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2015 Nonlinearity 28 3243

(http://iopscience.iop.org/0951-7715/28/9/3243)

IOP Publishing | London Mathematical Society

Nonlinearity

Nonlinearity **28** (2015) 3243–3261

doi:10.1088/0951-7715/28/9/3243

Darboux transformation and multi-dark soliton for *N*-component nonlinear Schrödinger equations

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Received 23 December 2013, revised 19 December 2014

Accepted for publication 15 July 2015

Published 5 August 2015

CrossMark

Recommended by Professor Athanassios S Fokas

Abstract

In this paper, we obtain a uniform Darboux transformation for multi-component coupled nonlinear Schrödinger (NLS) equations, which can be reduced to all previously presented Darboux transformations. As a direct application, we derive the single dark soliton and multi-dark soliton solutions for multi-component NLS equations with a defocusing case and a mixed focusing and defocusing case. Some exact single and two-dark solitons of three-component NLS equations are investigated explicitly. The results are meaningful for vector dark soliton studies in many physical systems, such as Bose–Einstein condensate, nonlinear optics, etc.

Keywords: Darboux transformation, dark soliton, N-component NLS equations

Mathematics Subject Classification: 37K15, 37K35, 35C08

(Some figures may appear in colour only in the online journal)

1. Introduction

It is well known that nonlinear Schrödinger (NLS)-type equations play a prominent role in nonlinear physical systems, such as nonlinear optics [1] and Bose–Einstein condensates [2]. In these physical systems, the nonlinear coefficient can be positive or negative, depending on the physical situations [3]. For example, the nonlinearity can be positive or negative when the interaction between the atoms is repulsive or attractive in Bose–Einstein condensates [2]. For nonlinear optics, it corresponds to the focusing or defocusing case. For the one-component system, many studies have been carried out [4] which demonstrate that the corresponding scalar NLS equation admits bright solitons [4], a breather [5, 6] and rogue wave [7, 8] in the focusing case, and dark solitons [9] in the defocusing case.

Since a variety of complex systems such as Bose–Einstein condensates, nonlinear optical fibres etc usually involve more than one component, the studies should be extended to multi-component NLS equation cases [10–12]. For the two-component coupled system with both focusing cases, the coupled NLS equations admit bright–bright solitons, and bright–dark, breather, rogue wave, bright–dark-breather and bright–dark-rogue wave solution [11–16]. With both defocusing cases, the coupled NLS equations admit bright–dark and dark–dark solitons, as well as breather solution [15–18]. With the defocusing and focusing coexisting case, the coupled NLS equations admit bright–bright solitons, bright–dark solitons, dark–dark solitons and breather solution [16, 19–21]. For the three-component NLS equations with all focusing, the 'four-petaled flower' structure rogue wave was presented recently by the Darboux transformation (DT) [22].

The DT method is a very effective and convenient way to derive kinds of localised waves, such as bright solitons, breather and rogue wave [11–14, 16]. However, the dark soliton can not be obtained by the classical DT method directly. The first time the dark soliton of the single NLS equation through the DT method was obtained occurred in 1996 [23]. The single dark soliton formula of the N-component NLS type equation appeared in 2006 and 2009 [24, 25]. The dark soliton of the inverse scattering method in the coupled NLS system was an open problem until 2006 [26]. Even the soliton solutions obtained in [26] can be degenerated into a scalar NLS equation. Therefore, it is still desirable to study how to obtain the multi-dark soliton through the DT method, which would be very meaningful for vector dark soliton studies in related physical systems.

In this paper, we would like to derive a simple multi-soliton formula for the dark soliton of the integrable N-component NLS system (7) through the generalising DT method. In order to present our work clearly and coherently, we revisit the methods from 1996 and 2006 [23, 24]. In 1996 [23], Mañas proposed a method to derive the dark soliton for defocusing the NLS equation. The Darboux matrix can be represented as

$$T = I + \frac{\mu_1 - \lambda_1}{\lambda - \mu_1}\frac{\Phi_1\Psi_1}{\Psi_1\Phi_1},$$

where μ_1 and λ_1 are real numbers, $\Phi_1 = (\phi_1, \bar{\phi}_1)^T$ is a solution with spectral parameter $\lambda = \lambda_1$, $\Psi_1 = (\psi_1, -\bar{\psi}_1)$ is a solution of conjugation system with spectral parameter $\lambda = \mu_1$. One fold DT could yields two dark solitons for defocusing the NLS equation. The symmetry relation is given as

$$T^\dagger(\bar{\lambda})\sigma_3 T(\lambda) = f(\lambda)\sigma_3,$$

where σ_3 is the third standard Pauli matrix and $f(\lambda)$ is a function of λ. However, even when trying our best, the Manãns' method cannot be applied to the multi-component case besides the degenerate case. Thus, we give up generalising the Manãns's method.

In [24], Degasperis and Lambardo presented a one-fold DT method for the dark soliton. The Darboux matrix is

$$D(x, t; \lambda) = I + \mathrm{i}\frac{\hat{z}\hat{z}^\dagger\Sigma}{(\lambda - \lambda_1)P_1}, \quad \hat{z}^\dagger\Sigma\hat{z} = 0, \quad P_1 = \int \hat{z}^\dagger\Sigma\hat{z}\,\mathrm{d}x, \tag{1}$$

where $\hat{z}$ is a solution with spectral parameter $\lambda = \lambda_1$ and Σ is a diagonal matrix with elements ± 1. Evidently, it is not convenient to calculate the integration. Besides, the integration restricts us from iterating the above DT (1) step by step. In order to look for a simpler method, we continue to search for some valuable information.

In 2009 [27], Cieśliński revisited different types of DT methods. He pointed out that the classical binary DT can be represented as a nilpotent gauge matrix. Indeed, we can see that the DT (1) is nothing but a nilpotent Darboux matrix. Thus, we can check whether or not the DT (1) can be converted into the classical binary DT. The answer is affirmative. First, we know that the most important property for the DT is the kernel for $D(x, t; \lambda)$:

$$\lim_{\lambda \to \lambda_1} D(x, t; \lambda_1)\hat{z}(\lambda) = 0.$$

On the other hand, we have the following equality

$$\lim_{\lambda \to \lambda_1} \frac{\mathrm{i}\hat{z}_1^{\dagger}\Sigma\hat{z}(\lambda)}{\lambda - \lambda_1} = -\int \hat{z}_1^{\dagger}\Sigma\hat{z}_1 \mathrm{d}x.$$

Thus, the DT (1) can be represented as the classical DT [29]

$$\hat{z}(\lambda) \to D(x, t; \lambda)\hat{z}(\lambda) = \hat{z}(\lambda) - \frac{\hat{z}_1\Omega(\hat{z}_1, \hat{z})}{\Omega(\hat{z}_1, \hat{z}_1)}, \quad \mathrm{d}\Omega(f, g) = f^{\dagger}\Sigma g \mathrm{d}x.$$

Generally speaking, the DT is considered as a special gauge transformation [28, 29]. That is the reason we underestimate the classical binary DT [29–31]. The binary DT was first proposed by Babich, Matveev and Salle in [30]. For details of the binary DT one can refer to reference [29] (and the references therein). Indeed, the binary DT is the consistent transformation for the AKNS system. With the binary DT method, we can reduce it to the Zakharov–Shabat dressing operator [32] or the loop group representation [33]. In particular, we can obtain the DT which can be used to derive the dark soliton very effectively and conveniently. It is worth mentioning that there have been some other methods for deriving the dark soliton for multi-component NLS equations recently, such as the algebraic-geometry reduction method, the KP equation reduction method and the dressing-Hirota method [19, 34, 35].

Here, we need to remark that the single step DT for focusing and defocusing multi-component NLS equations was given by Degasperis and Lombardo in 2006. However, they did not given a uniform representation. Very recently, before we submitted our paper, Tsuchida studied the subject by a Darboux–Bäcklund transformation [36], in which both bright-soliton solutions and dark-soliton solutions could be obtained, depending on the signs of the nonlinear terms. Indeed, their results are a generalisation of Park and Shin's results [37] and Degasperis and Lombardo's results [24, 25] based on a special calculation technique.

