state (u_*, v_*, w_*) defined in (1.7). In the case of $\gamma_1,\gamma_2>0$, we rewrite $P_3(\chi,\lambda_k)$ in (5.4) as follows:

$$\begin{aligned} P_3(\chi,\lambda_k)=&\lambda_k^3 d_1 d_2+\lambda_k^2(d_2 u_*+\xi b_1 u_* v_*)\\ &+\lambda_k(\gamma_1 u_* w_*\gamma_2 d_2+b_2 v_* w_* d_1+b_1 u_* v_*+\gamma_1 u_* v_* w_*\xi)\\ &+\lambda_k^2\chi(\phi_u^* d_2\gamma_1 u_* w_*+\phi_v^* d_1 b_2 v_* w_*+\phi_v^*\xi\gamma_1 u_* v_* w_*)\\ &+\lambda_k\chi u_* v_* w_*(b_2\phi_v^*+\gamma_1\phi_v^*-\phi_u^* b_1 b_2)+(b_2+\gamma_1-\gamma_2 b_1 b_2)u_* v_* w_*. \end{aligned} \tag{5.11}$$

Lemma 5.6 (Linear stability/instability: $\phi(u,v)=v$) *Let $\phi(u,v)=v$ and assume* (1.8) and (5.3) *hold. Then we have the following results:*

(1) (u_*, v_*, w_*) *is linearly stable provided*

$$\chi+\xi\le\frac{\widetilde{K_3}}{\gamma_1 u_* v_* w_*}\ \text{ and }\ d_2+1\ge\frac{\xi\gamma_1 u_*}{b_2}, \tag{5.12}$$

with $\widetilde{K_3}>0$ defined in (5.16).

(2) (u_*, v_*, w_*) *is linearly unstable provided $\chi>0$ large enough and one of the following conditions holds:*

$$\begin{cases} d_2+1>\frac{\xi\gamma_1 u_*}{b_2},\\ 0<\lambda_k<\frac{\gamma_1 u_*}{(d_2+1)b_2-\xi\gamma_1 u_*} \ \ \textit{for some } k\in\mathbb{N}^+, \end{cases} \tag{5.13}$$

or

$$d_2 + 1 \leq \frac{\xi \gamma_1 u_*}{b_2} \ \text{for all } k \in \mathbb{N}^+. \tag{5.14}$$

Proof Since $\phi(u, v) = v$, one has $\phi_v^* = 1$ and $\phi_u^* = 0$. Noting $b_2 + \gamma_1 - \gamma_2 b_1 b_2 > 0$, it follows from (5.11) that $P_3(\chi, \lambda_k) > 0$ for all $k \in \mathbb{N}$.

Since (1.8) and (5.3) hold, we derive from (5.6) that $K_1 > 0$ and $K_4 > 0$. Hence, to determine the sign of $H(\chi, \lambda_k)$, we only need to consider the values of K_2 and K_3. Using the facts $\phi_v^* = 1$ and $\phi_u^* = 0$, we rewrite K_2 and K_3 defined in (5.6) as follows:

$$K_2 = \widetilde{K_2} + \chi v_* w_* [(d_2 + 1)b_2 - \xi \gamma_1 u_*] \ \text{ and } \ K_3 = \widetilde{K_3} - (\chi + \xi)\gamma_1 u_* v_* w_*,$$

where $\widetilde{K_2} > 0$ and $\widetilde{K_3} > 0$ are defined by

$$\widetilde{K_2} := (d_1 d_2 + d_1)u_* + \xi(d_1 + d_2)b_1 u_* v_* + (d_1 + d_2 + 1)(d_2 + 1)u_* \tag{5.15}$$

and

$$\begin{aligned}\widetilde{K_3} := & (d_2 + 1)u_*^2 + (d_1 + 1)\gamma_1 \gamma_2 u_* w_* + (d_2 + 1)b_2 v_* w_* \\ & + (d_1 + d_2)b_1 u_* v_* + b_1 \xi u_*^2 v_*. \end{aligned} \tag{5.16}$$

Then we can derive from (5.12) that K_2 and K_3 are positive and hence $H(\chi, \lambda_k) > 0$ for all $k \in \mathbb{N}$, which implies that (u_*, v_*, w_*) is linearly stable by using Routh–Hurwitz criterion.

Next, we shall show that (u_*, v_*, w_*) is linearly unstable for large χ under conditions (5.13) or (5.14). To this end, we rewrite $H(\chi, \lambda_k)$ (see in (5.5)) as follows:

$$\begin{aligned} H(\chi, \lambda_k) = & \lambda_k^3 K_1 + \lambda_k^2 \widetilde{K_2} + \lambda_k \widetilde{K_3} + K_4 + \lambda_k \chi v_* w_* \left(\lambda_k [(d_2 + 1)b_2 - \xi \gamma_1 u_*] - \gamma_1 u_*\right) \\ & - \lambda_k \xi \gamma_1 u_* v_* w_*, \end{aligned} \tag{5.17}$$

where $\widetilde{K_2}$ and $\widetilde{K_3}$ are defined by (5.15) and (5.16), respectively.

Since λ_k and the value of (u_*, v_*, w_*) are independent of χ, then if (5.13) or (5.14) holds, we can find $\chi > 0$ large enough such that

$$H(\chi, \lambda_k) \leq 0,$$

and hence, the coexistence steady state (u_*, v_*, w_*) is linearly unstable by applying Routh–Hurwitz criterion again. □

Remark 5.7 Compared with the results obtained in Lemma 5.2 and Lemma 5.6, we found that the intraguild predation (i.e., $\gamma_1, \gamma_2 > 0$) plays an important role for the pattern formation.

