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# Darboux transformation for a two-component derivative nonlinear Schrödinger equation

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## Abstract

In this paper, we consider a two-component derivative nonlinear Schrödinger equation and present a simple Darboux transformation for it. By iterating this Darboux transformation, we construct a compact representation for the  $N$ -soliton solutions.

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## 1. Introduction

The nonlinear partial differential equations with multi-soliton solutions have been studied extensively. They are often widely applicable in physics and thus constitute very important equations in mathematical physics. The celebrated examples include the Korteweg–de Vries equation, sine-Gordon equation and nonlinear Schrödinger (NLS) equation and many others [1]. These systems, known as soliton or integrable equations, are also very rich in mathematical properties and the whole subject is closely related to other mathematical branches such as differential geometry, algebraic geometry, combinatorics, Lie algebras, etc [4].

Since integrable systems have remarkable mathematical properties and numerous physical applications, their generalizations or extensions have attracted attention of many researchers. One possible direction is the multi-component generalization. This sort of extensions may also be physically interested. The most famous example might be Manakov's two-component NLS equation, which now is one of the most important equations in theory of pulse propagation along the optical fibre [2].

Another interesting soliton equation is the derivative nonlinear Schrödinger (DNLS) equation

$$iq_t = -q_{xx} + \frac{2}{3} i \epsilon (|q|^2 q)_x,$$

which appeared in plasma physics (see [11, 14]), describing Alfvén wave propagation along the magnetic field. This equation was solved by inverse scattering transformation by Kaup and

Newell [9]. Much research has been conducted for it and many results have been achieved. We mention here that a simple-looking Darboux transform, obtained independently by Imai [7] and Steudel [15], enables one to get its explicit  $N$ -soliton solution. The two-component extension of the DNLS equation was constructed by Morris and Dodd [12]. It reads as

$$iq_{1t} = -q_{1xx} + \frac{2}{3}i\epsilon[ (|q_1|^2 + |q_2|^2)q_1 ]_x, \quad (1)$$

$$iq_{2t} = -q_{2xx} + \frac{2}{3}i\epsilon[ (|q_1|^2 + |q_2|^2)q_2 ]_x, \quad (2)$$

where  $\epsilon = \pm 1$ . The system is relevant in the theory of polarized Alfvén waves and the propagation of the ultra-short pulse. It was studied by means of inverse scattering transformation [12]. More recently, Hirota's direct method was developed and in particular two-soliton solutions were constructed for this system [8].

In the subsequent discussion, we take  $\epsilon = -1$  for convenience. The system (1)–(2) has the following zero-curvature representation:

$$\Phi_x = U\Phi, \quad (3)$$

$$\Phi_t = V\Phi, \quad (4)$$

where  $\Phi = (\phi_1, \phi_2, \phi_3)^T$ ,  $\zeta$  is the spectral parameter and

$$U = U_2\zeta^2 + U_1\zeta, \quad V = \zeta^4V_4 + \zeta^3V_3 + \zeta^2V_2 + \zeta V_1$$

with

$$\begin{aligned} U_2 &= \begin{pmatrix} -2i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & i \end{pmatrix}, & U_1 &= \begin{pmatrix} 0 & q_1 & q_2 \\ r_1 & 0 & 0 \\ r_2 & 0 & 0 \end{pmatrix}, & V_4 &= \begin{pmatrix} -9i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ V_3 &= \begin{pmatrix} 0 & 3q_1 & 3q_2 \\ 3r_1 & 0 & 0 \\ 3r_2 & 0 & 0 \end{pmatrix}, & V_2 &= \begin{pmatrix} -i(r_1q_1 + r_2q_2) & 0 & 0 \\ 0 & ir_1q_1 & ir_1q_2 \\ 0 & ir_2q_1 & ir_2q_2 \end{pmatrix}, \\ V_1 &= \begin{pmatrix} 0 & iq_{1x} + \frac{2}{3}(r_1q_1 + r_2q_2)q_1 & iq_{2x} + \frac{2}{3}(r_1q_1 + r_2q_2)q_2 \\ -ir_{1x} + \frac{2}{3}(r_1q_1 + r_2q_2)r_1 & 0 & 0 \\ -ir_{2x} + \frac{2}{3}(r_1q_1 + r_2q_2)r_2 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Then a straightforward calculation shows that the compatibility condition of (3)–(4) leads to a system which reduces to (1) and (2) under the condition  $r_k^* = -q_k$  ( $k = 1, 2$ ).

The purpose of this paper is to construct a compact representation of the  $N$ -soliton solution for the two-component DNLS equation. We shall take the Darboux transformation approach. Indeed, the original Darboux transformation, which is associated with the Sturm–Liouville equation, has been generalized to many other differential and difference equations. It turns out that this approach often leads to elegant representations in terms of determinants for solutions of nonlinear systems and thus constitutes an ideal method to construct  $N$ -soliton solutions (see [3, 5, 6, 10]). In particular, Darboux transformations for certain multi-component integrable equations have been studied in [13, 16].