In general, if we have a one-fold DT, it follows that the n-fold DT can be obtained by iteration. However, it is not fitted for the dark soliton's DT. For Degasperis and Lombardo's method, the difficulty of iterating the DT lies in how to calculate the integration. As the time of the iteration increases, we cannot solve the integration even though by computer software, since the integrands would become very complicated. Indeed, the same thing is met in Tsuchida's method, the first step DT $T(\lambda)$ with special function $\Phi_1(\lambda_1)$ for the linear spectral problem, we can deal with the integration by the limit technique. However, if we iterate the DT directly, we cannot use this calculation trick again, since there is no similar equation (29). And if we need to iterate the DT, we need to use a new special function $\Phi_2(\lambda_2)$ to calculate the limit

$$\lim_{\lambda \to \lambda_2} \frac{[T(\lambda_2)\Phi_2(\lambda_2)]^{\dagger}\Lambda T(\lambda)\Phi_2(\lambda)}{\lambda - \lambda_2}$$

to replace the integration of Degasperis and Lombardo's integration. However, it is almost impossible to calculate the limitation directly and is even more difficult than Degasperis and Lombardo's integration.

Based on above mentioned problems, we propose a systematical method to construct the multi-fold uniform DT based on Degasperis and Lombardo's method [24, 25] and Ciesĺinśki's method [27]. The aim of our work is two fold. First, we reduce the binary DT of the AKNS system to obtain a uniform transformation for the AKNS system, which can be reduced to all previously presented DTs. Second, we use the binary DT of the AKNS system to derive the dark soliton and multi-dark soliton for the N-component NLS equation (7). The multi-dark soliton for the N-component NLS equation (7) through the DT method is obtained for the first time. In section 2, we introduce some basic knowledge about the AKNS system. Then we give the binary DT for the AKNS system. Based on the binary DT, we reduce it to the different transformations of the AKNS system. In section 3, by the the uniform transformation and limit technique [8, 38, 39], we derive the single dark soliton and multi-dark soliton for N-component NLS equation (7). In order to give us a clear understanding of our formula, we plot the explicit dark soliton and two dark solitons picture of the three-component NLS equations. The final section consists of some conclusions and discussions.

2. The AKNS system and the binary Darboux transformation

In this section, we first recall some results about the AKNS system and its reduction for multi-component NLS equations. Second, we introduce the binary DT for the AKNS system. Finally, we reduce the binary DT into a different transformation by a conjugation equation and limit technique. Integration for a high-order solution is automatic through the limit technique.

2.1. The AKNS system

We recall the classical results about the AKNS system [40]. Let $a = \mathrm{diag}(a_1, a_2, \cdots, a_n)$ be a fixed nonzero diagonal matrix in $\mathrm{gl}(n, \mathbb{C})$, and denote

$$\begin{aligned}\mathrm{sl}(n)_a &= \{y \in \mathrm{sl}(n, \mathbb{C}) | [a, y] = 0\},\\ \mathrm{sl}(n)_a^{\perp} &= \{y \in \mathrm{sl}(n, \mathbb{C}) | \mathrm{tr}(zy) = 0 \quad \text{for any } z \in \mathrm{sl}(n)_a\}.\end{aligned}$$

Let $L^\infty(\mathbb{R}, \mathrm{sl}(n)_a^{\perp})$ denote the space of maps in the $L^\infty(\mathbb{R})$ class. For the spectral problem

$$\Phi_x = (a\lambda + u(x))\Phi, \tag{2}$$

when $\lambda \to \infty$, we have the formal asymptotical behaviour

$$\Phi \to \exp[a\lambda x].$$

Thus, we can suppose $\Phi = m(x;\lambda)\mathrm{e}^{a\lambda x}$, where $m(x;\lambda)$ is an analytical function and possesses the following formal expansion

$$m(x;\lambda) = I + m_1(x)\lambda^{-1} + m_2(x)\lambda^{-2} + \cdots.$$

Substituting it into (2), we can obtain

$$\Phi_x\Phi^{-1} = mam^{-1}\lambda + m_x m^{-1} = a\lambda + [m_1, a] + O(\lambda^{-1})$$

and $\Phi_x\Phi^{-1}$ being holomorphic in $\lambda \in \mathbb{C}$ implies that $\Phi_x\Phi^{-1} = a\lambda + [m_1, a]$.

Let $b \in \mathrm{gl}(n, \mathbb{C})$ such that $[b,a] = 0$, we have the allowing formal expansion of mbm^{-1}at $\lambda = \infty$,

$$mbm^{-1} \sim Q_{b,0} + Q_{b,1}\lambda^{-1} + Q_{b,2}\lambda^{-2} + \cdots .$$

Since $\Phi_x\Phi^{-1} = \lambda a + u$ and $\Phi b\Phi^{-1} = mbm^{-1}$, we can obtain that

$$[\partial_x - a\lambda - u, \Phi b\Phi^{-1}] = 0.$$

It follows that

$$(Q_{j,b}(u))_x + [u, Q_{b,j}(u)] = [Q_{b,j+1}(u), a]. \tag{3}$$

Write

$$Q_{b,j} = T_{b,j} + P_{b,j} \in \mathrm{sl}(n)_a + \mathrm{sl}(n)_a^{\perp}.$$

Then equation (3) gives

$$\begin{aligned} P_{b,j} &= \mathrm{ad}(a)^{-1}((P_{b,j-1})_x - \pi_1([u, Q_{b,j-1}])),\\ (T_{b,j})_x &= \pi_0([u, P_{b,j}]), \end{aligned} \tag{4}$$

where π_0 and π_1 denote the projection of $\mathrm{sl}(n, \mathbb{C})$ onto $\mathrm{sl}(n)_a^{\perp}$ and $\mathrm{sl}(n)_a^{\perp}$ with respect to $\mathrm{sl}(n, \mathbb{C}) = \mathrm{sl}(n)_a + \mathrm{sl}(n)_a^{\perp}$, respectively. In reference [33], they proved that if b is a polynomial of a, then $Q_{b,j}$ is an order-(j-1) polynomial differential operator in u.

Then we have the following proposition:

Proposition 1 (Terng and Uhlenbeck, [33]). *Suppose $u(\cdot, t) \in L^{\infty}(\mathbb{R}, \mathrm{sl}(n)_a^{\perp})$ for all t,*

$$[\partial_x - a\lambda - u, \partial_t + b\lambda^j + v_1\lambda^{j-1} + \cdots + v_j] = 0$$

for some $v_1, \cdots, v_j$, and $\lim_{x\to-\infty} v_k(x,t) = \lim_{x\to-\infty} Q_{b,k}(u(x,t))$ for all $1 \leqslant k \leqslant j$. Then we have $v_k = Q_{b,k}(u)$, and

$$u_t = (Q_{j,b}(u))_x - [u, Q_{b,j}(u)] = [Q_{b,j+1}(u), a]. \tag{5}$$

In what follows, we consider the reality conditions. The details of the reality conditions are given in [33]. A Lax pair $[\partial_x + A(x,t;\lambda), \partial_t + B(x,t;\lambda)] = 0$ is said to satisfy the reality condition if $\sigma(A(x,t;\bar{\lambda})) = A(x,t;\lambda)$ and $\sigma(B(x,t;\bar{\lambda})) = B(x,t;\lambda)$, where the overbar denotes the complex conjugation and σ is the complex conjugate linear Lie algebra involution in $\mathrm{sl}(n, \mathbb{C})$.