Next, we shall study the possible pattern formation in the system (1.3) with alarm-taxis in the sense of $\phi(u, v) = uv$.

Lemma 5.8 (Linear stability/instability*: $\phi(u, v) = uv$) Let $\phi(u, v) = uv$, $\chi > 0$ and $\xi \geq 0$. Assume* (1.8) and (5.3) *hold. Then it holds that:*

(1) *(u_*, v_*, w_*) is linearly stable provided*

$$b_2 u_* + \gamma_1 u_* - v_* b_1 b_2 \geq 0 \tag{5.18}$$

and

$$0 < \xi \leq \min\left\{ \frac{\widetilde{K_3}}{\gamma_1 u_* v_* w_*}, \frac{d_1 + 1}{u_*} + \frac{(d_2 + 1)b_2}{u_* \gamma_1} \right\}, \tag{5.19}$$

where $\widetilde{K_3} > 0$ defined in (5.16).

(2) *(u_*, v_*, w_*) is linearly unstable provided $\chi > 0$ large enough and one of the following conditions holds:*

$$\begin{aligned} & b_2 u_* + \gamma_1 u_* - v_* b_1 b_2 < 0 \ \text{ and} \\ & 0 < \lambda_{k_0} < \frac{|b_2 u_* + \gamma_1 u_* - v_* b_1 b_2|}{d_2 \gamma_1 + d_1 b_2 + u_* \xi \gamma_1} \textit{ for some } k_0 \in \mathbb{N}, \end{aligned} \tag{5.20}$$

or

$$\begin{aligned} & \xi > \frac{d_1 + 1}{u_*} + \frac{(d_2 + 1)b_2}{u_* \gamma_1} \text{ and} \\ & \lambda_{k_0} > \frac{v_* b_1 b_2}{|(d_1 + 1)\gamma_1 + (d_2 + 1)b_2 - u_* \xi \gamma_1|} \textit{ for some } k_0 \in \mathbb{N}. \end{aligned} \tag{5.21}$$

Proof From $\phi(u, v) = uv$, one has $\phi_v^* = u_*$ and $\phi_u^* = v_*$. Hence, we can derive that

$$\begin{aligned} & P_3(\chi, \lambda_k) \\ = & \lambda_k^3 d_1 d_2 + \lambda_k^2 (d_2 u_* + \xi b_1 u_* v_*) + \lambda_k (\gamma_1 u_* w_* \gamma_2 d_2 + b_2 v_* w_* d_1 \\ & + b_1 u_* v_* + \gamma_1 u_* v_* w_* \xi) \\ & + \lambda_k \chi u_* v_* w_* [\lambda_k (d_2 \gamma_1 + d_1 b_2 + u_* \xi \gamma_1) + (b_2 u_* + \gamma_1 u_* - v_* b_1 b_2)] \\ & + (b_2 + \gamma_1 - \gamma_2 b_1 b_2) u_* v_* w_*, \end{aligned} \tag{5.22}$$

and

$$\begin{aligned} H(\chi, \lambda_k) = & \lambda_k^3 K_1 + \lambda_k^2 \widetilde{K_2} + \lambda_k (\widetilde{K_3} - \xi \gamma_1 u_* v_* w_*) + K_4 \\ & + \lambda_k \chi u_* v_* w_* \left(\lambda_k [(d_1 + 1)\gamma_1 + (d_2 + 1)b_2 - u_* \xi \gamma_1] + v_* b_1 b_2 \right). \end{aligned} \tag{5.23}$$

Then if (5.18) and (5.19) hold, one can verify that $P_3(\chi, \lambda_k) > 0$ and $H(\chi, \lambda_k) > 0$ for each $k \in \mathbb{N}$, and hence, by applying Routh–Hurwitz criterion, we obtain that (u_*, v_*, w_*) is linearly stable.

On the contrary, if (5.20) holds, we can choose χ large enough such that $P_3(\chi, \lambda_k) < 0$. Thus, we derive from Routh–Hurwitz criterion that (u_*, v_*, w_*) is linearly unstable.

Similarly, if (5.21) holds, we have $H(\chi, \lambda_k) < 0$ for large χ, and hence, (u_*, v_*, w_*) is linearly unstable. □

Remark 5.9 Compared with the Lotka–Volterra food chain model (5.7) with $\phi(u, v) = uv$, the intraguild predation model (i.e., $\gamma_1, \gamma_2 > 0$) has richer dynamics. Specifically, the intraguild predation model has not only the potential of steady state bifurcations but also that of Hopf bifurcations.

Remark 5.10 The instability results of the intraguild predation model with $\phi(u, v) = uv$ indicate that the alarm-taxis mechanism can promote potential steady state bifurcations, which cannot be induced by the intraguild predation model with $\phi(u, v) = v$.

6 Spatiotemporal Patterns: Numerical Simulations

In this section, we shall give some numerical simulations to verify our theoretical analysis in Section 5. As shown in Lemma 5.4, Lemma 5.6 and Lemma 5.8, with suitable conditions, as long as $\chi > 0$ is large enough, pattern formations possibly occur for the system (1.3) even in the case of $\xi = 0$, which are verified in Figs. 1, 3a and 4b. Furthermore, with fixed $\chi > 0$, our numerical simulations show that the parameter ξ plays a very different effect for the system (1.3) between the cases that $\gamma_1 = \gamma_2 = 0$ and $\gamma_1, \gamma_2 > 0$. For the food chain model with alarm-taxis, ξ has a stabilization effect on the homogeneous steady state (see Fig. 2), while it has a destabilization effect in the food chain model with intraguild predation and prey-taxis (see Fig. 3). As for the food chain model with intraguild predation and alarm-taxis, the effects of ξ on pattern formations are more complicated. The system may subsequently undergo steady state bifurcations, no pattern formations and Hopf bifurcations as ξ increases from 0 to 4 and then to 45, see Fig. 4.