The paper is organized as follows. In the next section, we construct an elementary Darboux transformation for the general system (3)–(4), which naturally induces a Darboux transformation for the related conjugate system. Then, we combine two Darboux transformations together and find a two-fold Darboux transformation, which turns out to be the appropriate one for the reduction we are interested in. The reduction problem will

be tackled in section 3 and an elegant Darboux transformation will be given there for our two-component DNLS equation. In section 4, we iterate our Darboux transformation and give  $N$ -soliton solutions for the two-component DNLS equation in terms of determinants. The final section includes some discussions.

## 2. Darboux transformation in general

We now consider the general linear system (3)–(4) and manage to find a Darboux transformation for it. Our strategy is to find an appropriate Darboux transformation such that it can be easily reduced to the two-component DNLS case. To this end, we start with an elementary Darboux transformation

$$\hat{\Phi} = T_1 \Phi$$

with the Darboux matrix  $T_1 = \zeta T_{11} + T_{10}$ . After some calculations and analysis, we find that  $T_1$  has to take the following explicit form:

$$T_1 = \begin{pmatrix} a\zeta & c_1 & c_2 \\ c_3 & b\zeta & c\zeta \\ c_4 & d\zeta & e\zeta \end{pmatrix}, \quad (5)$$

where  $a, b, c, d$  and  $e$  are the functions of  $(x, t)$ , while  $c_1, c_2, c_3$  and  $c_4$  need to be constants. For convenience, we make the assumption

$$c_1 = c_3 = 1, \quad c_2 = c_4 = 0. \quad (6)$$

Since  $\text{Tr}(U) = 0$ ,  $\text{Tr}(V) = -9i\zeta^4$ , we may assume

$$\det(T_1) = \zeta \left( \frac{\zeta^2}{\zeta_1^2} - 1 \right), \quad (7)$$

where  $\zeta_1$  is a complex constant. Thus, the Darboux matrix  $T_1$  is singular at  $\zeta = \zeta_1$ . Next we associate the entries of  $T_1$  with a special solution of our linear systems (3)–(4). To this end, taking  $(\varphi_1, \varphi_2, \varphi_3)^T$  as a corresponding solution of the Lax pairs at  $\zeta = \zeta_1$  and requiring

$$T_1|_{\zeta=\zeta_1} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} = 0, \quad (8)$$

we obtain

$$a = -\frac{\varphi_2}{\zeta_1 \varphi_1}, \quad b = \frac{-\varphi_1 + \zeta_1 Q_2 \varphi_3}{\zeta_1 \varphi_2}, \quad c = -Q_2, \quad d = -\frac{\varphi_3}{\varphi_2}, \quad e = 1, \quad (9)$$

where  $Q_2$  is a potential:  $Q_{2,x} = q_2$ .

Now we have the following.

**Theorem 1.** Let  $(\varphi_1, \varphi_2, \varphi_3)^T$  be a particular solution of (3)–(4) at  $\zeta = \zeta_1$  and the matrix  $T_1$  be given by (5) with entries defined by (6) and (9). Then  $T_1$  is a Darboux matrix for the linear system (3)–(4), namely  $\hat{\Phi} = T_1 \Phi$  is a new solution of (3)–(4). The transformations between fields are given by

$$\begin{aligned} \hat{q}_1 &= \frac{(q_1 \varphi_2 + q_2 \varphi_3 - 3i\zeta_1 \varphi_1) \varphi_2}{\varphi_1^2}, & \hat{r}_1 &= \frac{(r_1 \varphi_1 + 3i\zeta_1 \varphi_2) \varphi_1 + (r_2 \varphi_2 - r_1 \varphi_3) \zeta_1 Q_2 \varphi_1}{\varphi_2^2}, \\ \hat{r}_2 &= \frac{(r_1 \varphi_3 - r_2 \varphi_2) \zeta_1 \varphi_1}{\varphi_2^2}, & \hat{q}_2 &= \frac{(\varphi_2 q_1 + \varphi_3 q_2 - 3i\zeta_1 \varphi_1) \zeta_1 Q_2 \varphi_2 - q_2 \varphi_1 \varphi_2}{\zeta_1 \varphi_1^2}, \end{aligned}$$

where hatted quantities are transformed variables.

**Proof.** What we need to do is to check that the equations

$$T_{1x} + T_1 U = \hat{U} T_1, \quad T_{1t} + T_1 V = \hat{V} T_1$$

hold, where

$$\hat{U} = \zeta^2 U_2 + \zeta \hat{U}_1, \quad \hat{V} = \zeta^4 V_4 + \zeta^3 \hat{V}_3 + \zeta^2 \hat{V}_2 + \zeta \hat{V}_1$$

and  $\hat{U}_1, \hat{V}_3, \hat{V}_2$  and  $\hat{V}_1$  are  $U_1, V_3, V_2, V_1$  with the corresponding entries  $r_1, r_2, q_1$  and  $q_2$  replaced respectively by  $\hat{r}_1, \hat{r}_2, \hat{q}_1$  and  $\hat{q}_2$ . Checking can be done by direct calculations.  $\square$

**Remark 1.** It is interesting to note that under this Darboux transformation, we also have an alternative representation for  $\hat{r}_2$ :  $d_x = \hat{r}_2$ .