In this paper, we merely consider the $u(N+1-k, k)$ hierarchy on $L^{\infty}(\mathbb{R}, u_a^{\perp})$ [33]. Here

$$u(N+1-k, k) = \{y \in \mathrm{gl}(N, \mathbb{C}) | y^{\dagger}\Lambda + \Lambda y = 0\},$$

where the symbol † denotes the hermite conjugation, $\Lambda = \mathrm{diag}(1, \Lambda_1)$ and

$$\Lambda_1 = \mathrm{diag}(\epsilon_1, \epsilon_2, \cdots, \epsilon_N), \tag{6}$$

$\epsilon_i = -1$, for $1 \leqslant i \leqslant k$; $\epsilon_i = 1$, for $k+1 \leqslant i \leqslant N$. Let $a = \mathrm{diag}(\mathrm{i}, -\mathrm{i}I_N)$. Then

$$u_a^{\perp} = \{y \in u(k, N+1-k) | [a, y] = 0\} = \left\{ \begin{pmatrix} 0 & \mathrm{i}q^{\dagger}\Lambda_1 \\ \mathrm{i}q & 0 \end{pmatrix} \Big| q \in \mathbb{C}^N \right\}.$$

The second flow in the $u(N+1-k, k)$ hierarchy on $L^{\infty}(\mathbb{R}, u_a^{\perp})$ with $b = a$ is the following N-component NLS equations:

$$\mathrm{i}\mathbf{q}_t + \frac{1}{2}\mathbf{q}_{xx} + \mathbf{q}\mathbf{q}^{\dagger}\Lambda_1\mathbf{q} = 0, \tag{7}$$

where

$$\mathbf{q} = (q_1, q_2, \cdots, q_N)^T,$$

which admits the following Lax pair

$$\begin{aligned} \Phi_x &= (\mathrm{i}\lambda\sigma_3 + \mathrm{i}Q)\Phi, \\ \Phi_t &= \left(\mathrm{i}\lambda^2\sigma_3 + \mathrm{i}\lambda Q - \frac{1}{2}(\mathrm{i}\sigma_3 Q^2 - \sigma_3 Q_x)\right)\Phi, \end{aligned} \tag{8}$$

where

$$Q = \begin{bmatrix} 0 & \mathbf{q}^{\dagger}\Lambda_1 \\ \mathbf{q} & \mathbf{0}_{N\times N} \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & \mathbf{0}_{1\times N} \\ \mathbf{0}_{N\times 1} & -\mathbf{I}_{N\times N} \end{bmatrix}.$$

If all $\epsilon_i = 1$, which corresponds to the focusing case, if all $\epsilon_i = -1$, which corresponds to the defocusing case, or otherwise the mixed case. In this paper, we would like to focus on the DT and multi-dark soliton solution for the N-component NLS system (7).

2.2. The binary DT for the AKNS system

We consider the binary DT for the AKNS system with symmetry reduction. First, we give some lemmas.

Lemma 1. *Suppose Φ_1 and Φ are the special vector solutions for system (8) at $\lambda = \lambda_1$ and λ, respectively, then we can have the following total differential*

$$\mathrm{d}\Omega(\Phi_1, \Phi) = \Phi_1^{\dagger}\Lambda\sigma_3\Phi\mathrm{d}x + [(\lambda + \bar{\lambda}_1)\Phi_1^{\dagger}\Lambda\sigma_3\Phi + \Phi_1^{\dagger}\Lambda Q\Phi]\mathrm{d}t. \tag{9}$$

In addition, we have

$$\Omega(\Phi_1, \Phi) = \frac{\Phi_1^{\dagger}\Lambda\Phi}{\mathrm{i}(\lambda - \bar{\lambda}_1)} + C. \tag{10}$$

If $\lambda_1 \in \mathbb{R}$, we have

$$\Omega(\Phi_1, \Phi_1) = \lim_{\lambda\to\lambda_1} \frac{\Phi_1^{\dagger}\Lambda\Phi}{\mathrm{i}(\lambda - \lambda_1)} + C, \tag{11}$$

where C is a complex constant.

Proof. Taking the complex conjugation to (8) both sides, we have

$$\begin{aligned} \Phi_{1,x}^{\dagger}\Lambda &= -\Phi_1^{\dagger}\Lambda[\mathrm{i}\bar{\lambda}_1\sigma_3 + \mathrm{i}Q], \\ \Phi_{1,t}^{\dagger}\Lambda &= -\Phi_1^{\dagger}\Lambda\left[\mathrm{i}\bar{\lambda}_1^2\sigma_3 + \mathrm{i}\bar{\lambda}_1 Q - \frac{1}{2}\sigma_3(\mathrm{i}Q^2 - Q_x)\right]. \end{aligned} \tag{12}$$

Left multiplying by $\Phi_1^\dagger\Lambda$ into both sides of (8) and right multiplying by Φ into both sides of (12), then we can obtain

$$\left[\frac{\Phi_1^\dagger\Lambda\Phi}{\mathrm{i}(\lambda-\bar{\lambda}_1)}\right]_x=\Phi_1^\dagger\Lambda\sigma_3\Phi,$$
$$\left[\frac{\Phi_1^\dagger\Lambda\Phi}{\mathrm{i}(\lambda-\bar{\lambda}_1)}\right]_t=(\lambda+\bar{\lambda}_1)\Phi_1^\dagger\Lambda\sigma_3\Phi+\Phi_1^\dagger\Lambda Q\Phi.$$

It follows that equations (9)–(11) are verified. □

In what follows, to keep the uniqueness of the constants C, we choose it as zero. Following the idea in the introduction, we can obtain that the one-fold binary DT for the N-component NLS equation (7) is

$$\begin{aligned}\Phi &\to \Phi[1]=\Phi-\frac{\Phi_1\Omega(\Phi_1,\Phi)}{\Omega(\Phi_1,\Phi_1)},\\ Q &\to Q[1]=Q-\mathrm{i}\left[\sigma_3,\frac{\Phi_1\Phi_1^\dagger\Lambda}{\Omega(\Phi_1,\Phi_1)}\right].\end{aligned}\tag{13}$$

In the following, we verify the validity of the above transformation (13).

Theorem 1. *Suppose Φ satisfies the system (8), and Φ_1 is a special solution for the Lax pair (8) at $\lambda=\lambda_1$, and $\Phi_1^\dagger\Lambda\Phi_1=0$ if $\lambda_1\in\mathbb{R}$, then we have*

$$\begin{aligned}\Phi[1]_x &= (\mathrm{i}\lambda\sigma_3+\mathrm{i}Q[1])\Phi[1],\\ \Phi[1]_t &= \left(\mathrm{i}\lambda^2\sigma_3+\mathrm{i}\lambda Q[1]-\frac{1}{2}(\mathrm{i}\sigma_3 Q[1]^2-\sigma_3 Q[1]_x)\right)\Phi[1].\end{aligned}\tag{14}$$

Proof. We first verify the first equation of (14). By lemma 1, we have $\Omega(\Phi_1,\Phi)=\frac{\Phi_1^\dagger\Lambda\Phi}{\mathrm{i}(\lambda-\bar{\lambda}_1)}$. It follows that

$$\begin{aligned}\Phi[1]_x &= (\mathrm{i}\lambda\sigma_3+\mathrm{i}Q)\Phi-\frac{(\mathrm{i}\lambda_1\sigma_3+\mathrm{i}Q)\Phi_1\Omega(\Phi_1,\Phi)}{\Omega(\Phi_1,\Phi_1)}-\frac{\Phi_1\Phi_1^\dagger\Lambda\sigma_3\Phi}{\Omega(\Phi_1,\Phi_1)}+\frac{\Phi_1\Phi_1^\dagger\Lambda\sigma_3\Phi_1\Omega(\Phi_1,\Phi)}{\Omega^2(\Phi_1,\Phi_1)}\\ &= \left[\mathrm{i}\lambda\sigma_3+\mathrm{i}\left(Q-\mathrm{i}\frac{\sigma_3\Phi_1\Phi_1^\dagger\Lambda}{\Omega(\Phi_1,\Phi_1)}+\mathrm{i}\frac{\Phi_1\Phi_1^\dagger\Lambda\sigma_3}{\Omega(\Phi_1,\Phi_1)}\right)\right]\left(\Phi-\frac{\Phi_1\Omega(\Phi_1,\Phi)}{\Omega(\Phi_1,\Phi_1)}\right),\end{aligned}$$

where the second equality uses the relation $\Phi_1^\dagger\Lambda\Phi_1=0$ if $\lambda_1\in\mathbb{R}$. And it is straightforward to verify the validity of the symmetry relation for $Q[1]$. Then the first equation of (14) is verified. Besides, we can obtain the following relation. Since $\Phi[1]=T\Phi$,

$$T=I-\frac{\Phi_1\Phi_1^\dagger\Lambda}{\mathrm{i}(\lambda-\bar{\lambda}_1)\Omega(\Phi_1,\Phi_1)},$$

it follows that

$$T_x+\mathrm{i}T(\lambda\sigma_3+Q)=\mathrm{i}(\lambda\sigma_3+Q[1])T.\tag{15}$$