Moreover, comparing the linear stability results in Lemma 5.2 with Lemma 5.6, we conclude that the intraguild predation is of importance for inducing pattern formations, which is verified in Fig. 3a, while it is still unclear whether taxis mechanisms or intraguild predation have essential effects on triggering pattern formations. As shown in Fig. 4a, there is no pattern formation in the case of $\chi = \xi = 0,\ \gamma_1, \gamma_2 > 0$. This fact along with Figs. 1, 3a and 4b demonstrates that signal taxis mechanism plays an indispensable and essential role in promoting spatially inhomogeneous patterns.

6.1 Food Chain Model with Alarm-Taxis: $\gamma_1 = \gamma_2 = 0$ and $\phi(u, v) = uv$

In this subsection, we shall give some numerical simulations to the system (1.3) with $\phi(u, v) = uv$ in the case of $\gamma_1 = \gamma_2 = 0$. To this end, we fix the value of the parameters in all simulations as follows:

$$d_1 = 0.1,\ d_2 = b_1 = b_2 = 1, \theta_1 = 0.1,\ \theta_2 = 0.7,\ \gamma_1 = \gamma_2 = 0,$$

which gives $(u_*, v_*, w_*) = (0.3, 0.7, 0.2)$ and $\theta_1 + b_1\theta_2 < 1$ as well as $2b_1\theta_2 > 1$. Hence, from Lemma 5.4, with the fact $H(\chi, \lambda_k) > 0$ we expect only the spatiotemporal

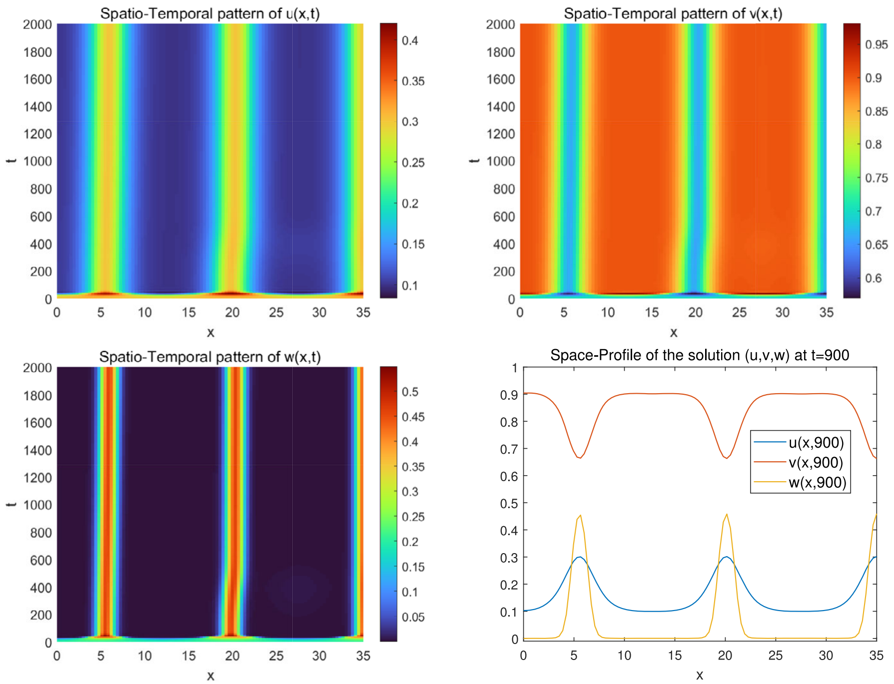


Fig. 1 Numerical simulation of spatiotemporal patterns generated by (1.3) with $\phi(u, v) = uv$ and $\gamma_1 = \gamma_2 = 0$. The parameter values are: $\chi = 80$, $\xi = 0$, $d_1 = 0.1$, $d_2 = b_1 = b_2 = 1$, $\theta_1 = 0.1$, $\theta_2 = 0.7$. The initial datum (u_0, v_0, w_0) is set as a small random perturbation of the homogeneous coexistence steady state $(0.3, 0.7, 0.2)$

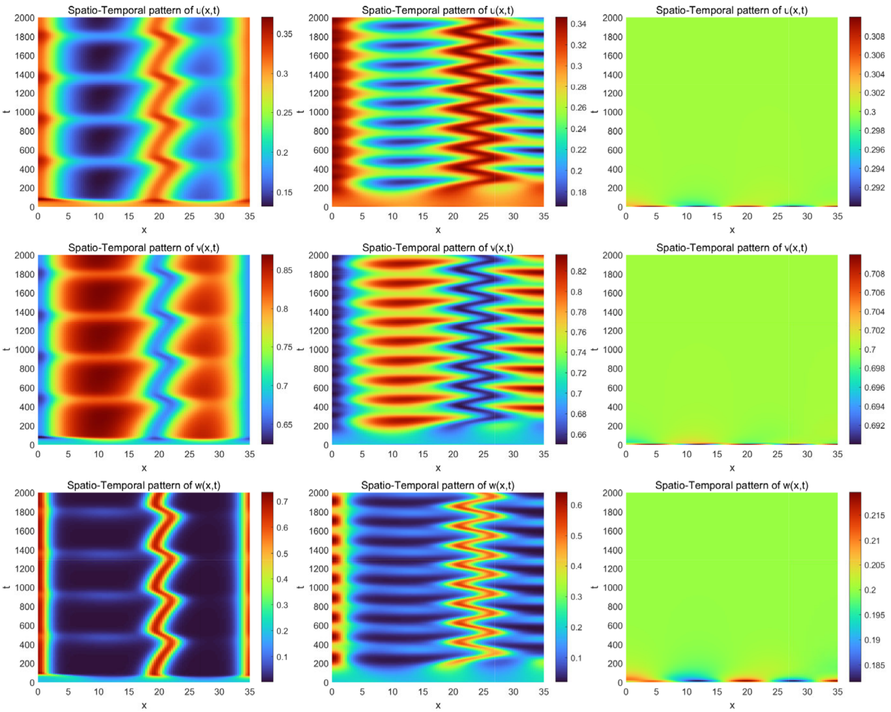
steady state (aggregation) pattern occurs when