To proceed, we note that the two-component DNLS equation also has the following Lax pairs:

$$-\Psi_x = \Psi U, \quad (10)$$

$$-\Psi_t = \Psi V, \quad (11)$$

where  $\Psi = (\phi_1, \phi_2, \phi_3)$  and  $U$  and  $V$  are the same as above. This linear problem actually is the conjugate problem of (3)–(4). A simple but useful observation is

**Lemma 1.** If the matrix  $T$  is a Darboux matrix of the original linear system (3)–(4), then  $T^{-1}$  is a Darboux matrix of the conjugate linear system (10)–(11).

**Proof.** Direct calculation.  $\square$

Now we consider the conjugate linear system and its Darboux transformation. The analysis goes as in the case of the original linear system. Taking  $(\chi_1, \chi_2, \chi_3)$  as a special solution of the system (10)–(11) at  $\zeta = \zeta_2$ , and constructing the matrix

$$T_2 = \begin{pmatrix} \hat{a}\zeta & 1 & 0 \\ 1 & \hat{b}\zeta & \hat{c}\zeta \\ 0 & \hat{d}\zeta & \zeta \end{pmatrix}, \quad (12)$$

where

$$\hat{a} = -\frac{\chi_2}{\zeta_2 \chi_1}, \quad \hat{b} = -\frac{\chi_1 + \zeta_2 R_2 \chi_3}{\zeta_2 \chi_2}, \quad \hat{c} = -\frac{\chi_3}{\chi_2}, \quad \hat{d} = R_2 \quad (13)$$

and

$$R_{2,x} = r_2,$$

we have

**Theorem 2.** The matrix  $T_2$  defined by (12) is an elementary Darboux matrix of the conjugate linear system (10)–(11) and the transformations between the field variables are given by

$$\begin{aligned} \hat{q}_1 &= \frac{q_1 \chi_1^2 + 3i\zeta_2 \chi_1 \chi_2 + \zeta_2 \chi_1 R_2 (q_1 \chi_3 - \chi_2 q_2)}{\chi_2^2}, & \hat{r}_1 &= \frac{(r_1 \chi_2 + r_2 \chi_3) \chi_2 - 3i\zeta_2 \chi_1 \chi_2}{\chi_1^2}, \\ \hat{r}_2 &= \frac{(3iR_2 \zeta_2^2 - r_2) \chi_1 \chi_2 - \zeta_2 \chi_2 R_2 (r_1 \chi_2 + r_2 \chi_3)}{\zeta_2 \chi_1^2}, & \hat{q}_2 &= \frac{(q_1 \chi_3 - q_2 \chi_2) \zeta_2 \chi_1}{\chi_2^2}. \end{aligned}$$

**Proof.** Direct calculation.  $\square$

Similar to remark 1, we have

**Remark 2.** An alternative formula for  $\hat{d}$  is  $\hat{d}_x = r_2$ . Thus,  $\hat{d} = d$ .

Finally we may have a combined Darboux transformation in the following manner: we take a particular solution  $\Phi_1 \equiv (\varphi_1, \varphi_2, \varphi_3)^T$  of (3)–(4) at  $\zeta = \zeta_1$  and a particular solution  $(\chi_1, \chi_2, \chi_3)$  of (10)–(11) at  $\zeta = \zeta_2$ . Then, with  $(\varphi_1, \varphi_2, \varphi_3)^T$  we may use theorem 1 and have a Darboux transformation whose Darboux matrix is  $T_1$ . At this stage,  $(\chi_1, \chi_2, \chi_3)$  is converted into a new solution  $\Psi_1 \equiv (\hat{\chi}_1, \hat{\chi}_2, \hat{\chi}_3) = (\chi_1, \chi_2, \chi_3)T_1^{-1}|_{\zeta=\zeta_2}$  for the conjugate linear system. This solution, with the help of theorem 2, enables us to construct a Darboux matrix  $T_2$  and take a Darboux transformation for the conjugate linear system, which in turn induces a transformation for the original linear system. Schematically it looks as

$$\Phi \xrightarrow[\text{seed : } \Phi_1]{T_1} \hat{\Phi} \xrightarrow[\text{seed : } \Psi_1]{T_2^{-1}} \Phi[1].$$

It is now easy to find the explicit formulae. Indeed, the three components of  $\Psi_1$  reads as

$$\begin{aligned}\hat{\chi}_1 &= \frac{\zeta_1 \zeta_2 \chi_1 \varphi_1 + \zeta_1^2 \chi_2 \varphi_2 + \zeta_1^2 \chi_3 \varphi_3}{(\zeta_1^2 - \zeta_2^2) \varphi_2}, \\ \hat{\chi}_2 &= \frac{\zeta_1^2 \chi_1 \varphi_1 + \zeta_1 \zeta_2 \chi_2 \varphi_2 + \zeta_1 \zeta_2 \chi_3 \varphi_3}{(\zeta_1^2 - \zeta_2^2) \varphi_1}, \\ \hat{\chi}_3 &= \frac{\zeta_1 (\zeta_1 \chi_1 \varphi_1 + \zeta_2 \chi_2 \varphi_2 + \zeta_2 \chi_3 \varphi_3)}{(\zeta_1^2 - \zeta_2^2) \varphi_1} Q_2 + \frac{\chi_3}{\zeta_2}.\end{aligned}$$