Expanding the matrix T with $T = I + \frac{T_1}{\lambda} + \frac{T_2}{\lambda^2} + \cdots$, substituting into equation (15), and comparing the coefficient of λ, we can get

$$Q[1]T_1 = T_1 Q + [T_2, \sigma_3] - \mathrm{i}T_{1,x}. \tag{16}$$

Finally, we consider the time evolution equation of (14). Since matrix T is a special gauge transformation, by direct calculation we have

$$\left[T_t + T\left(\mathrm{i}\lambda^2\sigma_3 + \mathrm{i}\lambda Q - \frac{1}{2}\sigma_3(\mathrm{i}Q^2 - Q_x)\right)\right]T^{-1} = \mathrm{i}\lambda^2\sigma_3 + \mathrm{i}\lambda Q[1] + V_1[1], \tag{17}$$

where $V_1[1] = -\frac{1}{2}(\mathrm{i}\sigma_3 Q^2 - \sigma_3 Q_x) + \mathrm{i}[T_2, \sigma_3] + \mathrm{i}(T_1 Q - Q[1]T_1)$. By equation (16), we have $V_1[1]^o = \frac{1}{2}\sigma_3 Q[1]_x$, and then we have $V_1[1]^d = -\frac{1}{2}\sigma_3 Q[1]^2$ by direct calculation. The superscripts ^{o} and ^{d} denote the off-diagonal and diagonal part of the block matrix, respectively. This completes the theorem. □

In the following, we consider the n-fold binary DT based on the above theorem.

Theorem 2. *Suppose we have n different solutions Φ_is for the Lax pair (8) at $\lambda = \lambda_i$ $(i = 1, 2, \cdots, n)$, and $\Phi_j^\dagger \Lambda \Phi_j = 0$ if $\lambda_j \in \mathbb{R}$, then the n-fold binary DT can be presented as*

$$\Phi \to \Phi[n] = \Phi - \Theta M^{-1}\Omega, \quad \Theta = [\Phi_1, \Phi_2, \cdots, \Phi_n], \tag{18}$$

where

$$M = \begin{bmatrix} \Omega(\Phi_1, \Phi_1) & \Omega(\Phi_1, \Phi_2) & \cdots & \Omega(\Phi_1, \Phi_n) \\ \Omega(\Phi_2, \Phi_1) & \Omega(\Phi_2, \Phi_2) & \cdots & \Omega(\Phi_2, \Phi_n) \\ \vdots & \vdots & \ddots & \vdots \\ \Omega(\Phi_n, \Phi_1) & \Omega(\Phi_n, \Phi_2) & \cdots & \Omega(\Phi_n, \Phi_n) \end{bmatrix}, \quad \Omega = \begin{bmatrix} \Omega(\Phi_1, \Phi) \\ \Omega(\Phi_2, \Phi) \\ \vdots \\ \Omega(\Phi_n, \Phi) \end{bmatrix}.$$

The transformation between the fields is

$$Q \to Q[n] = Q - \mathrm{i}[\sigma_3, \Theta M^{-1}\Theta^\dagger \Lambda]. \tag{19}$$

Proof. In the first place, we have

$$\begin{aligned}
\Phi[n]_x &= \Phi_x - \Theta_x M^{-1}\Omega - \Theta M^{-1}\Omega_x + \Theta M^{-1}M_x M^{-1}\Omega \\
&= (\mathrm{i}\lambda\sigma_3 + \mathrm{i}Q)\Phi - \mathrm{i}(Q\Theta + \sigma_3\Theta D)M^{-1}\Omega - \Theta M^{-1}\Theta^\dagger\Lambda\sigma_3\Phi + \Theta M^{-1}\Theta^\dagger\Lambda\sigma_3\Theta M^{-1}\Omega \\
&= (\mathrm{i}\lambda\sigma_3 + \mathrm{i}Q)\Phi - \mathrm{i}Q\Theta M^{-1}\Omega - \Theta M^{-1}\Theta^\dagger\Lambda\sigma_3\Phi + \Theta M^{-1}\Theta^\dagger\Lambda\sigma_3\Theta M^{-1}\Omega \\
&\quad + \mathrm{i}\sigma_3\Theta M^{-1}(-\lambda + D^\dagger)\Omega + \sigma_3\Theta M^{-1}\Theta^\dagger\Lambda\Phi - \mathrm{i}\sigma_3\Theta D^\dagger M^{-1}\Omega \\
&= (\mathrm{i}\lambda\sigma_3 + \mathrm{i}Q)\Phi - \mathrm{i}Q\Theta M^{-1}\Omega - \Theta M^{-1}\Theta^\dagger\Lambda\sigma_3\Phi + \Theta M^{-1}\Theta^\dagger\Lambda\sigma_3\Theta M^{-1}\Omega \\
&\quad - \mathrm{i}\lambda\sigma_3\Theta M^{-1}\Omega + \sigma_3\Theta M^{-1}\Theta^\dagger\Lambda\Phi - \sigma_3\Theta M^{-1}(\mathrm{i}MD - \mathrm{i}D^\dagger M)M^{-1}\Omega \\
&= (\mathrm{i}\lambda\sigma_3 + \mathrm{i}Q)\Phi - \mathrm{i}Q\Theta M^{-1}\Omega - \Theta M^{-1}\Theta^\dagger\Lambda\sigma_3\Phi + \Theta M^{-1}\Theta^\dagger\Lambda\sigma_3\Theta M^{-1}\Omega \\
&\quad - \mathrm{i}\lambda\sigma_3\Theta M^{-1}\Omega + \sigma_3\Theta M^{-1}\Theta^\dagger\Lambda\Phi - \sigma_3\Theta M^{-1}\Theta^\dagger\Lambda\Theta M^{-1}\Omega \\
&= (\mathrm{i}\lambda\sigma_3 + \mathrm{i}Q[n])\Phi[n],
\end{aligned}$$

where the third equality uses the relation $\mathrm{i}(-\lambda + D^\dagger)\Omega + \Theta^\dagger\Lambda\Phi = 0$, the fifth equality uses the relation $\Theta^\dagger\Lambda\Theta = \mathrm{i}(MD - D^\dagger M)$ and $D = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$. Thus, the spectral problem is valid. □

Based on the above theorem, we merely need to verify the spectral problem. The time evolution part is similar to the above theorem. Thus we omit it. □

2.3. The uniform transformation through the binary DT

In this subsection, we consider the reduction from the binary DT. For convenience, we merely consider the one-fold binary DT, since the n-fold binary DT is nothing but a n-times iteration of the one-fold DT. First, we consider how to reduce the binary DT into the Zakharov–Shabat dressing operator [4]. If the spectral parameters $\lambda_1 \neq \bar{\lambda}_1$, we use the relation

$$\begin{aligned} \left(\frac{\Phi_1^\dagger \Lambda \Phi_1}{\lambda_1 - \bar{\lambda}_1}\right)_x &= \mathrm{i}\Phi_1^\dagger \sigma_3 \Lambda \Phi_1, \\ \left(\frac{\Phi_1^\dagger \Lambda \Phi_1}{\lambda_1 - \bar{\lambda}_1}\right)_t &= \mathrm{i}[(\lambda_1 + \bar{\lambda}_1)\Phi_1^\dagger \Lambda \sigma_3 \Phi_1 + \Phi_1^\dagger \sigma_3 \Lambda Q \Phi_1]. \end{aligned} \tag{20}$$

It follows that

$$\Omega(\Phi_1, \Phi_1) = -\mathrm{i}\left(\frac{\Phi_1^\dagger \Lambda \Phi_1}{\lambda_1 - \bar{\lambda}_1}\right).$$

Then we have

$$\Phi[1] = \left(I + \frac{\bar{\lambda}_1 - \lambda_1}{\lambda - \bar{\lambda}_1}\frac{\Phi_1 \Phi_1^\dagger \Lambda}{\Phi_1^\dagger \Lambda \Phi_1}\right)\Phi, \quad Q[1] = Q + (\lambda_1 - \bar{\lambda}_1)\left[\sigma_3, \frac{\Phi_1 \Phi_1^\dagger \Lambda}{\Phi_1^\dagger \Lambda \Phi_1}\right].$$

By this transformation, we can obtain the bright soliton, and the breather and rogue wave solution. The high-order DT of this type was obtained in reference [8] by the limit technique.

