High-Order Soliton Solution of Landau–Lifshitz Equation

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The Landau–Lifshitz equation is analyzed via the inverse scattering method. First, we give the well-posedness theory for Landau–Lifshitz equation with the frame of inverse scattering method. The generalized Darboux transformation is rigorous considered in the frame of inverse scattering transformation. Finally, we give the high-order soliton solution formula of Landau–Lifshitz equation and vortex filament equation.

1. Introduction

The Landau–Lifshitz (L–L) equation [32]

$$\vec{S}_t = \vec{S} \times \vec{S}_{xx}, \quad \vec{S}(x,t) = (S^x, S^y, S^z)^T \in \mathbb{R}^3, \quad \vec{S}\cdot\vec{S} = 1, \tag{1}$$

describes nonlinear spin waves in an isotropic ferromagnet, where the symbols ^{T} and $\times$ mean the transpose and vector product respectively, $\vec{S}(x,t)$ is magnetization vector. Setting $\vec{S} = \vec{\gamma}_x$ and integrating (1) with respect to x, we can obtain another relative physical model—vortex filament equation (VFE) or localization induction equation

$$\vec{\gamma}_t = \vec{\gamma}_x \times \vec{\gamma}_{xx}, \;\; \vec{\gamma} = (\gamma^x, \gamma^y, \gamma^z)^T, \tag{2}$$

which is the simplest model of dynamics of Eulerian vortex filament, where space vector $\vec{\gamma}(x,t)$ represents the vortex filament, x is the arclength parameter, t is time. The model (2) was first derived by Da Rios, a student of Levi-Civita, in 1906 [37], and rediscovered by Arms and Hama in 1965 [4]. The model (2)

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also can be used to describe the flow of superfluids [39], to investigate the turbulent fluid [9, 44] and high-temperature superconductors [12].

It is well known that the inverse scattering method [1, 5, 23] is a powerful method to solve the cauchy problem of nonlinear integrable partial differential equation. In the past 40 years, the inverse scattering method had made great development in the field of mathematical physics. Initially the inverse scattering transformation utilizes Marchenko integral equation to reconstruct the potential function [23]. Afterwards Shabat used Riemann–Hilbert problem (RHP) to reconstruct the inverse scattering method [41]. In the last century, nineties, the RHP method had made important progress. For instance, the Deift–Zhou method [15–17] and initial-boundary problem [20, 21].

In the case of KdV equation, the poles of discrete spectrum must be simple, because the Lax operator is self-adjoint. However, to the focusing nonlinear Schrödinger (NLS) equations, the corresponding Lax operator is no longer self-adjoint. Thus it allows high-order pole, which corresponds to the high-order soliton. The scattering data are demanded for simple pole in the classical paper of Beals and Coifman [5]. Several years later, this restraint was removed by [40] and [49], respectively. However, they didn't give the exact soliton formula. The exact high-order soliton solution for NLS equation was given in [22] by the dressing method. The general soliton formula for NLS-type equation had been constructed by Shchesnovich and Yang [42, 43]. And, the high-order transmission coefficient by the Marchenko equation method was considered by Cohen and Kappeler [11]. Recently Aktosun et al. consider the high-order soliton solution of NLS equation with inverse scattering method by Gelfand-Levitan-Marchenko equation [3]. The exact second-order soliton solution of L–L equation was obtained by bilinear method in 1990 [6]. Besides the high-order pole, another interesting problem is the infinite pole and infinite soliton. To the best of our knowledge, the concept of infinite soliton was first provided by Zhou [49]. The explicit infinite soliton solution for KdV equation was rigorous and established by Gesztesy et al. [24]. The infinite soliton solution of NLS equation is obtained by Kamvissis [31]. Besides the high-order pole and infinite pole, the spectral singularity is also an obstacle to the inverse scattering method. This problem was first solved by Zhou [49] via the deformed RHP.

As well as the inverse scattering method, the Darboux–Bäcklund transformation is another powerful method to derive the multisoliton and other interesting physical solution. There are several methods to derive the Darboux transformation: for instance, state space method [25, 35, 38], the loop group method [46], and gauge transformation [26, 34]. The relation between different versions of Darboux–Bäcklund transformation had been indicated by Ciéliński [10]. Generally speaking, the Darboux transformation is merely a way to obtain the soliton solution in soliton theory. However, it has other utilization also. Deift and Trubowitz combined Darboux transformation with inverse scattering method for the Schrödinger spectral problem [14]. In this work, we would like to inherit their idea. The Darboux transformation can be used to deal with the initial boundary problem either [13, 21]. Besides, the Darboux transformation can be used for the analysis of orbitally stability property of soliton as well [36].

Finally, we recalled some results of L–L Equation (1). In 1977, Takhtajan used inverse scattering method to derive the two-soliton solution and infinite sets of constants for the first time [45]. The gauge equivalence between NLS equation and L–L equation was obtained by Zakharov and Takhtajan [47] in the frame of inverse scattering transformation. Indeed, this gauge transformation is another version of Hasimoto transformation essentially. The generalized Hasimoto transformation was rigorously considered with tools of differential geometry in [8]. Recently, Calini et al. considered the spectral stability property for soliton and periodical solution of VFE [7].

In this work, first we prove the global well-posedness for L–L equation with initial data in space $H^{2,1}(\mathbb{R})$ without discrete scattering data via RHP method. Second, we handle generalized Darboux transformation [27, 28] in a rigorous way with the frame of inverse scattering method. Via this method, the global solution and the general soliton solution formula of L–L equation is obtained. What need alludes is, for the evolution of discrete scattering data, we use the evolution of eigenfunction replaced with the proportionality coefficient. In this way, we can readily deal with the evolution of high-order spectrum.

This paper is organized as following. In Section 2, we give the scattering and inverse scattering analysis for L–L equation. To establish the well-posedness theory, we combine the gauge transformation and inverse scattering method. In Section 3, we give the Darboux transformation in the frame of inverse scattering. In Section 4, the explicit general soliton formula of L–L equation is constructed. The final section includes some discussions and remarks.

2. The scattering, inverse scattering, and well-posedness theory

It is well known that the KdV, MKdV, sine-Gordan, and NLS can be obtained from the AKNS hierarchy [1] by some symmetry reduction. The symmetry reduction is called the reality condition [46], which is also the solvable condition for the RHP. Thus in this section, we first recall the symmetry condition. Then we give the scattering, inverse scattering analysis, and gauge transformation theory to L–L equation.

The focusing NLS

$$\mathrm{i}q_t + q_{xx} + 2|q|^2 q = 0 \tag{3}$$

is the second flow of $su(2)$ (the fixed-point set of the involution $\sigma(y) = -y^{\dagger}$, where superscript "†" represents hermite conjugation) hierarchy, and turns out to be a compatibility condition for the following linear system

$$\begin{aligned}\Phi_x &= U(\lambda)\Phi, \quad U(\lambda) \equiv -\mathrm{i}\lambda\sigma_3 + Q, \\ \Phi_t &= V(\lambda)\Phi, \quad V(\lambda) \equiv -2\mathrm{i}\lambda^2\sigma_3 + 2\lambda Q - \mathrm{i}(Q^2 + Q_x)\sigma_3,\end{aligned} \tag{4}$$

here $Q = \begin{pmatrix} 0 & q(x,t) \\ -q^*(x,t) & 0 \end{pmatrix}$, superscript "*" represents complex conjugation, and σ_3 is standard Pauli matrix. It is readily seen that the matrices $U(\lambda)$ and $V(\lambda)$ possess the reality relation $U^\dagger(\lambda^*) = -U(\lambda)$ and $V^\dagger(\lambda^*) = -V(\lambda)$.

The L–L Equation (1) can be rewritten as

$$S_t = \frac{\mathrm{i}}{2}[S, S_{xx}], \quad S \in AO(2), \quad [A, B] \equiv AB - BA, \tag{5}$$

where

$$\begin{aligned} AO(2) &\equiv \{S | S^2 = I, S = S^\dagger, \text{and } \mathrm{tr}S = 0\}, S = \begin{pmatrix} S^z & S^- \\ S^+ & -S^z \end{pmatrix}, \\ S^\pm &= S^x \mp \mathrm{i}S^y, \end{aligned}$$

is also located in the $su(2)$ hierarchy. Equation (5) can be rewritten as the compatibility condition for the following system

$$\begin{aligned}\Psi_x &= -\mathrm{i}\lambda S\Psi, \\ \Psi_t &= W(\lambda)\Psi, \quad W(\lambda) \equiv -2\mathrm{i}\lambda^2 S + \lambda S S_x.\end{aligned} \tag{6}$$

It is ready to verify the reality condition $W^\dagger(\lambda^*) = -W(\lambda)$. The coefficient matrix of system (4) and (6) possess the same reality condition. Besides this, a gauge transformation can be related between these two linear systems, this fact was found by Zakharov and Takhtajan [47].

The aim in this section is to solve the cauchy problem of (5) with initial data

$$S(x, 0) = S_0(x), \quad |S_{0,x}| \in H^{1,1}(\mathbb{R}), \tag{7}$$

where $|\cdot|$ stands the matrix or vector norm $|A| = (\mathrm{tr}A^\dagger A)^{1/2}$, $H^{1,1}(\mathbb{R})$ is the weighted Sobolev space

$$H^{1,1}(\mathbb{R}) = \{f | f, f_x, xf \in L^2(\mathbb{R})\},$$

and boundary condition

$$\lim_{|x|\to\infty} S = \sigma_3. \tag{8}$$

Notation: We denote $e^{\mathrm{ad}\sigma_3}\cdot \equiv e^{\sigma_3} \cdot e^{-\sigma_3}$.

2.1. Scattering problem for spectral problem

The spectral problem for L–L Equation (5) is the first equation of (6). If we directly analyze the spectral problem (6), similar reason as the derivative NLS equation [33], it is not convenient to analyze the asymptotical behavior of analytical solution. Thus we use the gauge transformation. First, we establish the following lemma:

LEMMA 1. *If $S \in AO(2)$, then S can be decomposed into $S = g\sigma_3 g^\dagger$ uniquely, where g satisfies $g^\dagger g = I$, $g_x^\dagger g + \sigma_3 g_x^\dagger g \sigma_3 = 0$, $\lim_{x\to-\infty} g(x) = I$.*

Proof: We use the linear algebra method to construct the matrix g directly. Because matrix S is a unitary matrix, it can be diagonalizable. Using simple algebra, we can see that the eigenvalue of S is ± 1. Then S can be decomposed into

$$S = g_0 \exp(\mathrm{i}\theta\sigma_3)\sigma_3 \exp(-\mathrm{i}\theta\sigma_3) g_0^\dagger, \tag{9}$$

where θ is an undetermined real function and

$$g_0 = \begin{pmatrix} \sqrt{\dfrac{1+S^z}{2}} & -\dfrac{S^-}{\sqrt{2(1+S^z)}} \\ \dfrac{S^+}{\sqrt{2(1+S^z)}} & \sqrt{\dfrac{1+S^z}{2}} \end{pmatrix}.$$

To satisfy the condition $g_x^\dagger g + \sigma_3 g_x^\dagger g \sigma_3 = 0$, we can adjust the function θ. Directly calculating, we have

$$\left[e^{-\mathrm{i}\theta\sigma_3} g_0^\dagger\right]_x g_0 e^{\mathrm{i}\theta\sigma_3} = e^{-\mathrm{i}\theta \,\mathrm{ad}\sigma_3}(g_{0,x}^\dagger g_0) - \mathrm{i}\theta_x \sigma_3,$$

and

$$g_{0,x}^\dagger g_0 = \begin{pmatrix} \dfrac{S_x^z S^z + S_x^- S^+}{2(1+S^z)} & \dfrac{1}{2}\left(S_x^- - \dfrac{S^- S_x^z}{1+S^z}\right) \\ -\dfrac{1}{2}\left(S_x^+ - \dfrac{S^+ S_x^z}{1+S^z}\right) & \dfrac{S_x^z S^z + S_x^+ S^-}{2(1+S^z)} \end{pmatrix}.$$

If we demand

$$\mathrm{i}\theta_x = \frac{S_x^- S^+ + S_x^z S^z}{2(1+S^z)}, \quad \text{that is, } \theta_x = \frac{S^x S_x^y - S_x^x S^y}{2(1+S^z)},$$

then g satisfies the condition $g_x^\dagger g + \sigma_3 g_x^\dagger g \sigma_3 = 0$. Finally, to satisfy the condition $\lim_{x\to-\infty} g(x) = I$, we take

$$\theta(x) = \int_{-\infty}^{x} \frac{S^x S_x^y - S_x^x S^y}{2(1+S^z)} \mathrm{d}s.$$

This completes the proof. ■

Furthermore, we have

$$g_x^\dagger g \equiv Q,\ Q = \begin{pmatrix} 0 & q \\ -q^* & 0 \end{pmatrix},$$

$$q = \frac{1}{2}\left(S_x^- - \frac{S^- S_x^z}{1+S^z}\right)\exp\left(\mathrm{i}\int_{-\infty}^{x} \frac{S_x^x S^y - S^x S_x^y}{1+S^z}\mathrm{d}s\right),$$

and $4|q|^2 = (S_x^x)^2 + (S_x^y)^2 + (S_x^z)^2$. Via the relation $S = g\sigma_3 g^\dagger$, we have $S_x = g[\sigma_3, Q]g^\dagger$, and $S_{xx} = g([\sigma_3, Q_x] + [[\sigma_3, Q], Q])g^\dagger$. It follows that $4(|q_x|^2 + 4|q|^4) = (S_{xx}^x)^2 + (S_{xx}^y)^2 + (S_{xx}^z)^2$. Then ones obtain $q(x) \in H^{1,1}(\mathbb{R})$.