Using this seed solution, we find that the functions appeared in  $T_2$  in the present case read

$$\begin{aligned}\hat{a} &= -\frac{\varphi_2 (\zeta_1 \chi_1 \varphi_1 + \zeta_2 \chi_2 \varphi_2 + \zeta_2 \chi_3 \varphi_3)}{\zeta_2 \varphi_1 (\zeta_2 \chi_1 \varphi_1 + \zeta_1 \chi_2 \varphi_2 + \zeta_1 \chi_3 \varphi_3)}, \\ \hat{b} &= \frac{\varphi_3}{\varphi_2} Q_2 - \frac{\varphi_1 (\zeta_1 \zeta_2 \chi_1 \varphi_1 + \zeta_1^2 \chi_2 \varphi_2 + \zeta_2^2 \chi_3 \varphi_3)}{\zeta_1 \zeta_2 \varphi_2 (\zeta_1 \chi_1 \varphi_1 + \zeta_2 \chi_2 \varphi_2 + \zeta_2 \chi_3 \varphi_3)}, \\ \hat{c} &= -Q_2 + \frac{(\zeta_2^2 - \zeta_1^2) \chi_3 \varphi_1}{\zeta_1 \zeta_2 (\zeta_1 \chi_1 \varphi_1 + \zeta_2 \chi_2 \varphi_2 + \zeta_2 \chi_3 \varphi_3)}, \\ \hat{d} &= -\frac{\varphi_3}{\varphi_2}.\end{aligned}$$

The Darboux matrix we are seeking,  $T = T_2^{-1} T_1$ , after removing an overall factor  $\frac{\zeta_2^2}{(\zeta^2 - \zeta_2^2)}$  is

$$T = \begin{pmatrix} a\zeta^2 - 1 & c_1\zeta & c_2\zeta \\ c_3\zeta & b\zeta^2 - 1 & c\zeta^2 \\ c_4\zeta & d\zeta^2 & e\zeta^2 - 1 \end{pmatrix}, \quad (14)$$

where

$$a = \frac{D_2}{\zeta_1 \zeta_2 D_1}, \quad b = \frac{D_3}{\zeta_1 \zeta_2^2 D_2}, \quad e = \frac{D_4}{\zeta_1 \zeta_2^2 D_2}, \quad (15)$$

$$c = \frac{(\zeta_2^2 - \zeta_1^2) \chi_3 \varphi_2}{\zeta_1 \zeta_2^2 D_2}, \quad d = \frac{(\zeta_2^2 - \zeta_1^2) \chi_2 \varphi_3}{\zeta_1 \zeta_2^2 D_2}, \quad (16)$$

$$c_1 = \frac{(\zeta_2^2 - \zeta_1^2) \chi_2 \varphi_1}{\zeta_1 \zeta_2 D_1}, \quad c_2 = \frac{(\zeta_2^2 - \zeta_1^2) \chi_3 \varphi_1}{\zeta_1 \zeta_2 D_1}, \quad (17)$$

$$c_3 = \frac{(\zeta_2^2 - \zeta_1^2) \chi_1 \varphi_2}{\zeta_1 \zeta_2 D_2}, \quad c_4 = \frac{(\zeta_2^2 - \zeta_1^2) \chi_1 \varphi_3}{\zeta_1 \zeta_2 D_2}, \quad (18)$$

and

$$\begin{aligned} D_1 &= \zeta_1 \chi_1 \varphi_1 + \zeta_2 \chi_2 \varphi_2 + \zeta_3 \chi_3 \varphi_3, \\ D_2 &= \zeta_2 \chi_1 \varphi_1 + \zeta_1 \chi_2 \varphi_2 + \zeta_1 \chi_3 \varphi_3, \\ D_3 &= \zeta_1 \zeta_2 \chi_1 \varphi_1 + \zeta_2^2 \chi_2 \varphi_2 + \zeta_1^2 \chi_3 \varphi_3, \\ D_4 &= \zeta_1 \zeta_2 \chi_1 \varphi_1 + \zeta_1^2 \chi_2 \varphi_2 + \zeta_2^2 \chi_3 \varphi_3. \end{aligned}$$

The transformations between field variables can be reformed neatly:

$$q_1[1] = q_1 - c_{1,x}, \quad q_2[1] = q_2 - c_{2,x}, \quad (19)$$

$$r_1[1] = r_1 - c_{3,x}, \quad r_2[1] = r_2 - c_{4,x}, \quad (20)$$

and  $c_i$ 's are given by (17)–(18).

### 3. Reduction

In the last section, we constructed a combined or two-fold Darboux transformation for our linear system (3)–(4). The relevant Darboux matrix and field variable transformations are given by (14) and (19)–(20), respectively. What we are interested in is to present a Darboux transformation for the two-component DNLS equation and thus we have to do reduction. Next we will show that our Darboux transformation can be reduced easily to the interested case.

