Increasing the Regenerative Braking Energy for Railway Vehicles

Shaofeng Lu, Paul Weston, Stuart Hillmansen, Hoay Beng Gooi, and Clive Roberts

Abstract—Regenerative braking improves the energy efficiency of railway transportation by converting kinetic energy into electric energy. This paper proposes a method to apply the Bellman–Ford (BF) algorithm to search for the train braking speed trajectory to increase the total regenerative braking energy (RBE) in a blended braking mode with both electric and mechanical braking forces available. The BF algorithm is applied in a discretized train-state model. A typical suburban train has been modeled and studied under real engineering scenarios involving changing gradients, journey time, and speed limits. It is found that the searched braking speed trajectory is able to achieve a significant increase in the RBE, in comparison with the constant-braking-rate (CBR) method with only a minor difference in the total braking time. An RBE increment rate of 17.23% has been achieved. Verification of the proposed method using BF has been performed in a simplified scenario with zero gradient and without considering the constraints of braking time and speed limits. Linear programming (LP) is applied to search for a train trajectory with the maximum RBE and achieves solutions that can be used to verify the proposed method using BF. It is found that it is possible to achieve a near-optimal solution using BF and the solution can be further improved with a more complex search space. The proposed method takes advantage of robustness and simplicity of modeling in a complex engineering scenario, in which a number of nonlinear constraints are involved.

Index Terms—Bellman–Ford (BF) algorithm, computation efficiency improvement, dynamic programming, energy-saving strategy, train braking energy increment.

I. INTRODUCTION

ENERGY conservation is becoming more important for modern rail transportation. It is reported that traction energy accounts for 60%–70% of the total energy consumption in rail transport systems [1]. The traction energy is consumed to overcome the resistances and is transformed into kinetic energy and heat energy. Regenerative braking converts kinetic energy into electrical energy and thus reduces the total energy cost [2], [3].

A typical electric machine generally holds two working modes: motoring and regenerative. During motoring mode, the direction of motor rotating speed agrees with the direction of torque. On the contrary, when the direction of rotating speed opposes the direction of torque, the electric machine enters the regenerative mode. For a railway vehicle, during the regenerative braking mode, the torque reduces the motor speed and generates the electric power.

The regenerative braking energy (RBE) will be converted by power electronic devices into electric energy, which can be fed back into the electric power grid in the ac electric network, and is used by other adjacent running trains through the dc electric network or stored in the energy storage devices (ESDs). Otherwise, the regenerative energy can be converted into heat using a large resistance bank referred to as the “dynamic braking” [4].

Regenerative braking takes a key role in energy efficiency for the rail transport. It is affected by various parameters and the parameters in a dc railway case are summarized in [5]. In order to improve energy efficiency via recapturing more RBE, different methods have been adopted in past research.

• The integration of ESDs reduces the dependence on the power transmission network and increases the effective RBE to be stored. Research works proposed in various research papers have focused on optimizing the ESDs to improve energy efficiency [6]–[10].

• More regenerative energy can be also achieved by improving the receptivity of the dc network since the amount of recaptured regenerative energy is constrained by the network receptivity [11]. Timetable optimization, as one of the feasible methods, has been applied to improve the network receptivity to increase the recaptured RBE [12].

• Optimization of the braking effort control strategy can lead to an improvement in energy efficiency. Optimization on the braking force distribution strategy is reported in [13] and [14], to maximize the recreative braking energy. The research work proposed in [15] and [16] is focused on optimizing the braking torque based on the electric motor characteristics.