In addition, because $\lim_{|x|\to\infty} S = \sigma_3$, we have

$$\lim_{x\to+\infty} g^\dagger = g_\infty = \mathrm{diag}(a^*(0), a(0)),\ \ a(0) = e^{\mathrm{i}\theta(+\infty)}.$$

As a byproduct, we can obtain a conservation law. Indeed, we can see that $\lim_{|x|\to\infty} g_0 = I$. It follows that $\lim_{x\to+\infty} g = \exp[\mathrm{i}\theta(+\infty)\sigma_3]$, that is,

$$\int_{-\infty}^{+\infty} \frac{S_x^x S^y - S^x S_x^y}{1+S^z}\mathrm{d}s = 2\arg(a(0)). \tag{10}$$

Via the gauge transformation $\Phi = g^\dagger\Psi$, we have

$$\Phi_x = (-\mathrm{i}\lambda\sigma_3 + Q)\Phi, \tag{11}$$

which is a standard AKNS spectral problem. Thus it is convenient to make the scattering analysis. To write spectral problem (11) as the integral equation, we make the following transformation $\Phi^\pm(x,t) = m^{(\pm)}(x,t)e^{-\mathrm{i}\lambda x\sigma_3}$. Associated with asymptotical behavior, we have

$$m^{(\pm)}(x;\lambda) = I + \int_{\pm\infty}^{x} e^{-\mathrm{i}(x-y)\lambda\mathrm{ad}\sigma_3} Q(y)m^{(\pm)}\mathrm{d}y \equiv I + K_{Q,\lambda,\pm}m^{(\pm)}.$$

The properties of the above Jost solutions can be summarized as following:

PROPOSITION 1 ([2]). *Suppose $Q \in L^1(\mathbb{R})$, then $(m_1^{(-)}, m_2^{(+)})$ is analytic in the upper half plane $\{\lambda \in \mathbb{C}|\mathrm{Im}(\lambda) > 0\}$, and $(m_1^{(+)}, m_2^{(-)})$ is analytic in the lower half plane $\{\lambda \in \mathbb{C}|\mathrm{Im}(\lambda) < 0\}$. And, they are all continuous on the real line.*

Proof: First we prove

$$m_1^{(-)}(x;\lambda) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_{-\infty}^{x} \begin{pmatrix} 1 & 0 \\ 0 & e^{2\mathrm{i}(x-y)\lambda} \end{pmatrix}\begin{pmatrix} 0 & q(y) \\ -q^*(y) & 0 \end{pmatrix} m_1^{(-)}(y;\lambda)\mathrm{d}y \tag{12}$$

has a unique analytic solution in the upper half plane. It is readily obtained that the estimation from (12),

$$|m_1^{(-)}(x;\lambda)| \le 1 + \int_{-\infty}^{x} |Q||m_1^{(-)}(y;\lambda)|\mathrm{d}y. \tag{13}$$

To prove the solvability of (12), we iterate the series as following:

$$m_1^{(-)}(x;\lambda) = g_0 + \sum_{n=1}^{+\infty} g_n(x;\lambda), \tag{14}$$

where

$$g_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad g_{k+1} = \int_{-\infty}^{x} \begin{pmatrix} 0 & q(y) \\ -q^*(y)e^{2i\lambda(x-y)} & 0 \end{pmatrix} g_k(y)\mathrm{d}y.$$

We can see that

$$|g_1(x;\lambda)| \le \int_{-\infty}^{x} |Q(y)|\mathrm{d}y,$$

it follows that

$$|g_k(x;\lambda)| \le \frac{1}{k!}\left(\int_{-\infty}^{x} |Q(y)|\mathrm{d}y\right)^k$$

From the above estimate, the series (14) converges uniformly in the upper half plane, thus the solution $m_1^{(-)}$ is analytical in the upper half plane and can be continuous extended to the real line. In addition, we have an estimation

$$|m_1^{(-)}(x;\lambda)| \le \exp\left(\int_{-\infty}^{x} |Q(y)|\mathrm{d}y\right).$$

Via the inequality (13) and the Growall inequality, the uniqueness is proved.

We have the parallel results for $m_1^{(+)}$, $m_2^{(\pm)}$. This completes the proof. ■

COROLLARY 1. *If* $|S_x| \in L^1(\mathbb{R})$*, then the Jost solution* $\Psi^{\pm}$ *for spectral problem* (6) *can be obtained as* $\Psi^- = gm^{(-)}\exp(-i\lambda x\sigma_3)$*,* $\Psi^+ = gm^{(+)}g_\infty \exp(-i\lambda x\sigma_3)$*. Let* $n^{(-)} = gm^{(-)}$ *and* $n^{(+)} = gm^{(+)}g_\infty$*, then* $n^{(\pm)}$ *possess the analytic and continuity property as* $m^{(\pm)}$*. Finally, we have* $g^\dagger = m^{(-)}(x,t;\lambda = 0)$*.*

Proof: The first two arguments are direct results from above propositions. The last argument is valid for the existence and uniqueness of ODE. ■

In the following, we analyze the scattering matrix. By the Abel formula, we have $\det(m^{(\pm)}) = \det(n^{(\pm)}) = 1$. Thus we can define a matrix function $A(\lambda)$ for real λ with $\det(A(\lambda)) = 1$ and

$$m^{(+)} = m^{(-)}e^{-i\lambda x \,\mathrm{ad}\sigma_3}A(\lambda), \qquad A(\lambda) = \begin{pmatrix} a(\lambda) & -b^*(\lambda) \\ b(\lambda) & a^*(\lambda) \end{pmatrix}, \tag{15}$$

where

$$a(\lambda) = \det\left(m_1^{(+)}, m_2^{(-)}\right) = 1 - \textstyle\int_{\mathbb{R}} q(y)m_{21}^{(+)}\mathrm{d}y = 1 - \int_{\mathbb{R}} q^*(y)m_{12}^{(-)}\mathrm{d}y,$$

$$b(\lambda) = e^{-2ix\lambda}\det(m_1^{(-)}, m_1^{(+)}) = \textstyle\int_{\mathbb{R}} q^*(y)e^{-2i\lambda y}m_{11}^{(+)}\mathrm{d}y = \int_{\mathbb{R}} q^*(y)e^{-2i\lambda y}m_{11}^{(-)}\mathrm{d}y.$$

It follows that $A(0) = g_\infty^{-1}$ and

$$n^{(+)} = n^{(-)} e^{-\mathrm{i}\lambda x \operatorname{ad}\sigma_3} A_1(\lambda), \tag{16}$$

where $n^{(+)} = gm^{(+)} g_\infty$, $n^{(-)} = gm^{(-)}$ and $A_1(\lambda) \equiv A(\lambda) g_\infty$. In summary, we describe the above process with the following arrow diagram

$$(\Psi, S(x,0)) \xrightarrow{g} (\Phi, Q(x,0)) \xrightarrow{\text{scattering}} \left(A(\lambda), A(0) = g_\infty^{-1}\right).$$

According to the above propositions, we can obtain that $A(\lambda) - 1 \in H^k(\mathrm{d}\lambda)$ [50]. It follows that we can define a solution m normalized as $x \to +\infty$:

$$m = \begin{cases} m_+ = \left(m_1^{(-)}, m_2^{(+)}\right) \begin{pmatrix} (a^*(\lambda^*))^{-1} & 0 \\ 0 & 1 \end{pmatrix}, & \mathrm{Im}(\lambda) > 0, \\ \\ m_- = \left(m_1^{(+)}, m_2^{(-)}\right) \begin{pmatrix} 1 & 0 \\ 0 & a^{-1} \end{pmatrix}, & \mathrm{Im}(\lambda) < 0. \end{cases} \tag{17}$$

Then we could have the following decomposition:

$$m_+ = m_- e^{-\mathrm{i}\lambda x \operatorname{ad}\sigma_3} v, \ \ \lambda \in \mathbb{R}, \tag{18}$$

where $m_\pm = m^{(+)} e^{-\mathrm{i}\lambda x \operatorname{ad}\sigma_3} v_\pm, \ v \equiv v_-^{-1} v_+$,

$$v_+ = \begin{pmatrix} 1 & 0 \\ r(\lambda) & 1 \end{pmatrix}, \quad v_- = \begin{pmatrix} 1 & -r^*(\lambda) \\ 0 & 1 \end{pmatrix},$$

$$v = \begin{pmatrix} 1 + |r(\lambda)|^2 & r^*(\lambda) \\ r(\lambda) & 1 \end{pmatrix}, \quad r = -\frac{b(\lambda)}{a^*(\lambda)}.$$

To complete the RHP, we need the boundary condition [5]

$$m \to I \text{ as } \lambda \to \infty. \tag{19}$$

Thus Equations (17)–(19) constitute the normalized RHP (see [18]) with the constraint $r(0) = 0$. The similar manner, we define $\widetilde{m}_+ = m_+ [a^*(\lambda)]^{\sigma_3}$, and $\widetilde{m}_- = m_- [a(\lambda)]^{-\sigma_3}$. The solution $\widetilde{m}$ is normalized as $x \to -\infty$. And, $\widetilde{m}_+ = \widetilde{m}_- e^{-\mathrm{i}\lambda x \sigma_3} \widetilde{v}(\lambda)$, $\widetilde{v}(\lambda) = a^{\sigma_3} v [a^*]^{\sigma_3}$. Accordingly, define

$$n = \begin{cases} n_+ = (n_1^{(-)}, n_2^{(+)}) \mathrm{diag}(1/a_1^*(\lambda), 1), & \mathrm{Im}(\lambda) > 0, \\ \\ n_- = (n_1^{(-)}, n_2^{(+)}) \mathrm{diag}(1, 1/a_1(\lambda)), & \mathrm{Im}(\lambda) < 0, \end{cases} \quad a_1(\lambda) = \frac{a(\lambda)}{a(0)}.$$

Then we have the RH problem for $n_\pm$

$$n_+ = n_- e^{-\mathrm{i}\lambda x \operatorname{ad}\sigma_3} v_1, \quad n_\pm = g m_\pm g_\infty, \tag{20}$$

where

$$v_1 = \begin{pmatrix} 1+|r_1| & r_1^* \\ r_1 & 1 \end{pmatrix}, \quad r_1 = -\frac{a_1^*(\lambda)}{b_1(\lambda)}, \quad b_1(\lambda) = \frac{b(\lambda)}{a(0)}.$$

However, in this case, when $\lambda \to \infty$, $n \to gg_\infty$. Thus this RHP is not a normalized RHP (see [18]). For convenience, we merely consider the normalized RHP (17)–(19).