$$\chi \geq \chi_k^{\mathcal{S}_1}(\xi) := \frac{5}{21(4-\lambda_k)}\left(100\lambda_k^2 + 30(10+7\xi)\lambda_k + 224 + \frac{42}{\lambda_k}\right), \tag{6.1}$$

for some $k \in \mathbb{N}^+$ such that $0 < \lambda_k < 4$ and here $\chi_k^{\mathcal{S}_1}(\xi)$ is the root of $P_3(\chi, \lambda_k) = 0$ in (5.10). Taking $\Omega = (0, 10\pi)$, with allowable wavenumber satisfying $0 < \lambda_k = (k/10)^2 < 4$, we get the allowable unstable modes for $k = 1, 2, 3 \cdots, 18, 19$. We choose $\lambda_k = (5/10)^2$, then $\chi_k^{\mathcal{S}_1}(\xi)$ in (6.1) can be updated as

$$\chi_5^{\mathcal{S}_1}(\xi) = \frac{631 + 70\xi}{21}.$$

We first pick $\xi = 0$ to find a value $\chi_5^{\mathcal{S}_1}(0) \approx 30.0476$ for the possibility of pattern formations. As shown in Fig. 1, by letting $\chi = 80 > 30.0476$ and we can find the spatiotemporal pattern. Particularly, from Fig. 1, we obtain that the time evolutionary profiles of solutions are horizontal lines, which indicates that the bifurcation might be the steady state bifurcation. Moreover, the space profiles show that all species reach an inhomogeneous coexistence state in space (see the last picture in Fig. 1).



(a) $\chi = 80,\ \xi$=10 (b) $\chi = 80,\ \xi$=20 (c) $\chi = 80,\ \xi$=40

Fig. 2 Numerical simulation of spatiotemporal patterns for (1.3) with $\phi(u, v) = uv$. The fixed parameter values are: $d_1 = 0.1,\ d_2 = b_1 = b_2 = 1, \theta_1 = 0.1,\ \theta_2 = 0.7$ and $\gamma_1 = \gamma_2 = 0$. The initial datum (u_0, v_0, w_0) is set as a small random perturbation of the homogeneous coexistence steady state $(0.3, 0.7, 0.2)$.

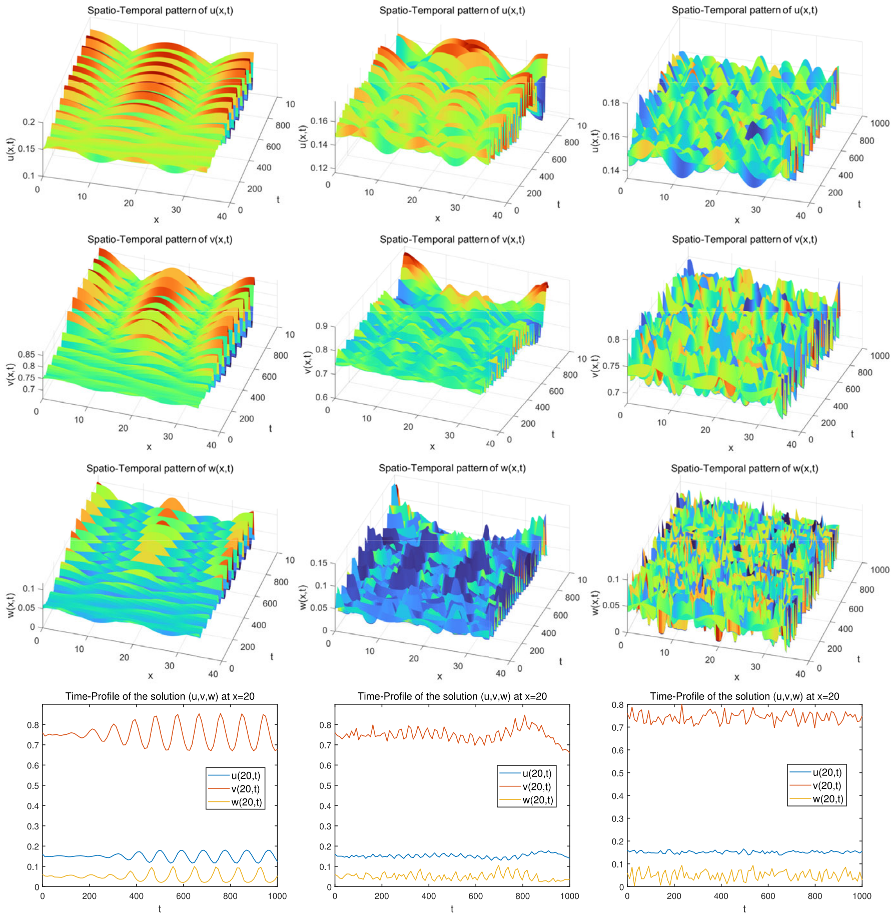
The expression in (6.1) implies that the critical value $\chi_k^{S_1}(\xi) > 0$ is increasing in terms of $\xi \geq 0$, and the spatiotemporal patterns generated due to any fixed large χ and fixed mode k will disappear as the value of $\xi \geq 0$ increases, which implies the prey-taxis has a stabilization effect on the homogeneous steady state. To verify this fact, we use numerical simulations to find that the spatiotemporal patterns gradually evolve into the spatially homogeneous patterns as ξ increases from 0 to 10, then to 20, and finally disappear at $\xi = 40$, see more details in Fig. 2.