The optimization methods of the braking effort control can be achieved by the train speed trajectory optimization to some extent since the train speed trajectory is the direct consequence of the applied torque of the motor. A recent paper has considered the RBE in their automatic train operation (ATO) speed profile design [17]. Optimal train control to locate the
energy-efficient train speed trajectory has been extensively studied over the past two decades [18]–[21]. Regenerative braking in these studies can be involved as part of the constraints for train control. The braking rate has also been taken into account for the speed profile optimization of a train with regenerative braking [22]. These papers only take the RBE as a part of the optimization objectives or constraints. The research work proposed in this paper, on the other hand, is focused on increasing the regenerative electric braking energy (RBE) via improving the train braking speed trajectory, which is referred to as the “braking trajectory.” While the maximization of RBE does not necessarily lead to the maximization of the total journey energy efficiency, this paper is motivated by the energy efficiency improvement to be made by the braking trajectory optimization. The research work proposed in this paper can be applied to evaluate the positive impact that the regenerative braking may have on an existing railway route with limited spaces for an optimization of the entire speed trajectory, e.g., a train required to operate aggressively. A suburban train has been modeled and the method of the modeling has been used in a number of previous works [23], [24]. It is assumed that the blended braking is applied where both electric and mechanical braking are available. A graphical search method, i.e., the Bellman–Ford (BF) algorithm [25], [26], is proposed to search for the optimal braking trajectory with the maximum RBE within the search space. The searched braking trajectory will be compared with the braking trajectory using the constant braking rate (CBR) to investigate the advantage of the proposed method. The proposed method will be further verified by the results achieved by linear programming (LP) in a simplified scenario.

The organization of this paper is presented as follows. Section I covers the background introduction, literature review, and research motivation. Section II covers the formulation of the objective function and its optimization system modeling. Section III introduces the optimization method: the BF algorithm. Section IV demonstrates case studies for a suburban train using BF. Section V covers the verification of our proposed method. Finally, the conclusion is drawn in Section VI.

II. SYSTEM MODELING

The braking process of a typical rail vehicle is modeled in a discrete manner shown in Fig. 1. To simplify the problem, it is assumed that the conversion from kinetic energy to electric energy is 100%. In Fig. 1, the braking candidate distance and speed are constrained by the forward and backward calculation using the maximum braking rate. Each train state is a combination of three variables: instant train speed \( v \) and braking time \( t \) and instant distance \( d \) from the final state shown in black circle in Fig. 1. Each gray round circle in Fig. 1 represents a group of train states with the same braking speed and distance but different braking time. Other train states should not have a zero speed or a speed exceeding the speed limit. The braking time for each train state since the final train state should not exceed the allowed braking time. If a train state is able to switch to another, the braking rate between these two states should be less than the maximum braking rate. In this paper, the train is allowed to keep its speed when it switches from one state to another, resulting in a maximum acceleration rate of 0 m/s\(^2\). If needed, the train can be allowed to re-motor and increase its speed in the braking process.

The modeling and optimization procedure is divided into the following three steps.

Step 1: Determine the braking candidate distance and speed. The braking process is divided into different subdivisions between the starting position and the ending position. The position to divide the total braking distance is referred to as the candidate position, and the speed on each candidate position is referred to as the candidate speed. The speed at each candidate position is constrained between the two braking speed trajectories by backward and forward calculations using the maximum braking rate. Both calculations should not exceed the speed limit, if any, and should be lower than zero speed.

Step 2: Generate the train states. The braking trajectory consists of a series of train-state switches and is generated using the following Lomonossoff’s equation in (1). A detailed introduction on the state-switch calculations has been covered in [23] and [27]. That is

\[
M \frac{dv}{dt} = F - (A + Bv + Cv^2) - Mg \sin(\alpha)
\]  

(1)

where,

- \( F \) is the tractive effort or braking effort if applicable within the maximum acceleration and deceleration rate;
- \( A, B, \) and \( C \) are the Davis coefficients;
- \( M' \) is the effective mass including rotary allowance;
- \( M \) is the tare mass;
- \( g \) is the acceleration due to gravity;
- \( v \) is the instantaneous train speed;
- \( t \) is the instantaneous time;
- \( \alpha \) is the slope angle.