When m has no spectral singularities, the scattering data can be represented as

$$\left\{m, e^{-\mathrm{i}\lambda x \mathrm{ad}\sigma_3} v(\lambda), \lambda \in \mathbb{R}; \quad e^{-\mathrm{i}\lambda x \mathrm{ad}\sigma_3} v_{\lambda'} \in V_{\lambda'}, \lambda' \in P\right\}, \tag{21}$$

where $\{v_{\lambda'}, \lambda \in P\}$ is the discrete part of the scattering data [50]. In the next section, we would like to deal with the discrete spectrum by generalized Darboux transformation. Thus one can set $P = \emptyset$. In this way, it is convenient to consider argument contour. The Zhou's method [49, 50] deals with the poles by adding small circle centered at the poles. The spectral singularity is solved by reconstructing a new RHP on $\Gamma = \mathbb{R} \cup S_\infty$. However, when the poles are located in the inside of S_∞. It is not convenient to define the new RHP. If we deal with poles or high-order poles by the generalized Darboux transformation, that problem will avoid automatically.

To describe the general case, we first consider the following equation:

$$\left(m_1^{(+)}, m_2^{(-)}\right) = I + \int_{x_0}^{x} e^{-\mathrm{i}(x-y)\lambda \mathrm{ad}\sigma_3} Q(y)(m_1^{(+)}(y), m_2^{(-)}(y))\mathrm{d}y, \tag{22}$$

where $x_0 = -\infty$ for the $(1, 2)$ and $(2, 2)$ entries, $x_0 = +\infty$ for the $(1, 1)$ and $(2, 1)$ entries. For the entry $(2, 2)$ of (22), using (15) we can obtain

$$\left(m_1^{(+)}, m_2^{(-)}\right) = \begin{pmatrix} 1 & 0 \\ 0 & a(\lambda) \end{pmatrix} + \int_{x_0'}^{x} e^{-\mathrm{i}(x-y)\lambda \mathrm{ad}\sigma_3} Q(y)\left(m_1^{(+)}(y), m_2^{(-)}(y)\right)\mathrm{d}y,$$

where $x_0' = +\infty$ for the $(1, 1)$, $(2, 1)$, and $(2, 2)$ entries, $x_0' = -\infty$ for the $(1, 2)$ entry. It follows that

$$\begin{aligned}\left(m_1^{(+)}, m_2^{(-)}\right) \begin{pmatrix} 1 & 0 \\ 0 & a(\lambda)^{-1} \end{pmatrix} = I + \int_{x_0'}^{x} & e^{-\mathrm{i}(x-y)\lambda \mathrm{ad}\sigma_3} Q(y)(m_1^{(+)}(y), m_2^{(-)}(y)) \\ & \times \begin{pmatrix} 1 & 0 \\ 0 & a(\lambda)^{-1} \end{pmatrix} \mathrm{d}y.\end{aligned}$$

Similar, we can obtain

$$(m_1^{(-)}, m_2^{(+)}) \begin{pmatrix} a^*(\lambda^*)^{-1} & 0 \\ 0 & 1 \end{pmatrix}$$

$$= I + \int_{-x_0'}^{x} e^{-\mathrm{i}(x-y)\lambda \mathrm{ad}\sigma_3} Q(y)(m_1^{(+)}(y), m_2^{(-)}(y)) \begin{pmatrix} a^*(\lambda^*)^{-1} & 0 \\ 0 & 1 \end{pmatrix} \mathrm{d}y.$$

Finally, we have

$$m(x,\lambda) = I + \int_{-x_0'\mathrm{sgn}(\mathrm{Im}\lambda)}^{x} e^{-\mathrm{i}(x-y)\lambda \mathrm{ad}\sigma_3} Q(y)m(y,\lambda)\mathrm{d}y, \tag{23}$$

which is the Fredholm integral equation. Hence by the analytical Fredholm theorem, it directly induces that solution m is meromorphic in $\mathbb{C}\backslash\mathbb{R}$.

The following results were similar as [50], thus here we merely give the main step and results.

Let $x_0 \in \mathbb{R}$ be such that $|q|_{L^1([x_0,+\infty))} < 1$. Using (23), we have a bounded solution $m^{(0)}$ normalized as $x \to +\infty$ for the potential $Q\chi_{(x_0,+\infty)}$. This solution does not have poles and spectral singularities. On the other hand define a solution

$$m^{(1)} = I - \int_{x}^{x_0} e^{-\mathrm{i}(x-y)\lambda \mathrm{ad}\sigma_3} Q(y)m^{(1)}(y,\lambda)\mathrm{d}y,$$

and another solution for $Q(x)$

$$m^{(2)}(x,\lambda) = m^{(1)}(x,\lambda)e^{-\mathrm{i}(x-x_0)\lambda \mathrm{ad}\sigma_3} m^{(0)}(x_0,\lambda).$$

This solution is consistent with $m^{(0)}$ at $x = x_0$, because of the existence and uniqueness property of ODE. It follows that $m^{(2)}$ is normalized as $x \to +\infty$. Because $m^{(1)}$ is entire in λ and $m^{(0)}(x,\cdot) \in \mathbf{A}H^k(\mathbb{C}\backslash\mathbb{R})$, then $m^{(2)}(x,\cdot) - I \in \mathbf{A}H^k(\mathbb{C}\backslash(\mathbb{R}\cup S_{R,r}))$, where $\mathbf{A}H^k(\Omega)$ denotes the space of functions analytic on Ω with H^k boundary values, $S_{R,r} = \{|\lambda| = R, |\lambda| = r\}$ for some $R > r > 0$.

Because a approaches 1 as $\lambda \to \infty$ and $a(0)$ as $\lambda \to 0$, they have no zero near $\lambda = \infty$ and $\lambda = 0$. Hence we use m near $\lambda = \infty$ and $\lambda = 0$, and $m^{(2)}$ elsewhere. Set $\Gamma = \mathbb{R}\cup S_{R,r}$, where $\Omega_+ = \Omega_1 \cup \Omega_4$ and $\Omega_- = \Omega_2 \cup \Omega_3$,

$$\Omega_1 = \{\lambda|\mathrm{Im}(\lambda) > 0, |\lambda| > R, \text{or } |\lambda| < r\}, \quad \Omega_4 = \{\lambda|\mathrm{Im}(\lambda) < 0, r < |\lambda| < R\},$$

and

$$\Omega_2 = \{\lambda|\mathrm{Im}(\lambda) < 0, |\lambda| > R, \text{or } |\lambda| < r\}, \quad \Omega_3 = \{\lambda|\mathrm{Im}(\lambda) > 0, r < |\lambda| < R\}.$$

Define $\mathbf{m} = m$ on $\Omega_1 \cup \Omega_2$, $\mathbf{m} = m^{(2)}$ on $\Omega_3 \cup \Omega_4$. It follows that $e^{-\mathrm{i}x\lambda \mathrm{ad}\sigma_3}\mathbf{v} = \mathbf{m}_-^{-1}\mathbf{m}_+$. Then we have the following theorem which can be established as the work of [50]

THEOREM 1 (Zhou, [50]).

(C1) The matrix $\mathbf{v}$ *admits a triangular factorization* $\mathbf{v} = \mathbf{v}_-^{-1}\mathbf{v}_+$, *where* $\mathbf{v}_\pm - I \in H^k(\partial\Omega_\pm)$, $\mathbf{v}_+|_{\partial\Omega_1} - I$ $(\mathbf{v}_+|_{\partial\Omega_4} - I)$ *is strictly lower* (*upper*)

triangular, and $\mathbf{v}_-|_{\partial\Omega_1} - I$ $(\mathbf{v}_-|_{\partial\Omega_3} - I)$ *is strictly upper (lower) triangular.*

(C2) There exists an auxiliary scattering matrix s *such that* $s_-^{-1}\mathbf{v}s_+ = \tilde{\mathbf{v}}_-^{-1}\tilde{\mathbf{v}}_+$ *for some invertible matrices* $\tilde{\mathbf{v}}_\pm \in I + H^k(\partial\Omega_\pm)$ *with* $\tilde{v}_\pm$ *having opposite triangularities of* $\mathbf{v}_\pm$.

(C3) The RH problem $(e^{-\mathrm{i}x\lambda \mathrm{ad}\sigma_3}\mathbf{v}, \Gamma)$ *is solvable for all* $x \in \Gamma$.

Because the symmetry property $Q^\dagger = -Q$, then

$$m(x;\lambda)m^\dagger(x;\lambda^*) = I, \qquad m^{(0)}(x;\lambda)m^{(0)\dagger}(x;\lambda^*) = I$$

it follows that

$$\mathbf{m}(x;\lambda)\mathbf{m}^\dagger(x;\lambda^*) = I.$$

Using this and the fact that the contour Γ is Schwarz-reflection-invariant with the orientation, we have the symmetry condition of $\mathbf{v}$ is

$$\mathbf{v}(\lambda) = \mathbf{v}^\dagger(\lambda^*). \tag{24}$$

This symmetry condition keeps the solvability of RHP [48].

Therefore we have established the scattering map

$$\mathbf{S} : S_{0,x} \mapsto \mathbf{v}(\lambda), \qquad H^{1,1} \to H_0^{1,1} \equiv H^{1,1} \cap \{\mathbf{v}(0) = I\}. \tag{25}$$

Following [18, 50], one can establish the following theorem. Because the proof is similar as [18, 50], we omit the explicit proof.

THEOREM 2. *If* $S_{0,x} \in H^{1,1}$, *then* $\mathbf{v}_\pm - I \in H_0^{1,1}$.

2.2. *Inverse scattering*

Suppose the scattering data are given, we can resolve the potential function $Q(x)$. For convenience, denote $w_x = e^{-\mathrm{i}\lambda x \mathrm{ad}\sigma_3}w$. Indeed, the RHP $(\mathbf{v}_x, \Gamma = \mathbb{R} \cup S_{R,r})$ is equivalent to the integral equation problem

$$\mu = I + C_{\mathbf{v}_{x\pm}}\mu, \qquad C_{\mathbf{v}_{x\pm}}\mu = C_\Gamma^+\mu(\mathbf{v}_{x+} - I) + C_\Gamma^-\mu(I - \mathbf{v}_{x-}),$$

where $\mathbf{v}_{x\pm} = e^{-\mathrm{i}\lambda x \mathrm{ad}\sigma_3}\mathbf{v}_\pm$,

$$C_\Gamma f = \frac{1}{2\pi\mathrm{i}}\int_\Gamma \frac{f(\zeta)}{\zeta - \lambda}\mathrm{d}\zeta,$$

$\lambda \notin \Gamma$, $\mu = m^{(+)}$. The symmetry condition for RHP $(\mathbf{v}_x, \Gamma = \mathbb{R} \cup S_{R,r})$ guarantees the existence and uniqueness of RHP [48].

Once this integral equation is solved, $\mathbf{m}$ can be constructed through

$$\mathbf{m} = I + C_\Gamma \mu e^{-\mathrm{i}x\lambda \mathrm{ad}\sigma_3}(\mathbf{v}_+ - \mathbf{v}_-) \tag{26}$$

and

$$Q = \mathrm{iad}\sigma_3 \mathbf{m}_{\infty,1} = -\frac{\mathrm{ad}\sigma_3}{\pi}\int_\Gamma \mu(\mathbf{v}_{x+} - \mathbf{v}_{x-})\mathrm{d}\lambda,$$

where we denote $\mathbf{m} = I + \mathbf{m}_{\infty,1}/\lambda + o(1/\lambda)$ as $\lambda \to \infty$. From simple calculation, we have the RHP

$$M_+ = M_- e^{-\mathrm{i}\lambda x\sigma_3}\mathbf{v}(\lambda), \quad M_\pm = m_{\pm,x} + \mathrm{i}\lambda[\sigma_3, m_\pm] - Qm_\pm.$$

Together with $M_\pm \in \partial C(L^2)$ [18], we have $M_\pm = 0$.