6.2 Food Chain Model with Intraguild Predation and Prey-Taxis: $\gamma_1, \gamma_2 > 0$ and $\phi(u, v) = v$

In this subsection, we shall give some numerical simulations to the system (1.3) with $\phi(u, v) = v$ and $\gamma_1, \gamma_2 > 0$. We fix the value of the parameters as follows:

$$d_1 = 0.1,\ d_2 = b_1 = b_2 = \gamma_2 = 1,\ \gamma_1 = 2,\ \theta_1 = 0.1,\ \theta_2 = 0.9.$$

Then the coexistence steady state is $(u_*, v_*, w_*) = (0.15, 0.75, 0.05)$. As discussed in Lemma 5.6, only Hopf bifurcations can occur by noting the fact $P_3(\chi, \lambda_k) > 0$.



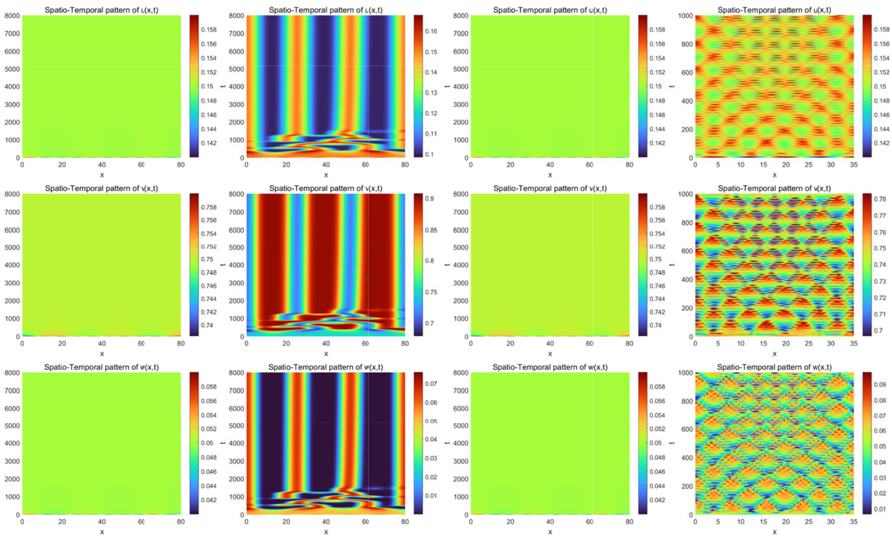
(a) $\chi = 100,\ \xi=0$ (b) $\chi = 100,\ \xi=10$ (c) $\chi = 100,\ \xi=20$

Fig. 3 Numerical simulation of spatiotemporal patterns generated by (1.3) with $\phi(u, v) = v$ and $\gamma_1, \gamma_2 > 0$. The parameter values are: $d_1 = 0.1,\ d_2 = b_1 = b_2 = \gamma_2 = 1,\ \gamma_1 = 2,\ \theta_1 = 0.1,\ \theta_2 = 0.9$. The initial datum (u_0, v_0, w_0) is set as a small random perturbation of the homogeneous coexistence steady state $(0.15, 0.75, 0.05)$.

Under the above parameters, we derive that $H(\chi, \lambda_k) = 0$ in (5.17) is equivalent to

$$\chi = \chi_k^{\mathcal{H}_1}(\xi) := \frac{9680\lambda_k^2 + (2640 + 495\xi)\lambda_k + \frac{54}{\lambda_k} + 1041 + 90\xi}{15(3 + 3\lambda_k\xi - 20\lambda_k)}, \tag{6.2}$$

which is positive provided $\lambda_k(20 - 3\xi) < 3$. Taking $\Omega = (0, 10\pi)$, the allowable wavenumber $\lambda_k = (k/10)^2$ satisfying $\lambda_k(20-3\xi) < 3$, then $k = 1, 2, 3$ are allowable unstable modes for any $\xi \geq 0$. Fixing $k = 2$ and (6.2) can be simplified as



(a) $\chi = 0, \xi{=}0$ (b) $\chi = 450, \xi{=}0$ (c) $\chi = 450, \xi{=}4$ (d) $\chi = 450, \xi{=}45$