The constraints between field variables are

$$r_1 = -q_1^*, \quad r_2 = -q_2^*$$

which should be invariant under Darboux transformation. Now we note that, for the solution  $(\varphi_1, \varphi_2, \varphi_3)^T$  of the linear system (3)–(4) at  $\zeta = \zeta_1$ ,  $(\varphi_1^*, \varphi_2^*, \varphi_3^*)$  is the solution of the conjugate linear system (10)–(11) at  $\zeta = \zeta_1^*$ . Therefore, we use it as our seed for the second-step Darboux transformation. Namely,

$$\Psi_1 = (\varphi_1^*, \varphi_2^*, \varphi_3^*), \quad \zeta_2 = \zeta_1^*.$$

With these considerations, it is easy to verify that

$$c_1^* = -c_3, \quad c_2^* = c_4;$$

therefore,

$$r_1[1] = -q_1[1]^*, \quad r_2[1] = -q_2[1]^*.$$

The final transformation is given neatly by

$$q_1[1] = q_1 - \frac{\zeta_1^{*2} - \zeta_1^2}{|\zeta_1^2|} \left( \frac{\varphi_1 \varphi_2^*}{|\varphi_1^2| \zeta_1 + \zeta_1^* (|\varphi_2^2| + |\varphi_3^2|)} \right)_x, \quad (21)$$

$$q_2[1] = q_2 - \frac{\zeta_1^{*2} - \zeta_1^2}{|\zeta_1^2|} \left( \frac{\varphi_1 \varphi_3^*}{|\varphi_1^2| \zeta_1 + \zeta_1^* (|\varphi_2^2| + |\varphi_3^2|)} \right)_x. \quad (22)$$

If we start with the vacuum solution  $q_1 = q_2 = 0$ , then the linear system (3)–(4) has a solution

$$\varphi_1 = e^{-2i\zeta_1^2 x - 9i\zeta_1^4 t}, \quad \varphi_2 = e^{i\zeta_1^2 x} = \varphi_3$$

which leads to

$$q_1[1] = q_2[1] = \frac{6\zeta_1^2 \text{Im}(\zeta_1^2) e^{-iR} [\sinh(I)(\zeta_1^{*2} - 2|\zeta_1^2|) + \cosh(I)(\zeta_1^{*2} + 2|\zeta_1^2|)]}{\zeta_1^* [\sinh(I)(\zeta_1^2 + 2|\zeta_1^2|) + \cosh(I)(\zeta_1^2 - 2|\zeta_1^2|)]^2},$$

where  $R = 3 \operatorname{Re}(\zeta_1^2)x + 9 \operatorname{Re}(\zeta_1^4)t$ ,  $I = 3 \operatorname{Im}(\zeta_1^2)x + 9 \operatorname{Im}(\zeta_1^4)t$ . It is nothing but a solution of the DNLS equation. To find more interesting ones we need to iterate our Darboux transformation and we will do so in the next section.

#### 4. Iterations: $N$ -fold Darboux matrix

The appealing feature of a Darboux transformation is that it often leads to a determinant representation for  $N$ -solitons. To this end, one has to do iteration. In this section, we consider the iteration problem for our Darboux transformation.

First, let us rewrite our Darboux matrix  $T$  given by (14) with the reductions in mind. Introduce a new matrix

$$N(\zeta) = \operatorname{diag} \left( \frac{\zeta \varphi_1}{\zeta_1 D}, \frac{\zeta \varphi_2}{\zeta_1 D^*}, \frac{\zeta \varphi_3}{\zeta_1 D^*} \right),$$

where  $D = D_1|_{\chi_j=\varphi_j^*, \zeta_2=\zeta_1^*}$ . Then, the Darboux matrix  $T$  takes the following form:

$$T = \frac{\zeta^2 - \zeta_1^{*2}}{\zeta_1^{*2}} + \frac{\zeta_1^{*2} - \zeta_1^2}{\zeta_1^{*2}} N(\zeta) \begin{pmatrix} \zeta \varphi_1^* & \zeta_1^* \varphi_2^* & \zeta_1^* \varphi_3^* \\ \zeta_1^* \varphi_1^* & \zeta \varphi_2^* & \zeta \varphi_3^* \\ \zeta_1^* \varphi_1^* & \zeta \varphi_2^* & \zeta \varphi_3^* \end{pmatrix}.$$

Now, on the one hand we have already known

$$T|_{\zeta=\zeta_1} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} = 0, \quad (23)$$

i.e. our seed  $\Phi_1 = (\varphi_1, \varphi_2, \varphi_3)^T$  lies in the kernel of the matrix  $T|_{\zeta=\zeta_1}$ . On the other hand, let us suppose

$$T|_{\zeta=\zeta_1^*} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \frac{\zeta_1^{*2} - \zeta_1^2}{\zeta_1^*} N(\zeta_1^*) \begin{pmatrix} \varphi_1^* \psi_1 + \varphi_2^* \psi_2 + \varphi_1^* \psi_3 \\ \varphi_1^* \psi_1 + \varphi_2^* \psi_2 + \varphi_1^* \psi_3 \\ \varphi_1^* \psi_1 + \varphi_2^* \psi_2 + \varphi_1^* \psi_3 \end{pmatrix} = 0 \quad (24)$$