If the spectral parameters $\lambda_1 = \bar{\lambda}_1$. In this case, we use the limit technique to deal with this problem. Suppose Ψ_1 and Φ_1 are two mutually dependent solutions for the Lax pair at $\lambda = \lambda_1$ such that $\Phi_1^\dagger \Lambda \Psi_1 \equiv C_1 = \mathrm{const} \neq 0$ and $\Phi_1^\dagger \Lambda \Phi_1 = 0$, set $\Theta_1(\nu) = \Phi_1(\nu) + \frac{\beta(\nu - \lambda_1)}{C_1}\Psi_1(\lambda_1)$, then we can obtain

$$\begin{aligned} \lim_{\nu \to \lambda_1}\left(\frac{\Phi_1^\dagger \Lambda \Theta_1(\nu)}{\nu - \lambda_1}\right)_x &= \mathrm{i}\Phi_1^\dagger \sigma_3 \Lambda \Phi_1, \\ \lim_{\nu \to \lambda_1}\left(\frac{\Phi_1^\dagger \Lambda \Theta_1(\nu)}{\nu - \lambda_1}\right)_t &= \mathrm{i}[2\lambda_1 \Phi_1^\dagger \Lambda \sigma_3 \Phi_1 + \Phi_1^\dagger \sigma_3 \Lambda Q \Phi_1]. \end{aligned} \tag{21}$$

It follows that

$$\Omega(\Phi_1, \Phi_1) = -\mathrm{i} \lim_{\nu \to \lambda_1}\left(\frac{\Phi_1^\dagger \Lambda \Theta_1(\lambda)}{\nu - \lambda_1}\right).$$

Then we have

$$\begin{aligned} \Phi[1] &= \lim_{\nu \to \lambda_1}\left(I + \frac{\lambda_1 - \nu}{\lambda - \lambda_1}\frac{\Phi_1 \Phi_1^\dagger \Lambda}{\Phi_1^\dagger \Lambda \Theta_1(\nu)}\right)\Phi, \\ Q[1] &= Q + \lim_{\nu \to \lambda_1}\left[\sigma_3, \frac{(\nu - \lambda_1)\Phi_1 \Phi_1^\dagger \Lambda}{\Phi_1^\dagger \Lambda \Theta_1(\nu)}\right]. \end{aligned} \tag{22}$$

To keep the non-singularity of the above transformation, we have

$$\beta + \lim_{\nu\to\lambda_1}\left(\frac{\Phi_1^{\dagger}\Lambda\Phi_1(\nu)}{\nu-\lambda_1}\right)\neq 0,\ \text{ for any } (x,t)\in\mathbb{R}^2.$$

In the following section, we would like to use the above transformation to derive the single dark soliton and multi-dark soliton for the *N*-component NLS equation (7).

3. The dark soliton and multi-dark soliton

In this section, we consider the application of the binary DT. A direct application is using the DT to derive some special solutions. By the DT (22), we can obtain the dark soliton and multi-dark soliton for the *N*-component NLS equation (7) in a simple way.

3.1. The single dark soliton for the N-component NLS equations

To obtain the dark soliton, we use the seed solutions

$$q_i = c_i\,\mathrm{e}^{\mathrm{i}\theta_i},\ \ \theta_i = a_i\,x - \left(\frac{{a_i}^2}{2} - \sum_{l=1}^{N}\epsilon_l {c_l}^2\right)t,\ \ i=1,2,\cdots,N. \tag{23}$$

In the first place, we need to solve the Lax pair equation (8) with the above seed solutions. In order to solve the Lax pair equation, we use the gauge transformation

$$D = \mathrm{diag}(1,\ \mathrm{e}^{\mathrm{i}\theta_1},\cdots\ \mathrm{e}^{\mathrm{i}\theta_N})$$

converts the variable coefficient differential equation into a constant coefficient equation. Then we can obtain

$$\begin{aligned}\Phi_{0,x} &= \mathrm{i}U_0\Phi_0, \quad \Phi = D\Phi_0,\\ \Phi_{0,t} &= \mathrm{i}\left(\frac{1}{2}U_0^2 + \lambda U_0 - \sum_{l=1}^{N}\epsilon_l {c_l}^2 - \frac{1}{2}\lambda^2\right)\Phi_0,\end{aligned} \tag{24}$$

where

$$U_0 = \begin{bmatrix}\lambda & C\Lambda_1\\ C^T & -\lambda I_N - A\end{bmatrix},\quad A=\mathrm{diag}(a_1,\cdots,a_N),\quad C=(c_1,\ \cdots,c_N). \tag{25}$$

In the following, we consider the property of the matrix U_0. First we can obtain the characteristic equation of matrix U_0:

$$\det(\mu - U_0) = 0. \tag{26}$$

Then we have the vector solution for (24):

$$\Phi_0 = \begin{bmatrix}(\mathrm{e})^{\mathrm{iX}}\\ (\lambda + a_1 + \mu)^{-1}\mathrm{e}^{\mathrm{i}X}\\ \vdots\\ (\lambda + a_N + \mu)^{-1}\mathrm{e}^{\mathrm{i}X}\end{bmatrix},\ X = \mu\,x + \left(\frac{1}{2}\mu^2 + \lambda\mu - \frac{1}{2}\lambda^2 - \sum_{l=1}^{N}\epsilon_l {c_l}^2\right)t.$$

Since the coefficient of the algebraic equation (26) is a real number, the roots of the equation (26) are either real roots or complex conjugation root pairs. Thus, if the number of real roots is less than the order of the algebraic equation, then there exists a complex conjugation root pair. The real roots of the algebraic equation can be found by the existence of the theorem of zero root. To obtain the dark soliton, we need to choose the pair of conjugate complex roots of the characteristic equation (26). If μ and $\bar{\mu}$ are the roots of the characteristic equation (26), then we have

$$\mu_j - \lambda_j - \sum_{l=1}^{N} \frac{\epsilon_l {c_l}^2}{\lambda_j + \mu_j + a_l} = 0, \tag{27}$$

and

$$\bar{\mu}_i - \lambda_i - \sum_{l=1}^{N} \frac{\epsilon_l {c_l}^2}{\lambda_i + \bar{\mu}_i + a_l} = 0, \tag{28}$$

where μ_j and $\bar{\mu}_i$ are the roots of the characteristic equation (26) with $\lambda = \lambda_j$ and $\lambda = \lambda_i$, respectively $(i, j = 1, 2, \cdots, n)$, the overbar denotes the complex conjugation. It follows that

$$\mu_j - \lambda_j - (\bar{\mu}_i - \lambda_i) + \sum_{l=1}^{N} \frac{\epsilon_l {c_l}^2[(\lambda_j + \mu_j) - (\lambda_i + \bar{\mu}_i)]}{(\lambda_j + \mu_j + a_l)(\lambda_i + \bar{\mu}_i + a_l)} = 0.$$

Then we can obtain that

$$\frac{\mu_j - \lambda_j - (\bar{\mu}_i - \lambda_i)}{[(\lambda_j + \mu_j) - (\lambda_i + \bar{\mu}_i)]} + \sum_{l=1}^{N} \frac{\epsilon_l {c_l}^2}{(\lambda_j + \mu_j + a_l)(\lambda_i + \bar{\mu}_i + a_l)} = 0. \tag{29}$$

Thus, the formula

$$\frac{\Phi_i^\dagger \Lambda \Phi_j}{\lambda_j - \lambda_i} = \frac{2e^{i(X_j - \bar{X}_i)}}{\lambda_j - \lambda_i + \mu_j - \bar{\mu}_i}, \tag{30}$$

where

$$\begin{aligned} \Phi_i &= D\Phi_0|_{\lambda=\lambda_i, \mu=\mu_i},\ \Phi_j = D\Phi_0|_{\lambda=\lambda_j, \mu=\mu_j} \\ X_j &= \left[\mu_j x + \left(\lambda_j \mu_j + \frac{1}{2}{\mu_j}^2 - \frac{1}{2}{\lambda_j}^2\right)t\right], \\ \bar{X}_i &= \left[\bar{\mu}_i x + \left(\lambda_i \bar{\mu}_i + \frac{1}{2}{\bar{\mu}_i}^2 - \frac{1}{2}{\lambda_i}^2\right)t\right]. \end{aligned}$$