Between two train states with different speeds \( v_1 \) and \( v_2 \) and distances \( d_1 \) and \( d_2 \), where \( v_2 > v_1 \) and \( d_2 > d_1 \), the braking rate \( a_{br} \) is assumed to be constant and can be calculated in

\[
a_{br} = \frac{v_2^2 - v_1^2}{2(d_2 - d_1)}.
\]  

(2)
The calculation of the total electric braking energy between every two train states will be further achieved in each minor iterative step. In each minor step, the total braking effort (TBE) \( F_{\text{tb}} \) is calculated using the braking rate \( a_{\text{br}} \) obtained in (2). Assume that a train is with speed at a minor distance of \( \Delta D \). The TBE \( F_{\text{tb}} \) is a combination of the electrical braking force \( F_{\text{eb}} \) and the mechanical braking force \( F_{\text{mb}} \). The electrical braking regenerative energy \( E_{\text{eb}} \) in each minor step is calculated in

\[
E_{\text{eb}} = F_{\text{eb}} \Delta D. \tag{3}
\]

Assume that \( F_{\text{eb}}^{\text{max}} \) is the maximum available electrical braking force at current speed \( v \). The electric braking force and the total braking force are represented by

\[
F_{\text{eb}} = \min (F_{\text{eb}}^{\text{max}}, F_{\text{tb}}) \tag{4}
\]
\[
F_{\text{tb}} = a_{\text{br}} \ast M'. \tag{5}
\]

In Fig. 2, a schematic diagram of braking efforts is shown. The black solid line represents the maximum electric braking effort (EBE); the blue dashed line stands for an instantaneous constant braking effort imposed on the train during braking; the red dotted line represents the maximum feasible EBE; and the gray patterned area indicates the electric braking operation range.

**Step 3:** Construct a directed weighted graph \( G = (V, E) \) for optimization of the shortest path. Using the method in the second step, one is able to calculate the RBE and time usage while a train is braking from one train state to another using the method in Step 2. In a graph \( G = (V, E) \), \( V \) is the set of vertices in a graph to represent the train speed state, and \( E \) is to represent the set of edge connections between the states. The value for the RBE for braking between two states is negative. Thus, the shortest path of series of edges with minimum values between two states will represent the braking trajectory in association with the maximum RBE. The detail of the shortest path search algorithm will be covered in the next section. It is worth mentioning that journey time constraints will be imposed, so that only the path within the demanded journey time window will be regarded as valid.

### III. BF Algorithm

The BF algorithm is commonly applied to compute single-source shortest paths in a weighted directed graph [25], [26], [28]. Compared with Dijkstra’s algorithm [29], BF is able to cope with graphs with negative edge weights. The worst case performance for BF is \( O(|V| \ast |E|) \), where \(|V| \) and \(|E| \) are the numbers of vertices and edges. Each vertex represents a train state and the distance between two vertices represents the RBE generated when the train switches from one state to another.

Before the illustration of BF (see Fig. 3), some definitions are made as follows.

- Let \( i \) and \( j \) be any vertices in the vertex set \( V : i, j \in V \).
- Let \( d[i] \) represent the shortest distance from the source for vertex \( i \), and it is set infinity for initialization.
- Let \( p[i] \) be the previous vertex for \( d[i] \); \( p[i] = 0 \) for the source vertex.
- Let \( w \) be the weight matrix, where \( w(i, j) \) is the edge for vertices \( i \) and \( j \) representing the electric braking energy for trains states switching from vertex \( i \) to \( j \).

For any vertex \( i \), if there exists another random vertex \( j \), through which \( i \) is able to achieve a shorter distance to the initial vertex than its current shortest path, the shortest path between \( j \) and the initial vertex will be replaced by the new path involving \( i \). This is called a relaxation process, and the \( \text{RELAX} \) function is used to perform this process. Note that the \( \text{RELAX} \) function will be repeated for each vertex, to ensure that the distance for each vertex will be reduced to the minimum as long as there are no negative cycles. Negative cycles are eliminated by only allowing one-direction generation of train braking states at Step 3.

The pseudocode for the \textsc{Bellman-Ford} and \textsc{Relax} function is shown as follows.