for a certain vector function  $\Psi_1 = (\psi_1, \psi_2, \psi_3)^T$ ; then, for  $\zeta_1 \neq \zeta_1^*$  one has to impose  $\varphi_1^* \psi_1 + \varphi_2^* \psi_2 + \varphi_1^* \psi_3 = 0$  or

$$\Phi_1^\dagger \Psi_1 = 0.$$

Obviously

$$\Psi_1 = (-\varphi_2^*, \varphi_1^*, 0)^T \quad \text{or} \quad \Psi_1 = (-\varphi_3^*, 0, \varphi_1^*)^T$$

meet the requirement

$$T|_{\zeta=\zeta_1^*} \Psi_1 = 0. \quad (25)$$

We observe that the conditions (23) and (25) can in turn be used to determine the nine quantities appeared in  $T$  uniquely.

Now we are ready to do iterations. Assume that we are given  $N$  distinct complex numbers  $\zeta_1, \zeta_2, \dots, \zeta_N$  such that  $\zeta_k^{*2} \neq \zeta_k^2$  ( $k = 1, 2, \dots, N$ ). We further assume that the vector

$$\Phi_k = (\varphi_1^{(k)}, \varphi_2^{(k)}, \varphi_3^{(k)})^T$$

is a solution of a linear equation at  $\zeta = \zeta_k$ , i.e.

$$[\partial_x - U(\zeta = \zeta_k)](\Phi_k) = 0, \quad [\partial_t - V(\zeta = \zeta_k)](\Phi_k) = 0,$$



and

$$\Psi_k^1 = (-\varphi_2^{(k)*}, \varphi_1^{(k)*}, 0)^T, \quad \Psi_k^2 = (-\varphi_3^{(k)*}, 0, \varphi_1^{(k)*})^T,$$

which satisfy the orthogonal conditions  $\Phi_k^\dagger \Psi_k^l = 0$ .

With these seed solutions, we define

$$T_k = \frac{\zeta^2 - \zeta_k^{*2}}{\zeta_k^{*2}} + \frac{\zeta_k^{*2} - \zeta_k^2}{\zeta_k^{*2}} N_k(\zeta) \begin{pmatrix} \zeta \varphi_1^{(k)}[k-1]^* & \zeta_k^* \varphi_2^{(k)}[k-1]^* & \zeta_k^* \varphi_3^{(k)}[k-1]^* \\ \zeta_i^* \varphi_1^{(k)}[k-1]^* & \zeta \varphi_2^{(k)}[k-1]^* & \zeta \varphi_3^{(k)}[k-1]^* \\ \zeta_i^* \varphi_1^{(k)}[k-1]^* & \zeta \varphi_2^{(k)}[k-1]^* & \zeta \varphi_3^{(k)}[k-1]^* \end{pmatrix},$$

where

$$D[k] = \zeta_k |\varphi_1^{(k)}[k-1]|^2 + \zeta_k^* (|\varphi_2^{(k)}[k-1]|^2 + |\varphi_3^{(k)}[k-1]|^2),$$

$$N_k(\zeta) = \text{diag} \left( \frac{\zeta \varphi_1^{(k)}[k-1]^*}{\zeta_i D[k]}, \frac{\zeta \varphi_2^{(k)}[k-1]^*}{\zeta_i D[k]^*}, \frac{\zeta \varphi_3^{(k)}[k-1]^*}{\zeta_i D[k]^*} \right),$$

and our notation is as follows:

$$\Phi_j[k] = \begin{pmatrix} \varphi_1^{(j)}[k] \\ \varphi_2^{(j)}[k] \\ \varphi_3^{(j)}[k] \end{pmatrix} = T_k T_{k-1} \cdots T_1|_{\zeta=\zeta_j} \begin{pmatrix} \varphi_1^{(j)} \\ \varphi_2^{(j)} \\ \varphi_3^{(j)} \end{pmatrix},$$

and  $\Phi_j[0] = \Phi_j$ .

The  $N$ -times iterated Darboux matrix is given by

$$T = T_N T_{N-1} \cdots T_1.$$

It is easy to see that, similar to equation (23), the following relations hold:

$$T_k T_{k-1} \cdots T_1|_{\zeta=\zeta_k} \Phi_k = T_k|_{\zeta=\zeta_k} \Phi_k[k-1] = 0 \quad (k = 1, 2, \dots, N).$$