With this formula, we can readily take limit $\lambda_j \to \lambda_i$, and it follows that

$$\lim_{\lambda_j \to \lambda_i} \frac{\Phi_i^\dagger \Lambda \Phi_j}{\lambda_j - \lambda_i} = \frac{2e^{i(X_i - \overline{X_i})}}{\mu_i - \overline{\mu_i}}, \tag{31}$$

which also implies that $\Phi_i^\dagger \Lambda \Phi_i = 0$. Then we come back to the DT (22). Through the above explicit expression and set $\beta = \frac{2e^{2\alpha\mu_{1I}}}{\mu_1 - \bar{\mu}_1}$, where $\mu_{1I} = \mathrm{Im}(\mu_1)$ and $\alpha \in \mathbb{R}$, we have

$$\lim_{\nu \to \lambda_1} \frac{\Phi_1^\dagger \Lambda \Theta_1(\nu)}{\nu - \lambda_1} = \frac{2[e^{i(X_1 - \bar{X}_1)} + e^{2\alpha\mu_{1I}}]}{\mu_1 - \bar{\mu}_1}. \tag{32}$$

Then the DT (22) can be constructed explicitly as

$$T = I - \frac{(\mu_1 - \bar{\mu}_1)\Phi_1\Phi_1^{\dagger}\Lambda}{2(\lambda - \lambda_1)(e^{i(X_1 - \bar{X}_1)} + e^{2\alpha\mu_{1I}})}. \tag{33}$$

It follows that the single dark soliton solutions for the N-component NLS equations (7) are

$$q_i[1] = c_i\left[1 - \frac{B_i}{2} + \frac{B_i}{2}\tanh(Y_1)\right]e^{i\theta_i}, \tag{34}$$

where

$$B_i = \frac{\mu_1 - \bar{\mu}_1}{\lambda_1 + a_i + \mu_1}, \quad i = 1, 2, \cdots, N,$$
$$Y_1 = -\mu_{1I}[x + (\lambda_1 + \mu_{1R})t + \alpha], \ \ \mu_{1I} = \mathrm{Im}(\mu_1), \ \ \mu_{1R} = \mathrm{Re}(\mu_1).$$

Without loss of generality, we suppose $\mu_{1I} > 0$. When $x \to -\infty$, we have

$$q_i[1] \to c_i\, e^{i\theta_i}.$$

When $x \to +\infty$, we have

$$q_i[1] \to c_i\, e^{i(\theta_i + \omega_i)}, \quad \ln\left(\frac{\lambda_1 + a_i + \bar{\mu}_1}{\lambda_1 + a_i + \mu_1}\right) = i\omega_i.$$

The centre of the dark soliton $q_i[1]$ is $\cdots$ along the line $x + (\lambda_1 + \mu_{1R})t + \alpha = 0$. The velocity of the dark soliton $|q_i[1]|^2$ is $v = -(\lambda_1 + \mu_{1R})$. The depth of cavity of $|q_i[1]|^2$ is

$$\frac{c_i^2\mu_{1I}^2}{(\lambda_1 + a_i + \mu_{1R})^2 + \mu_{1I}^2}.$$

In the following, we consider how to determine whether or not dark bound states exist [19]. Through the relation (29), we have

$$-\sum_{l=1}^{N}\frac{\epsilon_l {c_l}^2}{|\lambda_1 + \mu_1 + a_l|^2} = 1. \tag{35}$$

Indeed, through the expression for the single dark soliton (34), to obtain the velocity of the dark soliton, we need to know the parameter $\lambda_1 + \mu_1$. And the velocity of the soliton is controlled by $-(\lambda_1 + \mu_{1R})$. Thus, if we need to find the soliton with a velocity equal to zero, we merely have to solve the following equation.

$$-\sum_{l=1}^{N}\frac{\epsilon_l {c_l}^2}{a_l^2 + \mu_{1I}^2} = 1. \tag{36}$$

If $\epsilon_l = -1$ for all l, which corresponds to the defocusing case, then the function $F(\mu_{1I}) = \sum_{l=1}^{N}\frac{{c_l}^2}{a_l^2 + \mu_{1I}^2}$ is an increasing function in the positive half axis. Then the equation (36) merely has a positive solution. Thus, in the defocusing case, there exists no dark bound state. So, the dark bound state merely perhaps exists in the mixed case.

In what follows, we illustrate some exact examples of the single dark soliton. Since the velocity of the soliton possesses the exact physical meaning, we can obtain the soliton by the velocity $v = -(\lambda_1 + \mu_{1R})$. First we solve the following equation about μ_{1I}:

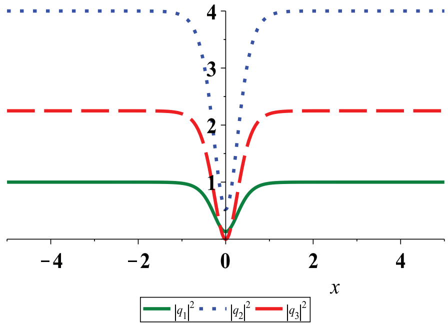

Figure 1. $t = 0$: Solid green line $|q_1|^2$, dotted blue line $|q_2|^2$, dashed red line $|q_3|^2$. The parameters are given in equation (37).

$$\sum_{l=1}^{N} \frac{-\epsilon_l {c_l}^2}{(a_l - v)^2 + \mu_{1I}^2} = 1.$$

Substituting $\mu_1 = -(v + \lambda_1) + \mathrm{i}\mu_{1I}^2$ into the characteristic equation (26), we can obtain an algebraic equation about λ_1. Solving the algebraic equation, we can obtain all of the parameters for the single dark soliton. For instance, we consider the three-component NLS equations with the defocusing case (i.e. $N = 3$, $\epsilon_1 = \epsilon_2 = \epsilon_3 = -1$). If we need to find the soliton with velocity $v = 0$, then we choose the parameters:

$$a_1 = 1,\ a_2 = -1,\ a_3 = 0,\ c_1 = 1,\ c_2 = 2,\ c_3 = \frac{3}{2},\ \ \alpha = 0,$$
$$\lambda_1 = -\frac{12}{33 + \sqrt{769}},\ \mu_1 = \frac{99 - 3\sqrt{769}}{80} + \frac{\mathrm{i}}{4}\sqrt{50 + 2\sqrt{769}}\,. \tag{37}$$

We can plot the picture of the single dark soliton by Maple (figure 1). Since the solitons are stationary, we merely plot the picture at $t = 0$.

3.2. The multi-dark soliton for the N-component NLS equations

In order to give the multi-dark soliton solution, we first adapt the binary DT with the limit technique. The n-fold binary DT (18) can be written with the following form

$$\Phi[n] = \Phi - \sum_{i=1}^{n} s_i \Omega(\Phi_i, \Phi). \tag{38}$$

Thus, we can suppose that

$$\Phi[n] = T_n \Phi, \quad T_n = I - \sum_{i=1}^{n} s_i \frac{\Phi_i^\dagger \Lambda}{\lambda - \lambda_i}. \tag{39}$$

The explicit expression for the Darboux matrix T_n can be determined by the following equations

$$\lim_{\lambda \to \lambda_j} T_n \left(\Phi_j + \frac{\beta_j}{C_j} (\lambda - \lambda_j) \Psi_j \right) = 0,$$

where Ψ_j is the mutually independent solution with Φ_j at $\lambda = \lambda_j$, $\beta_j = \frac{2}{\mu_j - \bar{\mu}_j} e^{2\alpha_j \mu_{jI}}$, $\mu_{jI} = \mathrm{Im}(\mu_j)$, $\alpha_j \in \mathbb{R}$ and $C_j \equiv \Phi_j^\dagger \Lambda \Psi_j = \text{const}$. By linear algebra, we have the following expression for T_n:

$$T_n = I - \Theta M^{-1}(\lambda - D)^{-1}\Theta^\dagger \Lambda, \quad \Theta = [\Phi_1, \Phi_2, \cdots, \Phi_n], \tag{40}$$

where

$$\begin{aligned} M &= \begin{bmatrix} \lim\limits_{\nu\to\lambda_1} \frac{\Phi_1^\dagger \Lambda \Phi_1(\nu)}{\nu - \lambda_1} + \beta_1 & \frac{\Phi_1^\dagger \Lambda \Phi_2}{\lambda_2 - \lambda_1} & \cdots & \frac{\Phi_1^\dagger \Lambda \Phi_n}{\lambda_n - \lambda_1} \\ \frac{\Phi_2^\dagger \Lambda \Phi_1}{\lambda_1 - \lambda_2} & \lim\limits_{\nu\to\lambda_2} \frac{\Phi_2^\dagger \Lambda \Phi_2(\nu)}{\nu - \lambda_2} + \beta_2 & \cdots & \frac{\Phi_2^\dagger \Lambda \Phi_n}{\lambda_n - \lambda_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\Phi_n^\dagger \Lambda \Phi_1}{\lambda_1 - \lambda_n} & \frac{\Phi_n^\dagger \Lambda \Phi_2}{\lambda_2 - \lambda_n} & \cdots & \lim\limits_{\nu\to\lambda_n} \frac{\Phi_n^\dagger \Lambda \Phi_n(\nu)}{\nu - \lambda_n} + \beta_n \end{bmatrix}, \\ D &= \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n). \end{aligned}$$