To prove the well-posedness of L–L Equation (5), we construct the gauge transformation

$$g(x) = m^{(-)}(x;\lambda = 0)^\dagger = \mathrm{diag}(a(0), a^*(0))m^{(+)}(x;\lambda = 0)^\dagger. \tag{27}$$

It is readily seen that $g(x)$ is an Hermite matrix, that is, $gg^\dagger = I$. And, the boundary condition is

$$\lim_{x\to-\infty} g(x) = I, \text{ and } \lim_{x\to+\infty} g(x) = \mathrm{diag}(a(0), a^*(0)). \tag{28}$$

PROPOSITION 2 ([47], ZAKHROV–TAKTAJAN). *If $q(x)$ belongs to $H^{1,1}(\mathbb{R})$ and satisfies the boundary conditions $\lim_{|x|\to\infty} q(x) = 0$ and scattering data restraint $r(0) = 0$, $g(x)$ satisfies Equations* (27) *and* (28)*, then the function $S(x) = g(x)\sigma_3 g^\dagger(x)$ satisfies the boundary condition* (8)*, and $|S_x(x)| \in H^{1,1}(\mathbb{R})$, $\Psi^\pm$ satisfies the spectral problem $\Psi_x = -\mathrm{i}\lambda S\Psi$.*

If we expand n in the neighborhood of 0, that is,

$$n = gmv^{-1}(0) = I + n_1\lambda + o(\lambda^2),$$

then we can resolve

$$S = \sigma_3 + \mathrm{i}n_{1,x}. \tag{29}$$

Via this resolvent formula, one can obtain a compact formula. Indeed, $mv^{-1}(0)$ satisfies the equation

$$(mv^{-1}(0))_x = -\mathrm{i}\lambda[\sigma_3, mv^{-1}(0)] + Qmv^{-1}(0).$$

Then we can expand $mv^{-1}(0)$ in the neighborhood of 0:

$$mv^{-1}(0) = g^{-1} + \lambda m_1(x,t) + o(\lambda^2),$$

where $m_{1,x} = -\mathrm{i}[\sigma_3, g^{-1}] + Qm_1$. Together with $n_1 = gm_1$, we have $S = g\sigma_3 g^\dagger$.

Similar as [18,50], together with the fact $4|q|^2 = \vec{S}_x^2$ and $4(|q_x|^2 + 4|q|^4) = \vec{S}_{xx}^2$ and Sobolev embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$, we can establish the following theorems:

THEOREM 3 ([50], ZHOU). *Under the conditions*

- $r \in H^k(\Gamma)$, $r(0) = 0$

- $W_\Gamma(1+|r|^2)=0$, *W stands for the winding-number constraint, and RHP* $(\mathbf{v}_x,\Gamma)$ *is solvable for all* $\lambda\in\Gamma$*, then we have* $|S_x|\in L^2((1+x^2)\mathrm{d}x)$.

THEOREM 4. *In addition, if* $\mathbf{v}_\pm - I, \tilde{\mathbf{v}}_\pm - I \in H_0^{1,1}$*, then* $|S_x|\in H^{1,1}$.

Thus the above theorems establish the local existence and uniqueness theorem for L–L Equation (5) in $H^{1,1}$ without discrete scattering data. If the initial data $|S_x(x,0)|\in H^{1,1}$ are without discrete scattering data, then the solution $S(x,t)$ is existent and unique in the local part of $t=0$.

2.3. Time evolution and global well-posedness without discrete scattering data

Up to now, we proved L–L Equation (5) is in global existence and unique in the space $H^{1,1}(\mathbb{R})$. To obtain the time evolution for scattering data, we use the time evolution part of Lax pair (4) or (6). However, the gauge transformation between two linear systems had been established in Ref. [47]. Thus, we merely need to analyze one of them. We still analyze the time evolution part of Lax pair (4). We know NLS Equation (3) is equivalent with the following compatibility condition

$$U_t - V_x + [U,V]=0.$$

Differential spectral problem (4) with t, together with compatibility condition, we have

$$(\Phi_t^\pm - V\Phi^\pm)_x = U(\lambda)(\Phi_t^\pm - V\Phi^\pm).$$

For arbitrary $t\in[0,\infty)$, by asymptotical analysis we can obtain

$$m_t^{(\pm)} = -2\mathrm{i}\lambda^2[\sigma_3, m^{(\pm)}] + [2\lambda Q - \mathrm{i}(Q^2+Q_x)\sigma_3]m^{(\pm)}. \tag{30}$$

PROPOSITION 3. *The evolution of the continuous scattering data is given by the following equation*

$$A_t = -2\mathrm{i}\lambda^2[\sigma_3, A].$$

Proof: Suppose we have

$$m^{(+)}e^{-\mathrm{i}\lambda x\sigma_3} = m^{(-)}e^{-\mathrm{i}\lambda x\sigma_3}A(\lambda).$$

By the Lebesgue dominated convergence theorem, it follows that

$$A(\lambda) = \lim_{x\to-\infty} e^{\mathrm{i}\lambda x\,\mathrm{ad}\sigma_3} m^{(+)}.$$

It is readily seen that

$$e^{\mathrm{i}\lambda x\,\mathrm{ad}\sigma_3} m_t^{(+)} = -2\mathrm{i}\lambda^2[\sigma_3, e^{\mathrm{i}\lambda x\,\mathrm{ad}\sigma_3}m^{(+)}] + e^{\mathrm{i}\lambda x\,\mathrm{ad}\sigma_3}[(2\lambda Q - \mathrm{i}(Q^2+Q_x)\sigma_3)m^{(+)}]$$

Taking the limit $x\to-\infty$ both sides, we obtain

$$A_t = -2\mathrm{i}\lambda^2[\sigma_3, A],$$

which completes the proof. ∎

Thus the RHP (18) becomes

$$m_+ = m_- e^{-\mathrm{i}\lambda(x+2\lambda t)\mathrm{ad}\sigma_3} v(\lambda). \tag{31}$$

And, $r(\lambda, t) = r(\lambda, 0)e^{4\mathrm{i}\lambda^2 t} \in H_0^{1,1}(\mathbb{R})$. Thus scattering data persists the solvability property. It follows that the global existence and uniqueness of L–L equation (5) without discrete scattering data are proved.

In the following, we consider the discrete scattering data evolution. First, we rewrite Equation (30) with the following equations:

$$\begin{aligned}(\Phi_1^+ e^{-2\mathrm{i}\lambda^2 t})_t &= V(\lambda)(\Phi_1^+ e^{-2\mathrm{i}\lambda^2 t}),\\ (\Phi_2^- e^{2\mathrm{i}\lambda^2 t})_t &= V(\lambda)(\Phi_2^- e^{2\mathrm{i}\lambda^2 t}).\end{aligned} \tag{32}$$

We know the discrete spectrum λ_i corresponds L^2 eigenfunction

$$\Phi_1^+(x, 0; \lambda_i) = \gamma_i \Phi_2^-(x, 0; \lambda_i), \ \ \lambda_i \in \mathbb{C}_-. \tag{33}$$

It follows that

$$\Phi_1^+(x, t; \lambda_i)e^{-2\mathrm{i}\lambda_i^2 t} = \gamma_i \Phi_2^-(x, t; \lambda_i)e^{2\mathrm{i}\lambda_i^2 t}.$$

If the discrete spectrum is multiple algebraic spectrum, we have

$$\begin{aligned}\frac{1}{j!}\left(\frac{d}{d\lambda}\right)^j [\Phi_1^+(x, 0; \lambda_i)] &= \frac{\gamma_i}{j!}\left(\frac{d}{d\lambda}\right)^j [\Phi_2^-(x, 0; \lambda_i)]\\ &\quad + \sum_{k=1}^{j} \frac{\beta_{i,k}}{(j-k)!}\left(\frac{d}{d\lambda}\right)^{j-k} [\Phi_2^-(x, 0; \lambda_i)],\\ &\quad j = 1, 2, \ldots, r_i.\end{aligned} \tag{34}$$

First, we can obtain the following equation:

$$\frac{1}{j!}\left[\frac{d^j}{d\lambda^j}\left(\Phi_1^+ e^{-2\mathrm{i}\lambda^2 t}\right)\right]_t = \frac{1}{j!}\sum_{l=0}^{j} C_j^l \left(\frac{d^l}{d\lambda^l}V(\lambda)\right)\left(\frac{d^{j-l}}{d\lambda^{j-l}}\left(\Phi_1^+ e^{-2\mathrm{i}\lambda^2 t}\right)\right), \tag{35}$$

where $C_j^l = \frac{j!}{l!(j-l)!}$. On the other hand, we have

$$\frac{\gamma_i}{j!}\left[\frac{d^j}{d\lambda^j}\left(\Phi_2^- e^{2\mathrm{i}\lambda^2 t}\right)\right]_t = \frac{\gamma_i}{j!}\sum_{l=0}^{j} C_j^l \left(\frac{d^l}{d\lambda^l}V(\lambda)\right)\left(\frac{d^{j-l}}{d\lambda^{j-l}}\left(\Phi_2^- e^{2\mathrm{i}\lambda^2 t}\right)\right),$$

$$\frac{\beta_{i,k}}{(j-k)!}\left[\frac{d^{j-k}}{d\lambda^{j-k}}\left(\Phi_2^- e^{2\mathrm{i}\lambda^2 t}\right)\right]_t$$

$$= \frac{\beta_{i,k}}{(j-k)!} \sum_{l=0}^{j-k} C_{j-k}^{l} \left(\frac{d^l}{d\lambda^l} V(\lambda) \right) \left(\frac{d^{j-k-l}}{d\lambda^{j-k-l}} \left(\Phi_2^- e^{2i\lambda^2 t} \right) \right),$$

$j = 1, 2, \ldots, r_i$, it follows that

$$\begin{aligned} &\left[\frac{\gamma_i}{j!} \frac{d^j}{d\lambda^j} \left(\Phi_2^- e^{2i\lambda^2 t} \right) + \sum_{k=0}^{j} \frac{\beta_{i,k}}{(j-k)!} \frac{d^{j-k}}{d\lambda^{j-k}} \left(\Phi_2^- e^{2i\lambda^2 t} \right) \right]_t \\ &= V(\lambda) \left[\frac{\gamma_i}{j!} \frac{d^j}{d\lambda^j} \left(\Phi_2^- e^{2i\lambda^2 t} \right) + \sum_{k=1}^{j} \frac{\beta_{i,k}}{(j-k)!} \frac{d^{j-k}}{d\lambda^{j-k}} \left(\Phi_2^- e^{2i\lambda^2 t} \right) \right] \\ &+ \sum_{l=0}^{j} \frac{1}{l!} \left(\frac{d^l}{d\lambda^l} V(\lambda) \right) \left[\frac{\gamma_i}{(j-l)!} \frac{d^{j-l}}{d\lambda^{j-l}} \left(\Phi_2^- e^{2i\lambda^2 t} \right) \right. \\ &\qquad \left. + \sum_{k=1}^{j-l} \frac{\beta_{i,k}}{(j-k-l)!} \frac{d^{j-l-k}}{d\lambda^{j-k-l}} \left(\Phi_2^- e^{2i\lambda^2 t} \right) \right]. \end{aligned} \tag{36}$$

By mathematical induction and existence and uniqueness of ordinary differential equation, we can obtain the time evolution relation

$$\left[\frac{1}{j!} \frac{d^j}{d\lambda^j} \left(\Phi_1^+ e^{-2i\lambda^2 t} \right) \right] = \left[\frac{\gamma_i}{j!} \frac{d^j}{d\lambda^j} \left(\Phi_2^- e^{2i\lambda^2 t} \right) + \sum_{k=0}^{j} \frac{\beta_{i,k}}{(j-k)!} \frac{d^{j-k}}{d\lambda^{j-k}} \left(\Phi_2^- e^{2i\lambda^2 t} \right) \right]. \tag{37}$$

3. The discrete spectrum and Darboux transformation

In this section, we use the Darboux transformation method to delete or add the discrete spectrum of L–L spectral problem (6). To derive the Darboux transformation for L–L Equation (5), we first give the Darboux transformation of NLS (3).