Furthermore, we recursively define

$$\Psi_k^l[0] = \Psi_k^l, \quad \Psi_k^l[j-1] = T_{j-1} T_{j-2} \cdots T_1|_{\zeta=\zeta_k^*} \Psi_k^l;$$

then we have

**Proposition 1.**  $\Phi_k^\dagger[k-1] \Psi_k^l[k-1] = 0$ .

**Proof.** We know  $\Phi_k^\dagger[0] \Psi_k^l[0] = 0$ . Let us suppose  $\Phi_k^\dagger[m] \Psi_k^l[m] = 0$  ( $0 \leq m < k-1$ ). Then, thanks to  $\Phi_k[m+1] = T_{m+1}|_{\zeta=\zeta_k} \Phi_k[m]$  and  $\Psi_k^l[m+1] = T_{m+1}|_{\zeta=\zeta_k^*} \Psi_k^l[m]$ , we have

$$\Psi_k^\dagger[m+1] \Phi_k^l[m+1] = \Psi_k^\dagger[m] T_{m+1}^\dagger|_{\zeta=\zeta_k} T_{m+1}|_{\zeta=\zeta_k^*} \Phi_k^l[m] = 0,$$

because  $T_{m+1}^\dagger|_{\zeta=\zeta_k} T_{m+1}|_{\zeta=\zeta_k^*} = \frac{1}{|\zeta_{m+1}^4|} (\zeta_k^{*2} - \zeta_{m+1}^{*2})(\zeta_k^{*2} - \zeta_{m+1}^2)$ . Therefore, the lemma follows from the mathematical induction.  $\square$

Based on proposition 1, we obtain

$$T_k T_{k-1} \cdots T_1|_{\zeta=\zeta_k^*} \Psi_k^l = T_k|_{\zeta=\zeta_k^*} \Psi_k^l[k-1] = 0, \quad (l = 1, 2).$$

Therefore, we have

$$T|_{\zeta=\zeta_k} \Phi_k = 0, \quad T|_{\zeta=\zeta_k^*} \Psi_k^1 = 0, \quad T|_{\zeta=\zeta_k^*} \Psi_k^2 = 0 \quad (26)$$

for  $k = 1, 2, \dots, N$ . We also note that our iterated Darboux matrix  $T$  is of the form

$$T = \sum_{k=0}^{2N} \zeta^k T_k = \sum_{n=1}^N \begin{pmatrix} a_{2n} \zeta^{2n} & c_1^{(2n-1)} \zeta^{2n-1} & c_2^{(2n-1)} \zeta^{2n-1} \\ c_3^{(2n-1)} \zeta^{2n-1} & b_{2n} \zeta^{2n} & c_{2n} \zeta^{2n} \\ c_4^{(2n-1)} \zeta^{2n-1} & d_{2n} \zeta^{2n} & e_{2n} \zeta^{2n} \end{pmatrix} + (-1)^N.$$

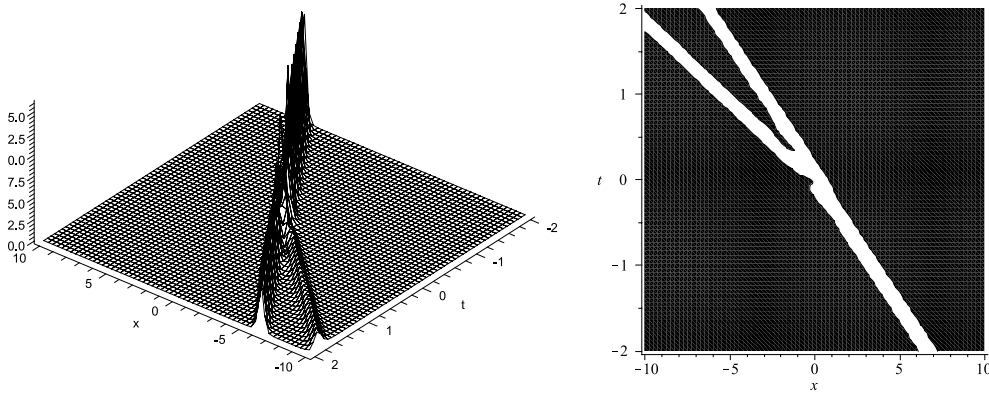


Figure 1.  $|q_1^2|$  component of the two-soliton solution and its contour plot.

The above coefficients can be found in  $T$  by solving the linear algebraic systems (26). The solution formulae are obtained from

$$q_1[N] = q_1 + \left( \frac{H_2}{H_1} \right)_x, \quad q_2[N] = q_2 + \left( \frac{H_3}{H_1} \right)_x,$$

where

$$H_1 = \begin{pmatrix} \zeta_1^{2N} \varphi_1^{(1)} & \zeta_1^{2N-1} \varphi_2^{(1)} & \zeta_1^{2N-1} \varphi_3^{(1)} & \dots & \zeta_1^2 \varphi_1^{(1)} & \zeta_1 \varphi_2^{(1)} & \zeta_1 \varphi_3^{(1)} \\ -\zeta_1^{2N*} \varphi_2^{(1)*} & \zeta_1^{2N-1*} \varphi_1^{(1)*} & 0 & \dots & -\zeta_1^{2*} \varphi_2^{(1)*} & \zeta_1^* \varphi_1^{(1)*} & 0 \\ -\zeta_1^{2N*} \varphi_3^{(1)*} & 0 & \zeta_1^{2N-1*} \varphi_1^{(1)*} & \dots & -\zeta_1^{2*} \varphi_3^{(1)*} & 0 & \zeta_1^* \varphi_1^{(1)*} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \zeta_N^{2N} \varphi_1^{(N)} & \zeta_N^{2N-1} \varphi_2^{(N)} & \zeta_N^{2N-1} \varphi_3^{(N)} & \dots & \zeta_N^2 \varphi_1^{(N)} & \zeta_N \varphi_2^{(N)} & \zeta_N \varphi_3^{(N)} \\ -\zeta_N^{2N*} \varphi_2^{(N)*} & \zeta_N^{2N-1*} \varphi_1^{(N)*} & 0 & \dots & -\zeta_N^{2*} \varphi_2^{(N)*} & \zeta_N^* \varphi_1^{(N)*} & 0 \\ -\zeta_N^{2N*} \varphi_3^{(N)*} & 0 & \zeta_N^{2N-1*} \varphi_1^{(N)*} & \dots & -\zeta_N^{2*} \varphi_3^{(N)*} & 0 & \zeta_N^* \varphi_1^{(N)*} \end{pmatrix},$$