**Algorithm 1 RELAX function**

```
if w(i, j) + d[i] < d[j] then
    d[j] ← w(i, j) + d[i]
    p(j) ← i
end if
```
Algorithm 2 **BELLMAN-FORD** function

```plaintext
for all \( i \) in \( V \) \n\hline 
\( d[i] \leftarrow \infty \) \hline 
Original distance from each vertex to the \nsource is infinity \n\hline 
\( p[i] \leftarrow i \) \hline 
Previous vertex being itself \n\hline 
end for 

for \( k := 1 \rightarrow (|V| - 1) \) do 
for all \((i,j)\) in \( E \) do 
RELAX \((i,j,w,d,p)\) 
end for 
end for 

for all \((i,j)\) in \( E \) do \hline 
\text{Identify the negative cycle in the graph} 
if \( w(i,j) + d[i] < d[j] \) then 
Return(\text{FALSE}) 
else 
\ldots 
end if 
end for 
```

It is well known that BF can become computationally infeasible if the search space grows large. In this paper, the following strategies are adopted to improve the computational efficiency, but nevertheless, a tradeoff between the optimality and computation efficiency should be maintained based on the system requirement.

- Parallel computation has been applied during the generation of states. For example, the information of the travelling time and RBE between two train states can be obtained in parallel.
- Heuristics can be applied to eliminate two identical states. For example, if two train states have the same travelling time from the initial state, and the same current speed and distance, the train state with the less RBE can be eliminated.
- The discretization of the state space can be applied. For example, since the journey time difference (in seconds) is insignificant, each state can be modeled with the unit second.

**IV. CASE STUDIES**

Case studies for a typical suburban train under a practical engineering scenario have been conducted. A suburban train will brake from a nominal speed until it finally stops. The total braking distance remains 1750 m in this paper. In addition to BF, the CBR method will also be applied. A random CBR of 0.5632 m/s² is proposed based on \( a = 0.5 \times \frac{v^2}{s} \), where \( v \) is the maximum train speed 44.4 m/s and \( s \) is the braking distance 1750 m. In fact, the CBR can be varied to generate different braking curves within the braking distance. The CBR method is implemented in a backward calculation using this CBR across the entire braking distance. The train will take a cruising operation, when a speed limit is imposed. The initial speed and total braking time will then be determined by CBR and will also be used for BF. Using this backward calculation, the initial speed and proposed journey time for both methods were obtained at 41.8 m/s and 80.86 s, respectively. Note that other constraints, such as the speed limits and gradients, are imposed as well. Finally, the results achieved by CBR will be compared with the results achieved by BF.

Fig. 4 shows the braking effort and resistance effort characteristics for a typical suburban train [30]. The Davis equation [31] is considered to account for the resistive force imposed on the train. The relative altitude profile is shown in Fig. 5.

All other essential parameters are listed in Tables I and II. Note that \( M \) and \( M' \) are the train mass and train effective mass (in tons), respectively. \( P_{\text{max}} \) is the maximum train generation power. \( v_{\text{max}} \) is the maximum allowed train speed. \( a_{\text{br}}^\text{max} \) is the maximum train braking rate. In Table II, the minimum distance interval is the minimum distance between two train states. The maximum distance interval will be adopted to ensure that the gradients and speed limits remain the same within one distance.
TABLE II

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Minimum distance interval (m)</td>
<td>100</td>
</tr>
<tr>
<td>Maximum distance interval (m)</td>
<td>200</td>
</tr>
<tr>
<td>Minor distance interval (m)</td>
<td>1</td>
</tr>
<tr>
<td>Minimum speed interval (m/s)</td>
<td>0.1</td>
</tr>
<tr>
<td>Minimum time interval (s)</td>
<td>1</td>
</tr>
</tbody>
</table>

TABLE III

<table>
<thead>
<tr>
<th>Simulation Results</th>
<th>$T_{CBR}$ (s)</th>
<th>$T_{BF}$ (s)</th>
<th>$E_{CBR}^{eb}$ (kWh)</th>
<th>$E_{BF}^{eb}$ (kWh)</th>
<th>Increment Rate (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>80.86</td>
<td>82.48</td>
<td>27.46</td>
<td>32.19</td>
<td>17.23</td>
</tr>
</tbody>
</table>

Fig. 6. Comparison of the braking speed trajectories by CBR and BF.

interval. The minor distance interval is the iterative simulation step to calculate the energy consumption between two train states. The minimum speed interval and the time interval are the minimum difference between the speed and time of every two states with two of the elements being the same. For example, if both the distance and speed are the same for two train states, the time difference between them should not be less than the minimum time interval.