By the equality (30) and (31) in the above subsection, together with transformation (19), then the n-dark soliton solutions for the equations (7) can be represented as follows:

$$q_i[n] = 2\frac{\begin{vmatrix} M & X^\dagger \\ \Theta_i & \frac{c_i}{2} \end{vmatrix}}{|M|} e^{i\theta_i}, \quad i = 1, 2, \cdots, N, \tag{41}$$

where

$$\begin{aligned} M &= \left(\frac{2[e^{i(X_j - \bar{X}_m)} + \delta_{mj}]}{(\lambda_j + \mu_j) - (\lambda_m + \bar{\mu}_m)} \right)_{1\leqslant m,j\leqslant n}, \\ X &= [e^{iX_1}, e^{iX_2}, \cdots, e^{iX_n}], \\ \Theta_i &= [\beta_{1,i} e^{iX_1}, \beta_{2,i} e^{iX_2}, \cdots, \beta_{N,i} e^{iX_n}], \end{aligned}$$

and

$$\delta_{mj} = \begin{cases} 0, & m \neq j, \\ e^{2\alpha_m \mu_{mI}}, & m = j, \end{cases} \quad \beta_{j,i} = \frac{1}{\lambda_j + a_i + \mu_j}, \quad j = 1, 2, \cdots, N.$$

Furthermore, the above formula (41) can be reduced as the following compact expression

$$q_i[n] = c_i \frac{|M_i|}{|M|} e^{i\theta_i}, \tag{42}$$

where

$$M_i = \left(\frac{2[e^{i(X_j - \bar{X}_m)} + \delta_{mj}]}{(\lambda_j + \mu_j) - (\lambda_m + \bar{\mu}_m)} - \frac{2\beta_{j,i}}{c_i} e^{i(X_j - \bar{X}_m)} \right)_{1\leqslant m,j\leqslant n}.$$

In what follows, we consider some dynamics for the two-dark solitons (42) of the three-component NLS equations. First, we consider the defocusing case $\epsilon_i = -1$, $i = 1, 2, 3$. By using the method in the above subsection (we choose velocity $v_1 = 1$ and $v_2 = -1$), the parameters are chosen as follows:

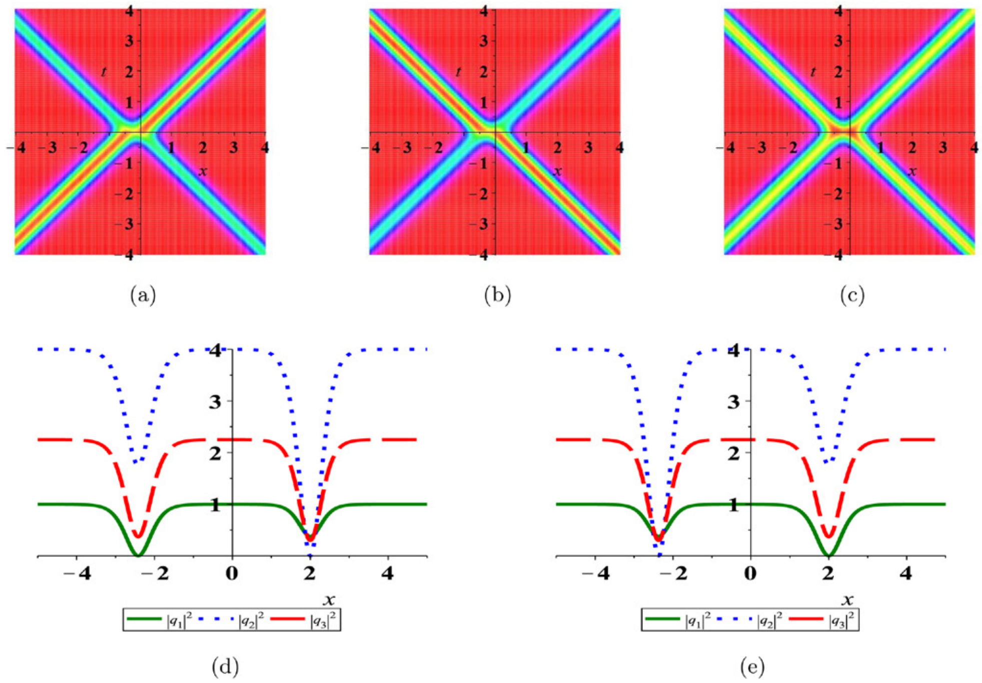

Figure 2. (a)–(c) Density plot of $|q_1|^2$, $|q_2|^2$ and $|q_3|^2$ respectively; (d), (e) solid green line $|q_1|^2$, dotted blue line $|q_2|^2$, dashed red line $|q_3|^2$. (d) $t = -2$, (e) $t = 2$. The parameters are given in equation (43).

$$\begin{aligned}
&a_1 = 1,\ a_2 = -1,\ a_3 = 0,\ c_1 = 1,\ c_2 = 2,\ c_3 = \frac{3}{2},\\
&\lambda_1 \approx -1.121\,588\,903,\ \lambda_2 \approx 0.743\,086\,1497,\ \alpha_1 = \alpha_2 = 0,\\
&\mu_1 \approx 0.121\,588\,9040 + 2.265\,094\,396\mathrm{i},\ \mu_2 \approx 0.256\,913\,8501 + 2.564\,117\,194\mathrm{i}.
\end{aligned} \tag{43}$$

Then we can show the dynamics of the two-dark soliton in figure 2. It is seen that the two-dark soliton in each component collide elastically. The 'phase shift' still emerges after the two-dark soliton interaction, which is similar to the bright soliton interactions in the scalar case.

Second, we consider the mixed focusing and defocusing cases $\epsilon_i = -1$, $i = 1, 2$ and $\epsilon_3 = 1$. In the first place, we consider the two-pole two-dark soliton. Since the characteristic equation (26) for the three-component NLS equations is a quartic equation, there perhaps exist two pairs of conjugate complex roots. These kinds of soliton cannot exist in the scalar or two-component NLS system, since the characteristic equation does not allow the existence of two pairs of conjugation complex roots. For instance, we choose the parameters as follows

$$\begin{aligned}
&a_1 = 1,\ a_2 = -1,\ a_3 = 0,\ c_1 = c_2 = c_3 = 1,\\
&\lambda_1 = \lambda_2 = 0,\ \mu_1 = \frac{\sqrt{2}}{2}(1 + \mathrm{i}),\ \mu_2 = -\frac{\sqrt{2}}{2}(1 + \mathrm{i}),\ \alpha_1 = \alpha_2 = 0.
\end{aligned} \tag{44}$$