Fig. 4 Numerical simulation of spatiotemporal patterns generated by (1.3) with $\phi(u, v) = uv$ and $\gamma_1, \gamma_2 > 0$. The parameter values are: $d_1 = 0.1,\ d_2 = b_1 = b_2 = \gamma_2 = 1,\ \gamma_1 = 2, \theta_1 = 0.1,\ \theta_2 = 0.9$. The initial datum (u_0, v_0, w_0) is set as a small random perturbation of the homogeneous coexistence steady state (0.15, 0.75, 0.05)

$$\chi_2^{\mathcal{H}_1}(\xi) = 61 + \frac{62386}{75(55 + 3\xi)}. \tag{6.3}$$

We first choose $\xi = 0$ to obtain a value $\chi_2^{\mathcal{H}_1}(0) \approx 76.124$ for possible pattern formations. As shown in Fig. 3a, with $\chi = 100 > 76.124$ in hand, we can find the spatiotemporal patterns. In particular, the time evolutionary profiles of solutions are periodically oscillatory, which indicates the bifurcation might be of Hopf bifurcation type (see the last picture in Fig. 3a). Moreover, the expression (6.3) indicates that for fixed unstable mode $k = 2$, the critical value $\chi_2^{\mathcal{H}_1}(\xi) > 0$ is decreasing about $\xi \geq 0$, which implies the prey-taxis might have a destabilization effect on patterns. This is an interesting phenomenon, which is different from the food chain model without intraguild predation.

To verify this fact, we take $\xi = 10$ and $\xi = 20$ and find that the patterns become unstable as ξ increases from 0 to 10 and then to 20, and the chaotic spatiotemporal patterns may happen, see Fig. 3c.

6.3 Food Chain Model with Intraguild Predation and Alarm-Taxis: $\gamma_1, \gamma_2 > 0$ and $\phi(u, v) = uv$

In this case, we fix the parameters as follows for simulations:

$$d_1 = 0.1,\ d_2 = b_1 = b_2 = \gamma_2 = 1,\ \gamma_1 = 2, \theta_1 = 0.1,\ \theta_2 = 0.9.$$

Then $(u_*, v_*, w_*) = (0.15, 0.75, 0.05)$. From Lemma 5.8, we know the steady state and Hopf bifurcations are both possible. Under the above parameters, we first derive from (5.22) and (5.23) in Lemma 5.8 that $P_3(\chi, \lambda_k) = 0$ and $H(\chi, \lambda_k) = 0$, which are equivalent to

$$\chi = \chi_k^{\mathcal{S}_2}(\xi) := \frac{1600\lambda_k^2 + 600\lambda_k(4 + 3\xi) + \frac{180}{\lambda_k} + 2100 + 180\xi}{27 - 27\lambda_k(7 + \xi)}, \tag{6.4}$$

and

$$\chi = \chi_k^{\mathcal{H}_2}(\xi) := \frac{77440\lambda_k^2 + 120\lambda_k(176 + 33\xi) + \frac{432}{\lambda_k} + 8328 + 180\xi}{54\lambda_k(\xi - 14) - 135}. \tag{6.5}$$

From (5.23) in Lemma 5.8, we know that if

$$0 \le \xi \le \min\left\{\frac{\widetilde{K_3}}{\gamma_1 u_* v_* w_*}, \frac{d_1 + 1}{u_*} + \frac{(d_2 + 1)b_2}{u_* \gamma_1}\right\} = \min\left\{\frac{347}{15} + \frac{3\xi}{2}, 14\right\} = 14,$$

then $H(\chi, \lambda_k) > 0$ for any $k \in \mathbb{N}$ and hence no Hopf bifurcation occurs, which motivates us to study the possibility of steady state pattern formation. To illustrate this case, we take $\Omega = (0, 10\pi)$, then from (5.20), the allowable unstable modes $k \in \mathbb{N}^+$ must satisfy $0 < \lambda_k = (k/10)^2 < \frac{1}{7+\xi}$.

We take $k = 3$ and $\xi = 0$, then (6.4) implies that

$$\chi_3^{\mathcal{S}_2}(0) \approx 433.329,$$

which is a value for possible pattern formations. As shown in Fig. 4b, choosing $\chi = 450 > 433.329$, we can find the spatiotemporal patterns. Particularly, the time evolutionary profiles of solutions are horizontal lines, which indicates the bifurcation might be the steady state bifurcation. Furthermore, for the fixed unstable mode $k = 3$, the bifurcations will disappear as ξ increases from 0 to 4, see Fig. 4c.

For relatively large $\xi > 14$, from Lemma 5.8 and the definition of $\chi_k^{\mathcal{H}_2}$ in (6.5), the Hopf bifurcations possibly occur as long as the allowable unstable modes $k \in \mathbb{N}^+$ satisfying $\lambda = (k/10)^2 > \frac{5}{2(\xi-14)}$. With $\chi = 450$ in hand, for the same unstable mode $k = 3$, we pick $\xi = 45$ to find the spatiotemporal patterns, see Fig. 4d.

Our results demonstrate that for the fixed large $\chi = 450$, as the parameter ξ increases, the steady state patterns (see Fig. 4b) evolve first into the constant state (see Fig. 4c) and then further develop into the Hopf bifurcation patterns (see Fig. 4d). Moreover, from Fig. 4a, we observe that no pattern formation occurs when $\chi = \xi = 0$ and $\gamma_1, \gamma_2 > 0$. This, together with Figs. 1, 3a, 4b and Lemma 5.2, indicates that the signal taxis mechanism plays an essential role in promoting pattern formation.