After the electric braking energy is calculated using the single-train simulator, a weighted connected graph will be created, and the implementation of BF is applied using the MatlabBGL library in MATLAB [32].

The total RBE achieved by CBR and BF are denoted by $E_{CBR}^{eb}$ and $E_{BF}^{eb}$, respectively. The total braking time by these two methods are denoted by $T_{CBR}$ and $T_{BF}$. BF searches a braking speed trajectory with various braking rates so that the RBE can be maximized within the search space.

Table III summarizes the simulation results. A comparison of the braking speed trajectories achieved by CBR and BF is shown in Fig. 6. The EBE and TBE by both methods are demonstrated in Fig. 7.

In Table III it is found that, using BF, an increase in RBE can be significantly achieved compared with the one by CBR. A slight deviation on the journey time for BF can be observed. In Fig. 7, the EBE in solid lines and the TBE in dashed lines are shown for both braking trajectories achieved by BF and CBR.

Fig. 7. Comparison of the braking effort by CBR and BF.

It has been demonstrated that BF takes advantage by incorporating the nonlinear constraints into the generation of states and is able to solve the RBE-maximization problem in a relatively convenient way. As will be demonstrated in Section V, the state discretization prevents BF from obtaining a globally optimal solution, but an increase in train states can be realized by reducing the minimum distance between every two train states and will lead to an increase in the achieved maximum RBE and to a longer computation time.

V. VERIFICATION OF THE PROPOSED METHOD

Here, a simplified case scenario, in which a train is braking on a level track with only nonpositive braking forces applied, will be discussed. With a reasonable approximation, LP can achieve an optimal braking trajectory with the maximum RBE. The results achieved by LP will be used to verify our proposed method. An RBE-maximization model is proposed and solved by LP. The details of the modeling are covered in Section V-A.

A. LP Modeling

As shown in the Appendix, it can be proved that, if the initial train speed is either too high or too low, an optimal solution for the RBE-maximization problem in a simplified scenario will only include part or all of the following three operations:
- maximum-braking operation;
- maximum-electric-braking operation;
- coasting operation.

Since the resistance exists along the entire braking operation, the optimal train braking speed trajectory will keep decreasing until a full stop. Assume that $v_1, v_2, \ldots, v_N$ are a set of speeds on the reducing braking speed trajectory of a train, with $v_N$ being zero and $v_1$ being the initial speed before braking. $N$ is a sufficiently large number and represents for the total number of the intermittent candidate speeds. $N$ is large enough to ensure that the speed change between two adjacent candidate speeds is so small that the average speed calculated based on these two speeds will be able to closely approximate the actual average speed with a satisfactory precision. A schematic for the discretized braking speed versus distance is shown in Fig. 8.
It is assumed that the difference between two adjacent braking speeds is constant. Therefore, as long as the initial and final speeds are known, all the intermittent speeds \( v_i \), \( i = 2, 3, \ldots, N - 1 \) will be known and can be calculated using

\[
v_i = v_1 - (I - 1) \frac{v_N - v_1}{N - 1}, \quad i = 2, 3, \ldots, N - 1.
\]

The average speed \( v_i^a \) between \( v_i \) and \( v_{i+1} \) is calculated by

\[
v_i^a = \frac{(v_i + v_{i+1})}{2}, \quad i = 1, 2, \ldots, N - 1.
\]

The average resistance between \( v_i \) and \( v_{i+1} \) is calculated by

\[
F_{i}^{re} = A + Bv_i^a + C (v_i^a)^2, \quad i = 1, 2, \ldots, N - 1
\]

where \( A, B, \) and \( C \) are the Davis coefficients.