3.1. Darboux transformation of NLS

The Darboux transformation for NLS is well known for us, we can readily establish the following theorem:

THEOREM 5 ([10,27,46]). *Assume we have N distinct parameters λ_1, λ_2, …, $\lambda_N \in \mathbb{C}_-$ and the corresponding special solution matrices $|y_1\rangle$, $|y_2\rangle$, …, $|y_N\rangle$, then the Darboux matrix can be represented as*

$$T_N = I - \left[\, |y_1\rangle,\, |y_2\rangle,\, \ldots,\, |y_N\rangle \,\right] M^{-1}(\lambda - S)^{-1} \begin{bmatrix} \langle y_1| \\ \langle y_2| \\ \vdots \\ \langle y_N| \end{bmatrix},$$

where $\mathbb{C}_-$ represents the lower half complex plane

$$M = \left(\frac{\langle y_i|y_j\rangle}{\lambda_j - \lambda_i^*}\right)_{1\le i\le N, 1\le j\le N},$$

$$S = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N),$$

and $|y_i\rangle = (m_1^{(+)}(\lambda_i), m_2^{(-)}(\lambda_i))e^{-\mathrm{i}\lambda_i x\sigma_3}C_i$, $C_i = (1, -\gamma_i)^T$ is a nonzero column vector.

LEMMA 2. *The matrix*

$$M = \left(\frac{\langle y_i|y_j\rangle}{\lambda_j - \lambda_i^*}\right)_{N\times N}$$

is a nonsingular.

Proof: Similar as Ref. [30]. ■

Indeed the essence of Darboux transformation is a kind of special gauge transformation. An important step is to find the seed solution for original spectral problem. Suppose we have a fundamental solution $\Phi(\lambda) = (\Phi_1(\lambda), \Psi_1(\lambda))$ of a spectral problem (11), the high-order Darboux transformation can be construct as following arrow diagram:

$$(\Phi_1, \Psi_1) \xrightarrow[\Phi_1(\lambda_1)\in \mathrm{Ker}(\mathrm{T}_0[1])]{T_0[1]}$$

$$\left(\Phi_1^{[1]}, \Psi_1^{[1]}\right) \xrightarrow[\Phi_1[1](\lambda_1)\in \mathrm{Ker}(\mathrm{T}_1[1])]{T_1[1]} \cdots$$

where $\Psi_1^{[1]} = T_0[1]\Psi_1$, $\Phi_1^{[1]} = [(T_0[1]\Phi_1)_\lambda + \beta_1 T_0[1]\Psi_1]|_{\lambda=\lambda_1}$,$\beta_1$ is a complex constants. We can see that the parameters β_1 is not convenient to calculate the exact solution. Indeed, we can absorb the parameter β_1 into Φ_1. We need the following lemma:

LEMMA 3. *Assume Φ_1 is a seed solution for* (11) *at $\lambda = \lambda_1$, and T is the Darboux transformation by Φ_1, Ψ_1 is another linear dependent solution with Φ_1, then $T\Psi_1$ is uniquely determined module a nozero constant.*

Proof: The Darboux matrix is

$$T = I + \frac{\lambda_1^* - \lambda_1}{\lambda - \lambda_1^*}\frac{\Phi_1\Phi_1^\dagger}{\Phi_1^\dagger\Phi_1},$$

directly calculating, it follows that

$$T\Psi_1 = \frac{\det(\Psi_1, \Phi_1)}{\Phi_1^\dagger\Phi_1}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\Phi_1^*.$$

However, by the Abel formula, $\det(\Psi_1, \Phi_1)_x = 0$. This completes the proof. ■

By the above lemma, we can see that the new seed function $\Phi_1[1]$ does not depend with exact form of Ψ_1. Thus we can choose function Ψ_1 arbitrarily. Thus, $\Phi_1[1]$ can be rewritten as

$$\Phi_1^{[1]} = \lim_{\xi\to 0}\frac{T_1[1](\lambda_1+\xi)(\Phi_1(\lambda_1+\xi)+\xi\beta_1\Psi_1(\lambda_1+\xi))}{\xi}.$$

Generally, we can obtain

$$\Phi_1^{[N-1]} = \lim_{\xi\to 0}\frac{T_{N-1}[1]\cdots T_1[1](\lambda_1+\xi)\left(\Phi_1(\lambda_1+\xi)+\sum_{i=1}^{N-1}\xi^i\beta_i\Psi_1(\lambda_1+\xi)\right)}{\xi^{N-1}}.$$

REMARK 1. *In Refs.* [27,28,30], *we use the relation*

$$\left[\exp\left(\sum_{i=1}^{N-1}\delta_i\xi^i\right)\right]_{[N-1]} = 1 + \sum_{i=1}^{N-1}\xi^i\beta_i,$$

where the symbol $_{[N-1]}$ represents the Taylor expansion truncate from ξ^{N-1}. And, δ_i can be determined by β_i through elementary Schur polynomial. When the spectral is branch spectral, ones need to make small modification to the above polynomial [27,28,30].

THEOREM 6. *Generalized Darboux matrix*

$$T_N = \prod_{i=1}^{s} T[i],\ \ N = \sum_{i=1}^{s} r_i,$$

where

$$T[i] = T_{r_i}[i]T_{r_i-1}[i]\cdots T_0[i],\ \ T_j[i] = \left(I + \frac{\lambda_i^* - \lambda_i}{\lambda - \lambda_i^*}P_i^{(j)}\right),\ \ j = 1, 2, \cdots, r_i,$$

$$T_0[i] = T[0] = I,\ \ P_i^{(j)} = \frac{|y_{i,j}\rangle\langle y_{i,j}|}{\langle y_{i,j}|y_{i,j}\rangle},$$

$$|y_{i,j}\rangle = \lim_{\xi\to 0} \frac{(T_{j-1}[i]\cdots T_0[i])(\lambda_i+\xi)\prod_{m=1}^{i-1} T[m](\lambda_i+\xi)}{\xi^{j-1}} \left(|y_i\rangle - \sum_{k=1}^{j-1}\xi^k \beta_{i,k}|x_i\rangle\right),$$

and $\beta_{i,0}=0,\ \ |y_i\rangle = \Phi_1^+(\lambda_i+\xi)-\gamma_i\Phi_2^-(\lambda_i+\xi),\ \ |x_i\rangle = \Phi_2^-(\lambda_i+\xi)$. *The function* $\Phi_1^+[N](\lambda_i)$ *is* $L^2(\mathbb{R})$ *eigenfunction for spectral problem* $L\Phi=\lambda\Phi$, *where* $L=\mathrm{i}\sigma_3(\partial_x - Q[N])$, $\Phi_1^+[N]=T_N\Phi_1^+$, *and*

$$Q[N] = Q + \mathrm{i}\left[\sigma_3, \sum_{i=1}^{s}\sum_{j=1}^{r_i}(\lambda_i^*-\lambda_i)P_i^{(j)}\right].$$

And, the eigenfunctions satisfy the following relation:

$$\frac{1}{j!}\frac{d^j}{d\lambda^j}(\Phi_1^+[N])|_{\lambda=\lambda_i} = \frac{\gamma_i}{j!}\frac{d^j}{d\lambda^j}(\Phi_2^-[N])|_{\lambda=\lambda_i} + \sum_{k=0}^{j}\frac{\beta_{i,k}}{(j-k)!}\frac{d^{j-k}}{d\lambda^{j-k}}(\Phi_2^-[N])|_{\lambda=\lambda_i},\ \ j=1,2,\ldots,r_i,$$

where $\Phi_2^-[N]=T_N\Phi_2^-$. *By above relations, its imply that* $\frac{d^j}{d\lambda^j}(\Phi_1^+[N])|_{\lambda=\lambda_i}$ *are the generalized eigenfunctions and belong to space* $L^2(\mathbb{R})$.

Proof: The generalized Darboux transformation is constructed in Ref. [27]. In the following, we derive the properties of eigenfunction. First, we expand the following function:

$$T_N(\lambda_i+\xi)\left(|y_i\rangle - \sum_{k=0}^{j-1}\xi^k\beta_{i,k}|x_i\rangle\right) = \sum_{k=0}^{+\infty} Q_k\xi^k,$$

where

$$Q_k = \frac{1}{k!}\frac{d^k}{d\lambda^k}(\Phi_1^+[N])|_{\lambda=\lambda_i} - \frac{\gamma_i}{k!}\frac{d^k}{d\lambda^k}(\Phi_2^-[N])|_{\lambda=\lambda_i} - \sum_{l=0}^{k}\frac{\beta_{i,l}}{(k-l)!}\frac{d^{k-l}}{d\lambda^{k-l}}(\Phi_2^-[N])|_{\lambda=\lambda_i},$$

and $k=0,1,\ldots,r_i-1$. By the construction of generalized Darboux transformation, we can obtain $Q_k=0$.

Because $\Phi_1^+[N](\lambda_i)\to 0$ exponentially as $x\to+\infty$ and $\Phi_2^-[N](\lambda_i)\to 0$ exponentially as $x\to-\infty$, we can deduce that $\Phi_1^+[N](\lambda_i)\in L^2(\mathbb{R})$. This completes the proof. ∎

THEOREM 7 ([43], LEMMA 4). *The above Darboux matrix can be represented as*

$$T_N = I - \begin{bmatrix} Y_1, & Y_2, & \ldots, & Y_s \end{bmatrix} M^{-1} D \begin{bmatrix} Y_1^\dagger \\ Y_2^\dagger \\ \vdots \\ Y_s^\dagger \end{bmatrix},$$

where

$$Y_i = \left[|z_i\rangle,\ |z_i\rangle^{(1)},\ \ldots \quad \frac{1}{(r_i-1)!}|z_i\rangle^{(r_i-1)} \right]_{\xi=0}, \qquad D = \operatorname{diag}(D_1, D_2, \ldots, D_s),$$

$$D_i = \begin{bmatrix} \frac{1}{\lambda-\lambda_i^*} & 0 & \cdots & 0 \\ \frac{1}{(\lambda-\lambda_i^*)^2} & \frac{1}{\lambda-\lambda_i^*} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \frac{1}{(\lambda-\lambda_i^*)^{r_i}} & \frac{1}{(\lambda-\lambda_i^*)^{r_i-1}} & \cdots & \frac{1}{\lambda-\lambda_i^*} \end{bmatrix},$$

$$M = \begin{bmatrix} M^{[11]} & M^{[12]} & \cdots & M^{[1s]} \\ M^{[21]} & M^{[22]} & \cdots & M^{[2s]} \\ \vdots & \vdots & \vdots & \vdots \\ M^{[s1]} & M^{[s2]} & \cdots & M^{[ss]} \end{bmatrix},$$

and symbol $^{(i)}$ *means the derivative with respect to* ξ*,*

$$|z_i(\xi)\rangle = |y_i(\lambda_i+\xi)\rangle + \sum_{k=1}^{r_i-1} \xi^k \beta_{i,k} |x_i(\lambda_i+\xi)\rangle,$$

$$M^{[ij]} = \left(M_{m,n}^{[ij]}\right)_{r_i \times r_j},$$

$$M_{m,n}^{[ij]} = \frac{1}{(m-1)!(n-1)!} \frac{\partial^{n-1}}{\partial \xi^{n-1}} \frac{\partial^{m-1}}{\partial (\xi^*)^{m-1}} \frac{\langle z_i | z_j \rangle}{\lambda_j - \lambda_i^* + \xi - \xi^*}.$$

Proof: Directly calculating, we can obtain

$$(T_N - I)_{lk} = -\frac{\det(M_1)}{\det(M)}, \quad M_1 = \begin{bmatrix} M & Y_k^\dagger \\ Y_l & 0 \end{bmatrix},$$

where Y_l means the l-th row of $[\gamma_1, \gamma_2, \ldots, \gamma_s]$. Taking the limits with respect to $\xi \to 0$ from above formula, we can obtain the results. ■

The above theorem we obtained through generalized Darboux transformation is consistent with the Lemma 4 in Ref. [43]. In the following, we consider the relation between Darboux transformation and scattering data.