7 Appendix: Linear Analysis

In this section, we are devoted to giving some basic linear analysis on the linear stability/instability of constant steady state for the system (1.3). To this end, we first linearize the system (1.3) at constant steady state (u_c, v_c, w_c) to obtain

$$\begin{cases} \Psi_t = \mathcal{A}\Delta\Psi + \mathcal{B}\Psi, & x \in \Omega, \ \ t > 0, \\ \nabla\Psi \cdot \nu = 0, & x \in \partial\Omega, \ \ t > 0, \\ \Psi(x, 0) = (u_0 - u_c, v_0 - v_c, w_0 - w_c)^{\mathcal{T}}, & x \in \Omega, \end{cases}$$

where $\mathcal{T}$ denotes the transpose matrix and

$$\Psi := \begin{pmatrix} u - u_c \\ v - v_c \\ w - w_c \end{pmatrix}, \ \ \mathcal{A} = \begin{pmatrix} d_1 & 0 & 0 \\ -\xi v_c & d_2 & 0 \\ -\chi w_c \phi_u^c & -\chi w_c \phi_v^c & 1 \end{pmatrix} \text{ and } \mathcal{B} = \begin{pmatrix} -u_c & -b_1 u_c & -\gamma_1 u_c \\ v_c & B_{22} & -b_2 v_c \\ \gamma_2 w_c & w_c & B_{33} \end{pmatrix},$$

with $\phi_u^c := \phi_u(u_c, v_c)$, $\phi_v^c := \phi_v(u_c, v_c)$ and

$$B_{22} := u_c - b_2 w_c - \theta_1 \ \text{ and } \ B_{33} := v_c + \gamma_2 u_c - \theta_2. \tag{7.1}$$

Let the sequence $\{\lambda_n\}_{n=0}^{\infty} : 0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \lambda_3 \ldots$ denotes the sequence of eigenvalues of $-\Delta$ under Neumann boundary condition. Then, the linear stability of (u_c, v_c, w_c) is determined by the eigenvalues of the matrix $(-\lambda_k \mathcal{A} + \mathcal{B})$, which satisfies the following characteristic equation

$$\mu^3 + P_1\mu^2 + P_2\mu + P_3 = 0,$$

where $P_i := P_i(\lambda_k)$ $(i = 1, 2, 3)$ are defined as below

$$\begin{aligned}
P_1(\lambda_k) :=& \lambda_k(d_1 + d_2 + 1) + u_c - B_{22} - B_{33}, \\
P_2(\lambda_k) :=& \lambda_k^2(d_1 d_2 + d_1 + d_2) + \lambda_k \{(d_2 + 1)u_c - (d_1 + 1)B_{22} - (d_1 + d_2)B_{33}\} \\
&+ \lambda_k(\chi\phi_u^c \gamma_1 u_c w_c + \chi\phi_v^c b_2 v_c w_c + \xi b_1 u_c v_c) \\
&+ \gamma_1\gamma_2 u_c w_c + b_2 v_c w_c + b_1 u_c v_c \\
&- u_c B_{22} - u_c B_{33} + B_{22} B_{33}, \\
P_3(\lambda_k) :=& \lambda_k^3 d_1 d_2 + \lambda_k^2(-d_1 d_2 B_{33} + d_2 u_c - d_1 B_{22}) \\
&+ \lambda_k^2(\chi\phi_u^c d_2 \gamma_1 u_c w_c + \chi\phi_v^c d_1 b_2 v_c w_c + \xi b_1 u_c v_c + \chi\phi_v^c \xi\gamma_1 u_c v_c w_c) \\
&+ \lambda_k(-u_c B_{22} - d_2 u_c B_{33} + d_1 B_{22} B_{33}) \\
&+ \lambda_k \left\{\gamma_1 u_c w_c(\gamma_2 d_2 - \chi\phi_u^c B_{22}) + b_2 v_c w_c(d_1 + \chi\phi_v^c u_c) \right. \\
&\left. + b_1 u_c v_c(1 - \xi B_{33}) + \gamma_1 u_c v_c w_c(\chi\phi_v^c + \xi) - \chi\phi_u^c b_1 b_2 u_c v_c w_c \right\} \\
&+ u_c B_{22} B_{33} - \gamma_1\gamma_2 u_c w_c B_{22} - b_1 u_c v_c B_{33} + (b_2 + \gamma_1 - \gamma_2 b_1 b_2) u_c v_c w_c.
\end{aligned} \tag{7.2}$$

Based on Routh–Hurwitz criterion (e.g., Appendix B.1 in Murray (2002)), the nonnegative constant steady states (u_c, v_c, w_c) are linearly stable if and only if for each $k \in \mathbb{N}$, it holds that

$$P_1 > 0, \quad P_3 > 0, \quad P_1 P_2 - P_3 > 0.$$

Calculating directly, one obtains

$$P_1 P_2 - P_3 =: \lambda_k^3 K_1^c + \lambda_k^2 K_2^c + \lambda_k K_3^c + K_4^c + \chi(\lambda_k^2 K_5^c + \lambda_k K_6^c),$$

where

$$\begin{aligned}
K_1^c :=& (d_1 d_2 + d_1 + d_2 + 1)(d_1 + d_2) > 0,\\
K_2^c :=& (d_1 d_2 + d_1)u_c + (d_1 + d_2)(-B_{33}) + (d_1 d_2 + d_2)(-B_{22}) + \xi(d_1 + d_2)b_1 u_c v_c\\
&+ (d_1 + d_2 + 1)\{(d_2 + 1)u_c - (d_1 + 1)B_{22} - (d_1 + d_2)B_{33}\},\\
K_3^c :=& (u_c - B_{22} - B_{33})\{(d_2 + 1)u_s - (d_1 + d_2)B_{33} - (d_1 + 1)B_{22}\}\\
&+ (d_2 + 1)B_{22}B_{33} - (d_1 + 1)u_c B_{33} - (d_1 + d_2)u_s B_{22}\\
&+ [(d_1 + 1)\gamma_2 - \xi]\gamma_1 u_c w_c + (d_2 + 1)b_2 v_c w_c\\
&+ (d_1 + d_2)b_1 u_c v_c + (u_c - B_{22})b_1 \xi u_c v_c,\\
K_4^c :=& -(B_{22} + B_{33})(B_{22}B_{33} + b_2 v_s w_s) - u_c(u_c - B_{22} - B_{33})(B_{22} + B_{33})\\
&+ (u_c - B_{33})\gamma_1\gamma_2 u_c w_c + (u_c - B_{22})b_1 u_c v_c - (\gamma_1 - \gamma_2 b_1 b_2)u_c v_c w_c.
\end{aligned}$$