The average maximum EBE between \( v_i \) and \( v_{i+1} \), which is denoted by \( F_{i}^{re,\max} \), can be obtained by linearly incorporating the braking characteristics shown in Fig. 4 using \( v_i^a \).

The distance between \( v_i \) and \( v_{i+1} \) is denoted by \( d_i, i = 1, 2, 3, \ldots, N - 1 \). The total distance constraints should be met as defined in

\[
D = \sum_{i=1}^{N-1} d_i
\]

where \( D \) is the total braking distance.

The average deceleration rate will be defined as

\[
a_{i}^{br} = \frac{v_{i+1}^2 - v_i^2}{2d_i}.
\]

\( a_{i}^{br} \) should be less than the maximum braking rate \( a_{br}^{\max} \), as shown in Table I. Hence, we can derive

\[
\frac{v_{i+1}^2 - v_i^2}{2d_i} \leq a_{br}^{\max}.
\]

It consequently yields

\[
d_i \geq \frac{v_{i+1}^2 - v_i^2}{2a_{br}^{\max}}.
\]
producing a discrete braking effort profile. In Table IV it is noted that BF is unable to achieve a solution as good as the solution achieved by LP, due to discretization.

In this paper, the minimum distance between two train states has been varied to change the total generated train states. A shorter minimum distance leads to more train states being generated. Generally, more train states will lead to more RBE, and the achieved solution will gradually approach a global optimal solution by searching all the possible train states. However, due to “the curse of dimensionality,” the train states will increase so greatly that optimization by BF will be practically impossible due to the limited computation capability. A sensitivity analysis has been conducted between the RBE achieved and the minimum distance between two train states for the case with an initial speed of 44.4 m/s. For comparison reasons, the RBE achieved by both LP and CBR are also demonstrated. The CBR can be simply obtained from the total braking distance and the initial braking speed. The results are shown in Fig. 13.

In Fig. 13 it is found that an increase in the minimum distance between every two train states will lead to less RBE. A minimum distance of 50 m between two train states will generate a total RBE of 37.60 kWh, which is also shown in Table IV, and a higher minimum distance will reduce the train states and RBE achieved. Compared with the results by CBR it is noted that, in this simplified case, BF is able to achieve a significant increase in the RBE. It is observed that BF is only able to achieve a suboptimal solution due to discretization of the train states, but a significant increase in RBE can be achieved, as compared with the one achieved by CBR, with desirable simplicity and robustness. Note that LP cannot be applied in nonlinear cases, the advantages of BF becomes significant when a train is required to brake along a distance with various speed limits and gradients.
VI. CONCLUSION

It is demonstrated in this paper that the RBE can be significantly increased for the searched train braking trajectory compared with the conventional braking trajectory using a nominal CBR. A shortest path search method, i.e., the BF algorithm, has been proposed to search for the braking trajectory to increase the RBE generated. BF has been applied in a case scenario with nonlinear constraints, such as speed limits, gradients, and braking time constraints. The optimization results demonstrate that a significant increase can be achieved compared with the one by CBR. The solutions by BF have also been verified by LP in a simplified case scenario. The obtained braking trajectory is suitable to provide a guidance during braking operation. Future works will be to develop an algorithm to improve the computation efficiency of the algorithm for the RBE optimization problem. Conclusions of this paper are drawn as follows.

- In a discretized train-state space, BF is able to search for the optimal braking trajectory with the maximum RBE under different constraints with a high level of robustness. A good tradeoff between the computation efficiency and the solution quality should be maintained.
- In a case scenario with speed limits, nonzero gradients, and a narrow allowable braking time window, BF is able to locate a train braking trajectory with a significant increase in RBE compared with the one achieved by CBR.
- In a simplified case scenario, an optimal solution can be achieved by LP and has been used as verification for BF. The RBE achieved by BF is slightly less than the RBE by LP. However, an increase in RBE can be obtained compared with the one achieved by CBR. The proposed method using BF takes advantage of its easiness and robustness on the application in a case scenario with nonlinear constraints.