PROPOSITION 4. *The Darboux matrix T_N transforms the scattering data $\{a(\lambda), b(\lambda)\}$ into*

$$\{\widetilde{a(\lambda)}, \widetilde{b(\lambda)}; \lambda_i, \gamma(\lambda_i), \beta_{i,k}\},$$

where

$$\widetilde{a(\lambda)} = a(\lambda)\prod_{i=1}^{s}\left(\frac{\lambda-\lambda_i}{\lambda-\lambda_i^*}\right)^{r_i}, \tag{38}$$

$$\widetilde{b(\lambda_i)} = b(\lambda_i).$$

Proof: Direct calculating, we obtain

$$a(\lambda) = \det\left(m_1^{(+)}, m_2^{(-)}\right),$$
$$b(\lambda) = \det(m_1^{(-)}e^{-\mathrm{i}\lambda x}, m_1^{(+)}e^{-\mathrm{i}\lambda x}).$$

It follows that the Darboux transformation

$$T_N\left(m_1^{(+)}, m_2^{(-)}\right) = (\widetilde{m_1}^{(+)}, \widetilde{m_2}^{(-)})$$

gives the first equation of (38). By symmetry relation, we have

$$m_2^{(+)} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}(m_1^{(+)}(\lambda^*))^*, \ \ m_1^{(-)} = -\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}(m_2^{(-)}(\lambda^*))^*.$$

It follows that

$$\widehat{T}_N(m_1^{(-)}, m_2^{(+)}) = (\widetilde{m_1}^{(-)}, \widetilde{m_2}^{(+)}), \quad \widehat{T}_N = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} T_N^*(\lambda^*)\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

By Theorem (6), we know that T_N is determined by spectral parameters λ_i, γ_i, $\beta_{i,k}$. Furthermore, we have

$$T_N \to \begin{pmatrix} 1 & 0 \\ 0 & \prod_{i=1}^{s}\left(\frac{\lambda-\lambda_i}{\lambda-\lambda_i^*}\right)^{r_i} \end{pmatrix}, \quad \widehat{T}_N \to \begin{pmatrix} \prod_{i=1}^{s}\left(\frac{\lambda-\lambda_i^*}{\lambda-\lambda_i}\right)^{r_i} & 0 \\ 0 & 1 \end{pmatrix}, \quad x \to +\infty,$$

$$T_N \to \begin{pmatrix} \prod_{i=1}^{s}\left(\frac{\lambda-\lambda_i}{\lambda-\lambda_i^*}\right)^{r_i} & 0 \\ 0 & 1 \end{pmatrix}, \quad \widehat{T}_N \to \begin{pmatrix} 1 & 0 \\ 0 & \prod_{i=1}^{s}\left(\frac{\lambda-\lambda_i^*}{\lambda-\lambda_i}\right)^{r_i} \end{pmatrix}, \quad x \to -\infty.$$

It follows that

$$\widetilde{b(\lambda)} = \lim_{x\to+\infty} \det(\widehat{T}_N m_1^{(-)} e^{-\mathrm{i}\lambda x}, T_N m_1^{(+)} e^{-\mathrm{i}\lambda x}) = b(\lambda). \qquad \blacksquare$$

The above proposition can be considered as adding the zeros of the scattering data $a(\lambda)$. The inverse process is to delete zeros of scattering data $a(\lambda)$, which can be established in Refs. [19,29].

3.2. Darboux transformation of L–L equation

The Darboux transformation for NLS was constructed earlier in detail. On the other hand, as we know, the Darboux transformation is a special gauge transformation. Based on these ideas, we could construct the Darboux transformation for L–L equation by combining the two gauge transformation mentioned above.

To give the Darboux transformation with a linear factional transformation or a simple element $L_{-}(\mathrm{GL}(2,\mathbb{C}))$, we use the loop group representation [46]. If matrix functions $\Phi^{\pm}$ satisfy

$$\begin{cases} \Phi_x^{\pm} = (-\mathrm{i}\lambda\sigma_3 + Q)\Phi^{\pm}, \\ \lim\limits_{x\to\pm\infty} \Phi^{\pm} = \exp(-\mathrm{i}\lambda\sigma_3 x), \end{cases}$$

then such $\Phi^{\pm}$ will be called the trivialization of potential function Q at $\pm\infty$. Similarly, if matrix functions $\Psi^{\pm}$ satisfy

$$\begin{cases} \Psi_x^{\pm} = -\mathrm{i}\lambda S\Psi^{\pm}, \\ \lim\limits_{x\to\pm\infty} \Psi^{\pm} = \exp(-\mathrm{i}\lambda\sigma_3 x), \end{cases}$$

then such $\Psi^{\pm}$ will be called the trivialization of function S at $\pm\infty$.

THEOREM 8. *Let S satisfies the boundary condition* (8), *and $\Psi^{\pm}$ are the trivialization of S at $\pm\infty$, respectively, and π is the projection of $\mathbb{C}^2$. For each $x \in \mathbb{R}$, set*

$$\Psi_1 = (\Psi_1^{+}(\lambda_1), \Psi_2^{-}(\lambda_1))\,(a(0), -\gamma(\lambda_1))^T, \quad \lambda_1 \in \mathbb{C}_{-},$$

$$\widehat{T} = I + \frac{\zeta_1^* - \zeta_1}{\zeta - \zeta_1^*}\pi,\ \ \pi = \frac{\Psi_1\Psi_1^{\dagger}}{\Psi_1^{\dagger}\Psi_1},\ \ \zeta = \lambda^{-1}.$$

Then

$$\widehat{S} = D_1(S + 2\mathrm{Im}(\zeta_1)\pi_x)D_1^{-1},\ \ D_1 = \mathrm{diag}\left(\frac{\lambda_1}{\lambda_1^*}, 1\right) \tag{39}$$

is the global solution for L–L equation defined on $\mathbb{R}^2$, *and*

$$\widehat{\Psi}^+ = D_1(\widehat{T}\Psi_1^+, \sigma\widehat{T}^*(\lambda^*)\sigma^{-1}\Psi_2^+)D_1^{-1},$$
$$\widehat{\Psi}^- = D_1(\sigma\widehat{T}^*(\lambda^*)\sigma^{-1}\Psi_1^-, \widehat{T}\Psi_2^-)D_1^{-1}$$

are the trivialization of $\widehat{S}$ *at* $+\infty$ *and* $-\infty$, *respectively, where*

$$\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Proof: Suppose the analytical matrix $\Phi_- = (\Phi_1^+, \Phi_2^-)$, $\Phi_+ = (\Phi_1^-, \Phi_2^+)$. The elementary Darboux transformation for spectral problem (11) is

$$T_- = I + \frac{\lambda_1^* - \lambda_1}{\lambda - \lambda_1^*}\frac{\Phi_1\Phi_1^\dagger}{\Phi_1^\dagger\Phi_1}, \quad T_+ = \sigma T_-^*(\lambda^*)\sigma^{-1},$$
$$\Phi_1 = (\Phi_1^+(\lambda_1), \Phi_2^-(\lambda_1))(1, -\gamma(\lambda_1))^T.$$

By the above Darboux transformation, we can obtain $\widehat{\Phi}_- = (\widehat{\Phi}_1^+, \widehat{\Phi}_2^-) = T_-\Phi_-$, $\widehat{\Phi}_+ = (\widehat{\Phi}_1^-, \widehat{\Phi}_2^+) = T_+\Phi_+$. It follows that $\widehat{\Phi}^+ = (\widehat{\Phi}_1^+, \widehat{\Phi}_2^+)$ and $\widehat{\Phi}^- = (\widehat{\Phi}_1^-, \widehat{\Phi}_2^-)$ are the trivialization of

$$\widehat{Q} = Q + \mathrm{i}(\lambda_1^* - \lambda_1)\left[\sigma_3, \frac{\Phi_1\Phi_1^\dagger}{\Phi_1^\dagger\Phi_1}\right],$$

at $\pm\infty$, respectively.

On the other hand, by gauge transformation, we obtain that

$$\Psi^- = g\Phi^-, \quad \Psi^+ = g\Phi^+\mathrm{diag}(a^*(0), a(0)),$$

are the trivialization of S at $\pm\infty$, respectively, where $g = [\Phi^-]^\dagger|_{\lambda=0}$. And,

$$\widehat{\Psi}^- = \widehat{g}\widehat{\Phi}^-, \quad \widehat{\Psi}^+ = \widehat{g}\widehat{\Phi}^+\mathrm{diag}(\widehat{a}^*(0), \widehat{a}(0)),$$

are the trivialization of $\widehat{S}$ at $\pm\infty$, respectively, where $\widehat{g} = [\widehat{\Phi}^-]^\dagger|_{\lambda=0}$ and $\widehat{a}(0) = \lambda_1/\lambda_1^* a(0)$, the function $\widehat{S}$ we will be given in the following. To obtain the relation between $\widehat{S}$ and S, we use the following analytical function:

$$\Psi_- = (\Psi_1^+, \Psi_2^-), \quad \widehat{\Psi}_- = (\widehat{\Psi}_1^+, \widehat{\Psi}_2^-).$$

It follows that

$$\begin{aligned}\widehat{g} &= \mathrm{diag}\left(a(0)\frac{\lambda_1}{\lambda_1^*}, 1\right)\left[\Phi_1^+|_{\lambda=0}, \Phi_2^-|_{\lambda=0}\right]^\dagger (T_-^\dagger|_{\lambda=0}) \\ &= D_1 g(T_-^\dagger|_{\lambda=0}).\end{aligned}$$

Together with the above equation, we have

$$\begin{aligned}\widehat{\Psi}_- &= \widehat{g}\left[\widehat{\Phi}_1^+(\lambda)\widehat{a}^*(0),\, \widehat{\Phi}_2^-(\lambda)\right]\\ &= D_1 g\left(T_-^\dagger|_{\lambda=0}\right)T_- g^\dagger \Psi_- D_1^{-1}\\ &= D_1 \widehat{T}\Psi_- D_1^{-1},\end{aligned}$$

where

$$\begin{aligned}\Psi_1 &= g(\Phi_1^+, \Phi_2^-)(1, -\gamma(\lambda_1))^T\\ &= (\Psi_1^+(\lambda_1), \Psi_2^-(\lambda_1))\,(a(0), -\gamma(\lambda_1))^T .\end{aligned}$$

Then we can obtain

$$\widehat{S} = D_1\left(S + \mathrm{i}(\zeta_1^* - \zeta_1)\pi_x\right) D_1^{-1}.$$

Then, $\widehat{\Psi}^{\pm}$ are the trivialization of $\widehat{S}$ at $\pm\infty$, respectively. Finally, the estimation

$$|\pi| \leq 1$$

implies that the solutions are global for $(x, t) \in \mathbb{R}^2$. ■

We define the matrix $\widehat{T}$ as the Darboux matrix of L–L Equation (5). In the following, we consider the N-fold Darboux transformation for L–L Equation (5). We give the following theorem:

LEMMA 4. *The N-fold Darboux transformation for L–L equation* (5) *can be represented as*

$$\widehat{T}_N = I - \left[\,|y_1\rangle,\, |y_2\rangle, \ldots,\quad |y_N\rangle\,\right] M^{-1}(\zeta - D)^{-1}\begin{bmatrix}\langle y_1|\\ \langle y_2|\\ \vdots\\ \langle y_N|\end{bmatrix},$$

where $|y_i\rangle = (\Psi_1^+(\lambda_i), \Psi_2^-(\lambda_i)))(a(0), -\gamma_i)^T$ *s are special solutions of Lax pair* (6) *at* $\lambda = \lambda_i$, $\gamma_i \in \mathbb{C}$,

$$M = \left(\frac{\langle y_i|y_j\rangle}{\zeta_j - \zeta_i^*}\right)_{1\leq i\leq N, 1\leq j\leq N},$$

and

$$D = \mathrm{diag}\left(\zeta_1^*, \zeta_2^*, \ldots, \zeta_N^*\right).$$

Proof: The N-fold Darboux transformation can be constructed by N times iteration of Darboux transformation, that is,

$$\widehat{T}_N = \widehat{T}[N]\widehat{T}[N-1]\ldots\widehat{T}[1],$$

where

$$\begin{aligned}\widehat{T}[i] \quad &= I + \frac{\zeta_i^* - \zeta_i}{\zeta - \zeta_i^*}\frac{\Psi_i[i-1]\Psi_i[i-1]^\dagger}{\Psi_i[i-1]^\dagger\Psi_i[i-1]},\\ \Psi_i[i-1] &= T[i-1]T[i-2]\ldots T[1]|y_i\rangle|_{\zeta=\zeta_i}.\end{aligned}$$

Because of the residue of $\widehat{T}_N$, we can write the above Darboux transformation $\widehat{T}_N$ with the following linear fractional transformation:

$$\widehat{T}_N = I + \sum_{i=1}^{N}\frac{P_i}{\zeta - \zeta_i^*},$$

where P_is are 2×2 matrices with rank equals 1. Thus, we can suppose $P_i = |x_i\rangle\langle y_i|$. Because P_is are uniquely determined by the iteration, thus if $\langle y_i|$s are determined, then $|x_i\rangle$s are uniquely determined.