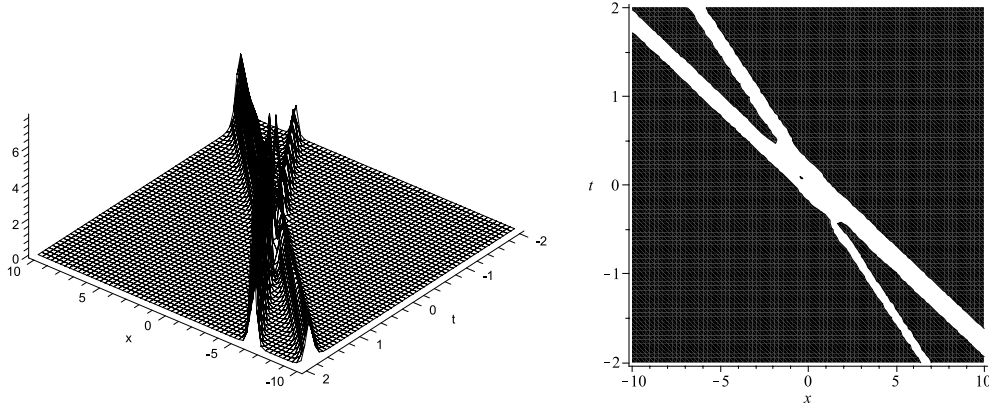
and  $H_2$  and  $H_3$  are  $H_1$  with the  $3N - 1$ st column and the  $3N$ th column replaced by  $L_1$  respectively, where

$$L_1 = (-\varphi_1^{(1)}, \varphi_2^{(1)*}, \varphi_3^{(1)*}, \dots, -\varphi_1^{(N)}, \varphi_2^{(N)*}, \varphi_3^{(N)*})^T.$$

To demonstrate the usefulness our solution formulae, we calculate solutions for the two-component DNLS equation. Selecting

$$\begin{aligned} \zeta_1 &= 1 + \frac{1}{3}i, & \zeta_2 &= 1 + \frac{2}{3}i, & \Phi_1 &= (e^{-2i\zeta_1^2 x - 9i\zeta_1^4 t}, 0, e^{i\zeta_1^2 x})^T, \\ \Phi_2 &= (e^{-2i\zeta_2^2 x - 9i\zeta_2^4 t}, e^{i\zeta_2^2 x}, e^{i\zeta_2^2 x})^T, \end{aligned}$$

and substituting them into (4) we could have the solutions. Figures 1 and 2 show these solutions by plotting  $|q_1^2|$  and  $|q_2^2|$ . It is interesting to point out that while the second figure exhibits standard two-soliton scattering, the first one demonstrates a fission process.



**Figure 2.**  $|q_2^2|$  component of the two-soliton solution and its contour plot.

## 5. Conclusion

Above we found a Darboux transformation for the two-component DNLS equation and obtained a closed formula for its solutions. We remark that our Darboux transformation can be easily generalized to the multi-component case. In fact, the Darboux matrix in this case is

$$T = \frac{\zeta^2 - \zeta_1^{*2}}{\zeta_1^{*2}} + \frac{\zeta_1^{*2} - \zeta_1^2}{\zeta_1^{*2}} N(\zeta) \begin{pmatrix} \zeta \varphi_1^* & \zeta_1^* \varphi_2^* & \zeta_1^* \varphi_3^* & \cdots & \zeta_1^* \varphi_n^* \\ \zeta_1^* \varphi_1^* & \zeta \varphi_2^* & \zeta \varphi_3^* & \cdots & \zeta \varphi_n^* \\ \zeta_1^* \varphi_1^* & \zeta \varphi_2^* & \zeta \varphi_3^* & \cdots & \zeta \varphi_n^* \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \zeta_1^* \varphi_1^* & \zeta \varphi_2^* & \zeta \varphi_3^* & \cdots & \zeta \varphi_n^* \end{pmatrix},$$

where

$$N(\zeta) = \text{diag} \left( \frac{\zeta \varphi_1}{\zeta_1 D}, \frac{\zeta \varphi_2}{\zeta_1 D^*}, \dots, \frac{\zeta \varphi_n}{\zeta_1 D^*} \right)$$

with

$$D = \zeta_1 |\varphi_1^2| + \zeta_1^* |\varphi_2^2| + \cdots + \zeta_1^* |\varphi_n^2|$$

and solution formulae may be derived.

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