APPENDIX

OPTIMALITY ANALYSIS USING PMP

Similar to Section V, no speed limits and journey time constraints are considered for a train on a zero-gradient track in this section. Only braking force can be applied in mechanical or electrical braking. This implies that the speed of train will be at least decreased by resistance. An optimality analysis for a braking trajectory to achieve the maximum braking energy is conducted using the Pontryagin’s Maximum Principle (PMP). Similar studies have been proposed for the optimization of the entire train trajectory in papers [18]–[21]. The motion of train during braking is modeled by

\[\frac{dt}{ds} = \frac{1}{v}\]  
\[M' \frac{dv}{ds} = f_{eb} + f_{mb} - f_r\]  

where \(t\) is the time; \(s\) is the distance; \(v\) is the train speed during braking; \(f_{eb}\), \(f_{mb}\), and \(f_r\) are the electric braking force, mechanical braking force, and resistance, respectively. Since zero gradient is considered, the force due to the gravity is omitted. \(f_{eb}\) is the electric braking force between zero and the maximum negative braking force \(f_{eb}^{max}\) since it is assumed that no motoring is allowed during this simplified braking process with a speed that keeps decreasing by resistance and possible braking force. \(f_{eb}^{max}\) depends on the current speed. \(f_{mb}\) is a nonpositive mechanical braking force limited by the maximum braking rate. \(M'\) is the train effective mass accounting for the rotary inertia. \(f_r\) is the force depending on the current speed; specifically, it can be presented by

\[f_r(v) = A + Bv + Cv^2\]  

where \(A\), \(B\), and \(C\) are the Davis coefficients.

The instantaneous RBE \(E_{eb}\) recovered is represented by

\[\frac{dE_{eb}}{ds} = -f_{eb}\]  

where the braking operation with a negative \(f_{eb}\) increases the RBE.

Assume that the total braking distance is \(S\) with an initial speed of \(v_1\). The boundary conditions are shown as

\[t(0) = 0 \quad t(S) < \infty\]  
\[v(0) = v_1 \quad v(S) = 0.\]  

Based on the PMP, the Hamiltonian is defined as

\[H = \frac{dE_{eb}}{ds} + \lambda_1 \frac{dv}{ds} + \lambda_2 \frac{dt}{ds}\]  
\[= -f_{eb} + \frac{\lambda_1}{M'v}(f_{eb} + f_{mb} - f_r) + \frac{\lambda_2}{v}\]  
\[= \left(\frac{\lambda_3}{M'v} - 1\right)f_{eb} + \frac{\lambda_1}{M'v}f_{mb} + \frac{\lambda_3}{M'v}(-f_r) + \frac{\lambda_2}{v}.\]  

\(\lambda_1\) and \(\lambda_2\) are the two control inputs. Let \(\mu\) denote the term \(\lambda_1/M'v\). Based on PMP, in order to maximize the Hamiltonian, the following observations can be made.

- If \(\mu > 1\), both \(f_{eb}\) and \(f_{mb}\) should be zero. This corresponds to a coasting operation.
- If \(\mu \in (0,1)\), \(f_{eb}\) should be as negative as possible, while \(f_{mb}\) should be zero. \(f_{eb} = f_{eb}^{max}\) if the resulted braking rate does not exceed the maximum braking rate. This operation is called the “full-electric-braking operation.”
- If \(\mu < 0\), both \(f_{eb}\) and \(f_{mb}\) should be as negative as possible. \(f_{eb}\) should increase first. The maximum braking rate constraints should be met. This operation is called the “full-braking operation.”
- If \(\mu = 0\) or \(\mu = 1\), a singular mode of train control will result.