On the other hand, we know

$$\widehat{T}^{-1} = \widehat{T}^\dagger(\zeta^*) = I + \sum_{i=1}^{N}\frac{P_i^\dagger}{\zeta - \zeta_i}.$$

By the residue relation of $\widehat{T}_N\widehat{T}_N^{-1} = I$, we have

$$|y_j\rangle + \sum_{i=1}^{N}|x_i\rangle\frac{\langle y_j|y_i\rangle}{\zeta_j - \zeta_i^*} = 0, \quad i, j = 1, 2, \ldots, N.$$

In addition, because $\mathrm{Rank}(\widehat{T}_N(\zeta_i)) = 1$, we can suppose

$$\mathrm{Ker}(\widehat{T}_N(\zeta_i)) = |y_i\rangle = (\Psi_1^+(\lambda_i), \Psi_2^-(\lambda_i)))(a(0), -\gamma_i)^T.$$

By simple linear algebra, we can obtain the N-fold Darboux transformation for L–L equation. This completes the proof. ■

In the following, we consider the high-order algebraic poles for the scattering problem. Similar as the above section, we can obtain the following theorem:

THEOREM 9. *The generalized Darboux matrix for L–L Equation* (5) *can be represented as*

$$\widehat{T}_N = I - YM^{-1}DY^\dagger,$$

where

$$\begin{aligned}Y &= \begin{bmatrix} Y_1, & Y_2, \ldots, & Y_s\end{bmatrix},\\ Y_i &= \left[\, |z_i\rangle, \quad |z_i\rangle^{(1)}, \ldots, \quad \frac{1}{(r_i-1)!}|z_i\rangle^{(r_i-1)}\right]|_{\xi=0},\end{aligned}$$

$$D = \mathrm{diag}(D_1, D_2, \ldots, D_s),$$

$$D_i = \begin{bmatrix} \dfrac{-\lambda\lambda_i^*}{\lambda-\lambda_i^*} & 0 & \cdots & 0 \\ \dfrac{-\lambda^2}{(\lambda-\lambda_i^*)^2} & \dfrac{-\lambda\lambda_i^*}{\lambda-\lambda_i^*} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \dfrac{-\lambda^2}{(\lambda-\lambda_i^*)^{r_i}} & \dfrac{-\lambda^2}{(\lambda-\lambda_i^*)^{r_i-1}} & \cdots & \dfrac{-\lambda\lambda_i^*}{\lambda-\lambda_i^*} \end{bmatrix},$$

$$M = \begin{bmatrix} M^{[11]} & M^{[12]} & \cdots & M^{[1s]} \\ M^{[21]} & M^{[22]} & \cdots & M^{[2s]} \\ \vdots & \vdots & \vdots & \vdots \\ M^{[s1]} & M^{[s2]} & \cdots & M^{[ss]} \end{bmatrix},$$

and

$$|z_i(\xi)\rangle = |y_i(\lambda_i+\xi)\rangle + \sum_{k=1}^{r_i-1} \xi^k \beta_{i,k} |x_i(\lambda_i+\xi)\rangle,$$

$$M^{[ij]} = \left(M^{[ij]}_{m,n}\right)_{r_i\times r_j},$$

$$M^{[ij]}_{m,n} = \frac{1}{(m-1)!(n-1)!}\frac{\partial^{n-1}}{\partial\xi^{n-1}}\frac{\partial^{m-1}}{\partial(\xi^*)^{m-1}}\frac{\langle z_i(\xi^*)|z_j(\xi)\rangle}{(\lambda_j+\xi)^{-1}-(\lambda_i^*+\xi^*)^{-1}},$$

and symbol $^{(i)}$ *means the derivative with respect to* ξ*, the transformations between the field functions are*

$$(S[N])_{kl} = S_{kl} + \mathrm{i}\left(\frac{A_{kl}}{\det(M)}\right)_x, \quad A_{kl} = \det\begin{bmatrix} M & Y[l]^\dagger \\ Y[k] & 0 \end{bmatrix}, \tag{40}$$

where $|x_i\rangle$ *is the linear dependent solution with* $|y_i\rangle$*,* $Y[i]$ *denotes the* i*-th row of the matrix* Y *and the subscript* $_{kl}$ *represents the* k*-th row and* l*-th column element.*

By simple linear algebra, we can obtain the following compact soliton formula:

$$(S[N])_{kl} = S_{kl} + \mathrm{i}\left(\frac{\det(M_{kl})}{\det(M)}\right)_x, \quad M_{kl} = M - Y[l]^\dagger Y[k]. \tag{41}$$

REMARK 2. *Integrating the above expression,*

$$\int (S[N])_{kl}\mathrm{d}x = \int S_{kl}\mathrm{d}x + \mathrm{i}\left(\frac{\det(M_{kl})}{\det(M)} - 1\right), \quad M_{kl} = M - Y[l]^{\dagger}Y[k]. \tag{42}$$

it follows that the soliton formula for VFE (2)

$$\gamma^{x}[N] = \mathrm{Re}\left(\int (S[N])_{12}\mathrm{d}x\right),$$

$$\gamma^{y}[N] = \mathrm{Im}\left(\int (S[N])_{12}\mathrm{d}x\right),$$

$$\gamma^{z}[N] = \int (S[N])_{11}\mathrm{d}x.$$

Finally, we give the transformation between (Ψ_{-}, S) and $(\Psi_{-}[N], \widehat{S}[N])$

$$\Psi_{-} \to \Psi_{-}[N] = D_N \widehat{T}_N \Psi_{-} D_N^{-1}, \tag{43}$$
$$S \to \widehat{S}[N] = D_N S[N] D_N^{-1},$$

where (Ψ_{-}, S) represents wave function and potential function without discrete scattering data, $(\Psi_{-}[N], \widehat{S}[N])$ represents wave function and potential function possess discrete scattering data and

$$D_N = \mathrm{diag}\left(\prod_{i=1}^{s}\left(\frac{\lambda_i}{\lambda_i^*}\right)^{r_i}, 1\right).$$

Because D_N is a trivial gauge transformation, we omit it in the process of obtaining exact solution.

And, the Darboux matrix $\widehat{T}_N$ is determined uniqueness by the parameters λ_i, γ_i, and $\beta_{i,k}$ and is nonsingular for $(x, t) \in \mathbb{R}^2$. By the transformation (43), if S is globally existent and unique, it follows that $D_N S[N] D_N^{-1}$ is globally existent and unique. Thus the global existence and uniqueness of the L–L Equation (5) is proved.

THEOREM 10. *If*

$$S_0 \in \left\{ S | S_x \in H^{1,1}, S \in AO(2), \lim_{|x|\to\infty} S = \sigma_3 \right\},$$

then the solutions of L–L Equation (5) *are globally existent and unique.*

4. High-order soliton solution

In this section, we consider the high-order soliton solution for L–L Equation (5). The mixed rational and exponential function solution (or high-order soliton) is obtained. Besides, we give the explicit expression for high-order soliton solution of L–L Equation (5) and VFE (2).

For the classical integrable L–L Equation (1), the single soliton or the N-soliton and the interaction of N-soliton have been studied in detail by the Riemann–Hilbert method [19]. In our case, we derive the Darboux transformation of L–L Equation (5) by the gauge transformation. With the Darboux transformation, we obtain a simple generalized soliton solution formula for L–L Equation (5).

Single soliton can be generated by Darboux transformation from the vacuum solution. In this case, there is no reflection coefficient. Then the RHP (26) can be solved evidently, that is, $\mathbf{m} = I$. Then we have $Q = 0$ and $S = \sigma_3$. To obtain the pure soliton solution, we use the Darboux transformation to yield the discrete spectrum. The vector functions Ψ_1^+ and Ψ_2^- can be represented as

$$\Psi_1^+ = \begin{pmatrix} e^{-\mathrm{i}\lambda x} \\ 0 \end{pmatrix}, \quad \Psi_2^- = \begin{pmatrix} 0 \\ e^{\mathrm{i}\lambda x} \end{pmatrix},$$

and $a(0) = 1$, $\gamma(t;\lambda_1) = -c_1 e^{4\mathrm{i}\lambda_1^2 t}$. Then the standard single soliton for L–L Equation (5) can be obtain by formula (39), that is,

$$\begin{pmatrix} S^z & S^- \\ S^+ & -S^z \end{pmatrix}, \quad S^+ = (S^-)^*, \tag{44}$$

where

$$\begin{aligned} S^z &= 1 - \frac{2b^2}{a^2+b^2}\mathrm{sech}^2(A), \quad A = 2b(x + 4at + x_0), \\ S^- &= \frac{2b}{a^2+b^2} e^{\mathrm{i}B}\mathrm{sech}(A)[b\tanh(A) + \mathrm{i}a], \\ B &= -2ax + 4[b^2 - a^2]t - \varphi_1. \end{aligned}$$

By definition

$$a = \mathrm{Re}(\lambda_1), \quad b = \mathrm{Im}(\lambda_1), \quad x_0 = -\frac{\ln|c_1|}{2\mathrm{Im}(\lambda_1)}, \quad \varphi_0 = \arg(c_1).$$

It follows that the single soliton of VFE (2) are

$$\begin{aligned} \gamma^x &= \frac{-b}{a^2+b^2}\mathrm{sech}(A)\cos(B), \\ \gamma^y &= \frac{-b}{a^2+b^2}\mathrm{sech}(A)\sin(B), \\ \gamma^z &= x - \frac{b}{a^2+b^2}(1 + \tanh(A)). \end{aligned}$$

To obtain the high-order soliton solution, we first give the following lemma:

LEMMA 5. *If $A(\xi)$ possesses the series expansions*

$$A(\xi)=\sum_{n=0}^{+\infty}\gamma_n\xi^n,$$

then we have the following series expansions:

$$\frac{A(\xi)\overline{A(\xi)}}{\bar{\lambda}_1-\lambda_1+(\bar{\xi}-\xi)}=\frac{1}{\bar{\lambda}_1-\lambda_1}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(\sum_{j=0,i\le j,}^{n}\ \sum_{i=0,j-i\le n-m}^{m}\times\frac{(-1)^{n-j-m+i}C_{n-j}^{m-i}\gamma_i\bar{\gamma}_{j-i}}{(\bar{\lambda}_1-\lambda_1)^{n-j}}\right)\xi^m\bar{\xi}^{n-m}. \tag{45}$$