Also

$$K_5^c := (d_1 + 1)\phi_u^c \gamma_1 u_c w_c + (d_2 + 1)\phi_v^c b_2 v_c w_c - \phi_v^c \xi \gamma_1 u_c v_c w_c, \tag{7.3}$$

and

$$\begin{aligned}
K_6^c :=& (u_c - B_{33})\phi_u^c \gamma_1 u_c w_c + (-B_{22} - B_{33})\phi_v^c b_2 v_c w_c\\
&+ \phi_u^c b_1 b_2 u_c v_c w_c - \phi_v^c \gamma_1 u_c v_c w_c.
\end{aligned} \tag{7.4}$$

Proof of Proposition 5.1 For the corresponding ODE system of (1.3), it has been proved in Hsu et al. (2015) that the constant steady state (u_c, v_c, w_c) is linearly stable under the following conditions:

$$(u_c, v_c, w_c) = \begin{cases} (1, 0, 0), & \text{if } \theta_1 > 1 \text{ and } \theta_2 > \gamma_2,\\ \left(\theta_1, \frac{1-\theta_1}{b_1}, 0\right), & \text{if } \theta_1 < 1 \text{ and } \theta_2 > \frac{b_1\gamma_2 - 1}{b_1}\theta_1 + \frac{1}{b_1},\\ \left(\frac{\theta_2}{\gamma_2}, 0, \frac{\gamma_2 - \theta_2}{\gamma_1\gamma_2}\right), & \text{if } \theta_2 < \gamma_2 \text{ and } \theta_2 < \frac{\gamma_1\gamma_2}{b_2 + \gamma_1}\theta_1 + \frac{b_2\gamma_2}{b_2 + \gamma_1}. \end{cases} \tag{7.5}$$

Under the conditions (7.5), we can derive from (7.1) that $B_{22} \le 0$ and $B_{33} \le 0$, which gives $K_j^c > 0$ $(j = 1, 2, 3, 4)$.

For the prey-only steady state $(1, 0, 0)$ or semi-coexistence steady state $\left(\theta_1, \frac{1-\theta_1}{b_1}, 0\right)$, one obtains $w_c = 0$, which together with the facts $B_{22} \le 0$ and $B_{33} \le 0$ substituted into P_3 in (7.2) implies that for any $k \in \mathbb{N}$

$$\begin{aligned}0 < P_3 = &\lambda_k^3 d_1 d_2 + \lambda_k^2(-d_1 d_2 B_{33} + d_2 u_c - d_1 B_{22} + \xi b_1 u_c v_c)\\ &+ \lambda_k[-u_c B_{22} - d_2 u_c B_{33} + d_1 B_{22} B_{33} + b_1 u_c v_c(1 - \xi B_{33})]\\ &+ u_c B_{22} B_{33} - b_1 u_c v_c B_{33}.\end{aligned}$$

Since $w_c = 0$, according to the definitions in (7.3)-(7.4), one has $K_5^c = K_6^c = 0$, which together with $K_i^c > 0$ $(i = 1, 2, 3, 4)$ implies $P_1 P_2 - P_3 > 0$. Hence, by Routh–Hurwitz criterion, the prey-only steady state E_1 and the semi-coexistence steady state E_{12} are linearly stable.

As for $E_{13} := \left(\frac{\theta_2}{\gamma_2}, 0, \frac{\gamma_2-\theta_2}{\gamma_1\gamma_2}\right)$, one has $v_c = 0$ which, together with $\phi_u \ge 0$, gives

$$K_5^c = (d_1 + 1)\gamma_1 \chi \phi_u^c u_c w_c \ge 0 \text{ and } K_6^c = (u_c - B_{33})\chi \phi_u^c \gamma_1 u_c w_c \ge 0.$$

Using the facts $K_i^c > 0$ $(j = 1, 2, 3, 4)$ again, one obtains $P_1 P_2 - P_3 > 0$ for each $k \in \mathbb{N}$. On the other hand, noting the facts $B_{22} \le 0$, $B_{33} \le 0$, $v_c = 0$ and $\phi_u^c \ge 0$, $\phi_v^c \ge 0$, from (7.2), we get that

$$\begin{aligned}0 < P_3 :=&\lambda_k^3 d_1 d_2 + \lambda_k^2(-d_1 d_2 B_{33} + d_2 u_c - d_1 B_{22} + \chi \phi_u^c d_2 \gamma_1 u_c w_c)\\ &+ \lambda_k\{-u_c B_{22} - d_2 u_c B_{33} + d_1 B_{22} B_{33} + (\gamma_2 d_2 - \chi \phi_u^c B_{22})\gamma_1 u_c w_c\}\\ &+ u_c B_{22} B_{33} - \gamma_1 \gamma_2 u_c w_c B_{22}.\end{aligned}$$

Therefore, $\left(\frac{\theta_2}{\gamma_2}, 0, \frac{\gamma_2-\theta_2}{\gamma_1\gamma_2}\right)$ is linearly stable by applying Routh–Hurwitz criterion. Then we complete the proof of Proposition 5.1.

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