Two adjoint equations are presented as

\[\frac{d\lambda_1}{ds} = -\frac{\partial H}{\partial v}\]  
\[\frac{d\lambda_2}{ds} = -\frac{\partial H}{\partial t} = 0.\]  

\(v\) is a variable that depends on \(s\), and together with (21), (15) and (16), we derive

\[\frac{d\mu}{ds} = \frac{1}{M'v^3}\left(\lambda_2 + \mu v^2 f'_r(v)\right)\]  

where \(f'_r(v)\) is the derivative of (17), i.e., \(f'_r(v) = B + 2Cv\).
In a singular operation mode, \( \mu = 1 \) or \( \mu = 0 \); hence, \( d\mu/ds = 0 \). Given that \( \lambda_2 \) is constant based on (22), it can be derived that

\[
-V^2 f'_r(V) = \lambda_2
\]

where \( V \) is a constant speed during both singular modes. The nonzero resistance defined by (17) is always against the motion of train, and only nonpositive electric or mechanical braking forces can be imposed. In order to keep a constant speed, the speed should be less than zero, so that the negative braking force can be equal to the resistance but in an opposite direction. In addition, \( V \) cannot be between 0 and \( -(B/C) \) because the resistance cannot be less than \( A \). It also means that the singular mode due to \( \mu = 0 \) is infeasible. Within the proposed modeling context, if \( \mu = 0, V = 0 \) and the train is unable to keep a zero speed with a constant negative resistance \( A \) and nonpositive braking force. Therefore, \( f'_r(V) < -B <= 0 \) for \( V < -(B/C) \). A schematic is shown in Fig. 14, for illustration of the feasible range of the speed during a singular mode due to \( \mu = 1 \).

A variable \( \xi \) is defined as

\[
\xi = -V^2 f'_r(V) \quad \frac{v^2}{f'_r(v)}.
\]

It is noted that \( \xi < 0 \) since \( f'_r(V) < 0 \) and \( f'_r(v) > 0 \). By rearranging (23), we derive

\[
d\mu = \frac{1}{M'v^3} \left( \lambda_2 + \mu v^2 f'_r(v) \right)
\]

\[
= \frac{\lambda_2}{M'v^3} + \frac{\mu}{v^2 f'_r(v)}
\]

\[
= \frac{f'_r(v)}{vM'} (\mu - \xi).
\]

It is noted that \( (f'_r(v)/vM') > 0 \), given that \( v > 0, f'_r(v) > 0 \), and \( M' > 0 \). There are four possible cases with regard to the initial value of \( \mu \).

**Case 1** If initially \( \mu \in (1, \infty), \mu > \xi \) and \( (d\mu/ds) > 0 \), \( \mu \) will keep increasing and the train will keep coasting until the end.

**Case 2** If initially \( \mu \in (0, 1), \mu > \xi \) and \( (d\mu/ds) > 0 \). As a result, the full-electric-braking operation should apply. \((d\mu/ds) > 0, \mu \) will keep increasing to be more than 1, and thus, a coasting operation will be applied until the end.

**Case 3** If initially \( \mu \in (\xi, 0), \mu > \xi \) and the full-braking operation should be applied. \( \mu \) will also keep increasing, and operations in **Case 2** and **Case 1** will apply subsequently.

**Case 4** If initially \( \mu \in (-\infty, \xi), \mu < \xi \) and the operations in **Case 3** will apply. However, in this case, \((d\mu/ds) < 0, \mu \) will keep decreasing, and the full-braking operation always applies. Hence, the train will keep braking with maximum braking force until the end.

It is noted that **Case 1** and **Case 4** cannot occur in most cases. Only coasting and full-braking operation will hardly achieve a desirable braking trajectory. Without considering the journey time and speed limits, the optimal braking control on a level track will be with an initial \( \mu \in (\xi, 0) \) or \( \mu \in (0, 1) \). Therefore, an optimal braking operation involves the full-braking operation, the full-electric-braking operation, and a coasting operation subsequently. These three operations may not necessarily coexist, but the operation order will remain the same, i.e., a coasting operation comes after the full-electric-braking operation, and the full-electric-braking operation comes after the full-braking operation. If a train has a very high initial speed and cannot brake using the full-braking operation, no feasible solution exists. Similarly, if a train has a very low speed and cannot maintain a positive speed before reaching the next station, no feasible solution exists as well. In such a case, motoring operations will be necessary to achieve a feasible braking trajectory.

**REFERENCES**