Proof: Indeed this lemma can be proved by directly calculating. To be convenience for reading, we give the details of calculating:

$$\begin{aligned}
\frac{A(\xi)\overline{A(\xi)}}{\bar{\lambda}_1-\lambda_1+(\bar{\xi}-\xi)}&=\frac{1}{\bar{\lambda}_1-\lambda_1}\left(\sum_{n=0}^{+\infty}\gamma_n\xi^n\right)\left(\sum_{n=0}^{+\infty}\bar{\gamma}_n\bar{\xi}^n\right)\sum_{k=0}^{+\infty}\left(\frac{\xi-\bar{\xi}}{\bar{\lambda}_1-\lambda_1}\right)^k\\
&=\frac{1}{\bar{\lambda}_1-\lambda_1}\sum_{n=0}^{+\infty}\left(\sum_{k=0}^{n}\gamma_k\bar{\gamma}_{n-k}\xi^k\bar{\xi}^{n-k}\right)\sum_{k=0}^{\infty}\left(\frac{\xi-\bar{\xi}}{\bar{\lambda}_1-\lambda_1}\right)^k\\
&=\frac{1}{\bar{\lambda}_1-\lambda_1}\sum_{n=0}^{+\infty}\sum_{j=0}^{n}\left(\frac{1}{(\bar{\lambda}_1-\lambda_1)^{n-j}}\sum_{l=0}^{j}\gamma_l\bar{\gamma}_{j-l}\xi^l\bar{\xi}^{j-l}\right.\\
&\qquad\left.\sum_{k=0}^{n-j}(-1)^{n-j-k}C_{n-j}^{k}\xi^k\bar{\xi}^{n-j-k}\right)\\
&=\frac{1}{\bar{\lambda}_1-\lambda_1}\sum_{n=0}^{+\infty}\sum_{j=0}^{n}\left(\frac{1}{(\bar{\lambda}_1-\lambda_1)^{n-j}}\sum_{m=0}^{n}\left(\sum_{i=0,i\le j,j-i\le n-m}^{m}(-1)^{n-j-(m-i)}\right.\right.\\
&\qquad\left.\left.\times C_{n-j}^{m-i}\gamma_i\bar{\gamma}_{j-i}\xi^m\bar{\xi}^{n-m}\right)\right)\\
&=\frac{1}{\bar{\lambda}_1-\lambda_1}\sum_{n=0}^{+\infty}\sum_{m=0}^{n}\left(\sum_{j=0,i\le j,}^{n}\ \sum_{i=0,j-i\le n-m}^{m}\right.\\
&\qquad\left.\frac{(-1)^{n-j-(m-i)}C_{n-j}^{m-i}\gamma_i\bar{\gamma}_{j-i}}{(\bar{\lambda}_1-\lambda_1)^{n-j}}\right)\xi^m\bar{\xi}^{n-m}. \qquad\blacksquare
\end{aligned}$$

LEMMA 6. *The expansion*

$$B(\xi) = (\lambda_1 + \xi)e^{-\mathrm{i}(\lambda_1+\xi)(x+2(\lambda_1+\xi)t)} = \sum_{i=0}^{\infty} \beta_i \xi^i, \quad \beta_i = \lambda_1 \widehat{\beta_i} + \widehat{\beta_{i-1}},$$

$$C(\xi) = (\lambda_1 + \xi)e^{\mathrm{i}(\lambda_1+\xi)(x+2(\lambda_1+\xi)t)} \sum_{i=0}^{\infty} \alpha_i \xi^i = \sum_{i=0}^{\infty} \delta_i \xi^i, \; \delta_i = \sum_{k=0}^{i} \alpha_k (\lambda_1 \widetilde{\beta_i} + \widetilde{\beta_{i-1}}),$$

where

$$\widehat{\beta_i} = \begin{cases} e^{-\mathrm{i}\lambda_1(x+2\lambda_1 t)} \sum_{j=0}^{n} \dfrac{\alpha^{2j}}{(2j)!} \dfrac{(-1)^{n-j}\beta^{n-j}}{(n-j)!}, & k = 2n, \quad n \geq 0, \\ e^{-\mathrm{i}\lambda_1(x+2\lambda_1 t)} \sum_{j=0}^{n} \dfrac{\alpha^{2j+1}}{(2j+1)!} \dfrac{(-1)^{n-j+1}\beta^{n-j}}{(n-j)!}, & k = 2n+1, \end{cases}$$

$$\widetilde{\beta_i} = \begin{cases} e^{-\mathrm{i}\lambda_1(x+2\lambda_1 t)} \sum_{j=0}^{n} \dfrac{\alpha^{2j}}{(2j)!} \dfrac{\beta^{n-j}}{(n-j)!}, & k = 2n, \quad n \geq 0, \\ e^{-\mathrm{i}\lambda_1(x+2\lambda_1 t)} \sum_{j=0}^{n} \dfrac{\alpha^{2j+1}}{(2j+1)!} \dfrac{\beta^{n-j}}{(n-j)!}, & k = 2n+1, \end{cases}$$

$\widehat{\beta_{-1}} = 0$, $\widetilde{\beta_{-1}} = 0$, $\alpha = \mathrm{i}(x + 4\lambda_1 t)$ *and* $\beta = 2\mathrm{i}t$.

Proof: By simple algebra, we have

$$(\lambda_1 + \xi)e^{-\mathrm{i}(\lambda_1+\xi)(x+2(\lambda_1+\xi)t)} = e^{-\mathrm{i}\lambda_1(x+2\lambda_1 t)}(\lambda_1 + \xi)\,(\cosh(\alpha\xi) + \sinh(\alpha\xi))\, e^{\beta\xi^2}.$$

It follows that we can obtain the expansion. ■

By above two lemmas, we have expansion

$$\begin{aligned} &\frac{B(\xi)\overline{B(\xi)} + C(\xi)\overline{C(\xi)}}{\bar{\lambda}_1 - \lambda_1 + (\bar{\xi} - \xi)} \\ &= \frac{1}{\bar{\lambda}_1 - \lambda_1} \sum_{n=0}^{\infty} \sum_{m=0}^{n} A_{m,n-m} \xi^m \bar{\xi}^{n-m}, \end{aligned} \tag{46}$$

where

$$A_{m,n-m} = \left(\sum_{j=0, i \leq j,}^{n} \sum_{i=0, j-i \leq n-m}^{m} \frac{(-1)^{n-j-m+i} C_{n-j}^{m-i} (\beta_i \bar{\beta}_{j-i} + \delta_i \bar{\delta}_{j-i})}{(\bar{\lambda}_1 - \lambda_1)^{n-j}} \right)$$

and

$$M_{k,m} = \frac{\partial^{m+k}}{\partial \xi^m \partial \bar{\xi}^k} \left(\frac{B(\xi)\overline{B(\xi)} + C(\xi)\overline{C(\xi)}}{\bar{\lambda_1} - \lambda_1 + (\bar{\xi} - \xi)} \right)_{\xi=0}$$
$$= \frac{1}{\bar{\lambda_1} - \lambda_1} \sum_{j=0, i\le j,}^{m+k} \sum_{i=0, j-i\le k}^{m} (-1)^{k+i-j} C_{m+k-j}^{m-i} \frac{\beta_i \bar{\beta}_{j-i} + \delta_i \bar{\delta}_{j-i}}{(\bar{\lambda_1} - \lambda_1)^{m+k-j}}.$$

Then we can obtain the following theorem:

THEOREM 11. *The N-th order soliton solution of L–L Equation* (5) *and VFE* (2) *can be represented as*

$$\begin{aligned} S^z[N] &= 1 + \mathrm{i}\left(\frac{\det(M^z)}{\det(M)}\right)_x, \quad M^z = (M_{k,m} - \bar{\beta}_k \beta_m)_{1\le k,m\le N}, \\ M &= (M_{k,m})_{1\le k,m\le N}, \\ S^-[N] &= \mathrm{i}\left(\frac{\det(M^-)}{\det(M)}\right)_x, \quad M^- = (M_{k,m} - \bar{\delta}_k \beta_m)_{1\le k,m\le N}. \end{aligned} \tag{47}$$

and

$$\gamma^x[N] = \mathrm{Re}\left(\mathrm{i}\frac{\det(M^-)}{\det(M)} - \mathrm{i}\right), \tag{48}$$

$$\gamma^y[N] = \mathrm{Im}\left(\mathrm{i}\frac{\det(M^-)}{\det(M)} - \mathrm{i}\right), \tag{49}$$

$$\gamma^z[N] = x + \mathrm{i}\left(\frac{\det(M^z)}{\det(M)} - 1\right),$$

respectively.

By the above formula, we can readily obtain the high-order soliton solution for VFE (2) equation. Specially, we take parameters $\lambda_1 = b\mathrm{i}$, $\gamma = 1$, and $\beta_{1,1} = c + \mathrm{i}d$, then we can obtain the second-order soliton solution for L–L equation (1) and VFE (2):

$$\gamma^x = \frac{-4[\cosh(2bx) + b(2x+d)\sinh(2bx)]\cos(4b^2t) - 4b(8bt - c)\cosh(2bx)\sin(4b^2t)}{2b^2(d+2x)^2 + 2b^2(8bt-c)^2 + \cosh(4bx) + 1},$$

$$\gamma^y = \frac{-4[\cosh(2bx) + b(2x+d)\sinh(2bx)]\sin(4b^2t) + 4b(8bt - c)\cosh(2bx)\cos(4b^2t)}{2b^2(d+2x)^2 + 2b^2(8bt-c)^2 + \cosh(4bx) + 1},$$

$$\gamma^z = x - \frac{2}{b}\left(1 + \frac{\sinh(4bx) - 2b(d+2x)}{2b^2(d+2x)^2 + 2b^2(8bt-c)^2 + \cosh(4bx) + 1}\right),$$

$$S^x = \gamma^x_x, \; S^y = \gamma^y_x, \; S^z = \gamma^z_x.$$

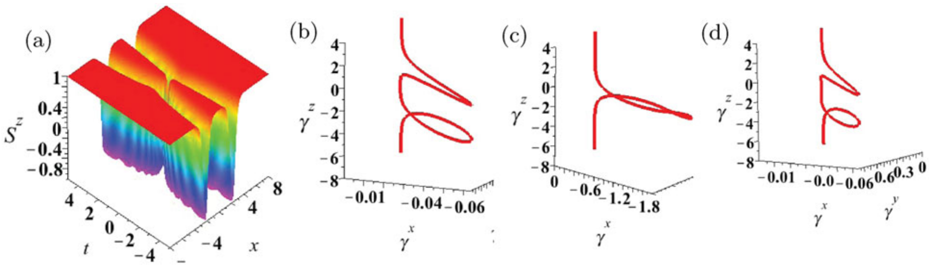

Figure 1. (a) Second-order soliton for L–L equation S^z and VFE γ^x, γ^y, and γ^z. Parameters $\lambda_1 = \mathrm{i}$, $\gamma_1 = 1$, $\beta_{1,1} = 0$: (b) $t = -4\pi$, (c) $t = 0$, and (d) $t = 4\pi$.

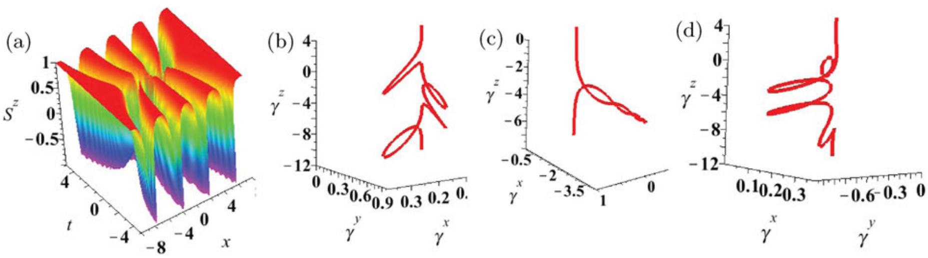

Figure 2. (a) Fourth-order soliton for L–L equation S^z and VFE γ^x, γ^y, and γ^z. Parameters $\lambda_1 = \mathrm{i}$, $\gamma_1 = 1$, $\beta_{1,1} = 0$, $\beta_{1,2} = 0$, and $\beta_{1,3} = 0$: (b) $t = -4\pi$, (c) $t = 0$, and (d) $t = 4\pi$.

Finally, we give the dynamics of L–L Equation (1) and VFE (2) by plotting a picture. The second-order soliton solution and fourth-order solution are showed with special parameters (Figures 1 and 2). It is seen that high-order soliton possesses the similar structure as multisoliton solution. The difference is that the velocity of high-order soliton is no longer a constant.

5. Conclusion and discussion

In conclusion, we analyze the L–L equation by inverse scattering method and generalized Darboux transformation. The generalized Darboux transformation [27, 28] is a general version for the Darboux transformation in [10, 46]. These results are self-contained. Besides, we remark that the general soliton solution for VFE (2) can be readily obtained by our formula (40). We intend to research the long-time asymptotics of high-order soliton in a subsequent publication.

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