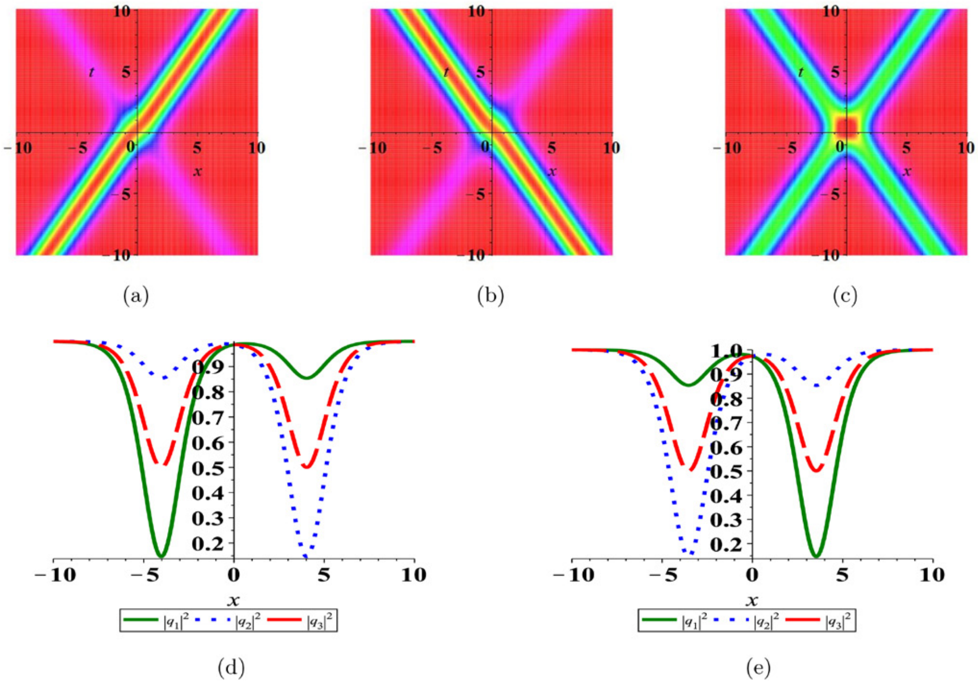

Figure 3. (a)–(c) Density plot of $|q_1|^2$, $|q_2|^2$ and $|q_3|^2$; (d), (e) solid green line $|q_1|^2$, dotted blue line $|q_2|^2$, dashed red line $|q_3|^2$, (d) $t = -5$, (e) $t = 5$. The parameters are given in equation (44).

The figures are given in figure 3. However, it is seen that there is no evidently different dynamics behaviour with the ordinary two-dark soliton solution.

Then we consider the two bound state of the three-component NLS equations, namely, the two solitons have the same velocity. The parameters are chosen by using the method in the above subsection. First, we choose the background parameters a_i, c_i and $v = 0$, we can obtain two different values $\mathrm{Im}(\mu_1), \mathrm{Im}(\mu_2)$. And then substituting them into the characteristic equation (26), we can obtain the two different spectral parameters λ_1, λ_2. Since the parameters α_1, α_2 depend on the initial position of the soliton, we choose different values to distinguish two solitons. For instance, we choose the parameters as follows

$$
\begin{aligned}
&a_1 = 1,\ a_2 = -1,\ a_3 = 0,\ c_1 = 2,\ c_2 = c_3 = 1,\\
&\lambda_1 = \frac{3}{2}\frac{\sqrt{5}-1}{3\sqrt{5}-5},\ \lambda_2 = \frac{3}{2}\frac{\sqrt{5}+1}{3\sqrt{5}-5},\ \alpha_1 = \frac{5}{\sqrt{5}-1},\ \alpha_2 = \frac{-5}{\sqrt{5}+1},\\
&\mu_1 = -\frac{15+3\sqrt{5}}{20} + \frac{\mathrm{i}}{2}(\sqrt{5}-1),\ \mu_2 = -\frac{15-3\sqrt{5}}{20} + \frac{\mathrm{i}}{2}(\sqrt{5}+1),
\end{aligned}
\tag{45}
$$

The dynamical evolution of the corresponding bound state dark solitons are shown in figure 4.

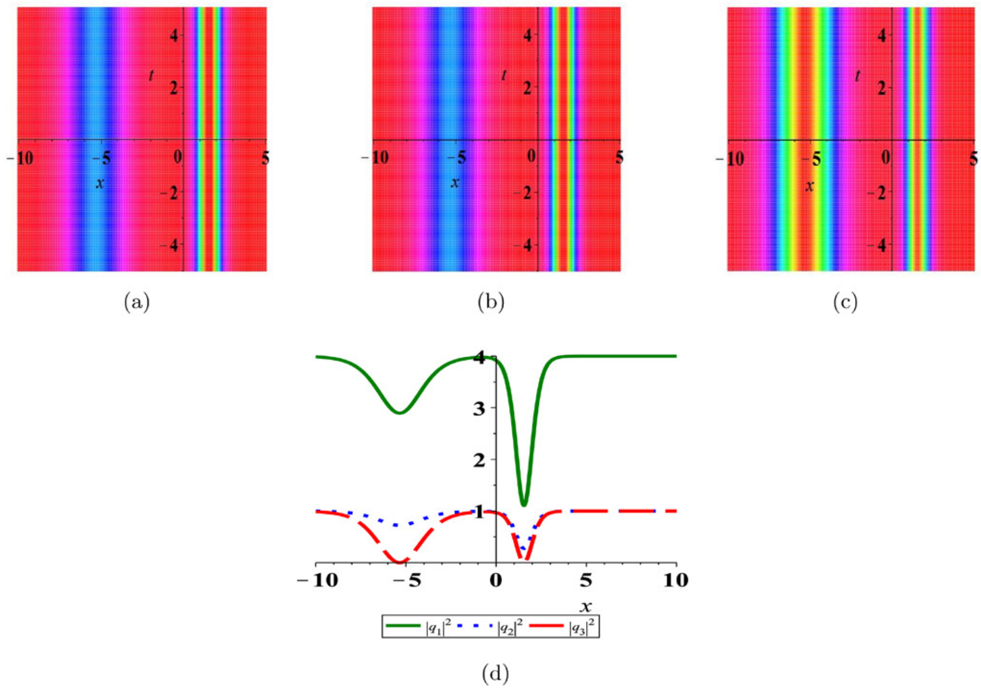

Figure 4. (a)–(c) Density plot of $|q_1|^2$, $|q_2|^2$ and $|q_3|^2$; (d) $t = 0$: solid green line $|q_1|^2$, dotted blue line $|q_2|^2$, dashed red line $|q_3|^2$. The parameters are given in equation (45).

4. Conclusions and discussions

In this paper, we obtain the uniform transformation for the *N*-component NLS equations, which can be used to derive multi-dark soliton solutions and many other types of localised wave solutions conveniently. To our knowledge, the transformation has a two-fold meaning, as follows.

First, the DT is related to the inverse scattering transformation, which is a method of solving the initial value problem of integrable PDE. The inverse scattering method of the coupled NLS equations is an open problem in soliton theory. In 2006, Abolowitz *et al* solved this problem with the special background [26]. The DT method presented here provides a way of solving this problem, at least for the discrete spectrum without restricting the background.

There is another open question in the well-known book by Faddeev and Takhtajan (p 145, end of the second paragraph). The authors deem that the solution to the Riemann problem with zeros cannot be expressed as a product of Blaschke–Potapov factors and a solution of the regular Riemann problem with same continuous spectrum data. Indeed, by the above binary DT, we can construct the $L^2(\mathbb{R})$ eigenfunction and the potential function for the spectral problem:

$$L\Phi = \lambda_1\Phi, \quad L = -\mathrm{i}\sigma_3\partial_x - \sigma_3 Q, \quad Q = \begin{bmatrix} 0 & -\bar{q} \\ q & 0 \end{bmatrix}, \ q \to \mathrm{e}^{\mathrm{i}(-c^2 t \pm \theta_\pm)} \text{ as } x \to \pm\,\infty,$$

here

$$\Phi = \frac{\Phi_1}{2[e^{2Y_1}+1]}, \quad q = c\left(1 - \frac{B_1}{2} + \frac{B_1}{2}\tanh(Y_1)\right)e^{-ic^2t},$$

where $\theta_\pm$ is the asymptotical phase and

$$B_1 = \frac{2i\mu_1}{\lambda_1 + i\mu_1}, \cdots, \; Y_1 = -\mu_1(x + \lambda_1 t + \alpha), \cdots, \; \mu_1 = \sqrt{c^2 - \lambda_1^2}, \; -c < \lambda_1 < c, \; \alpha \in \mathbb{R},$$

$$\Phi_1 = \begin{bmatrix} 1 \\ (\lambda_1 + i\mu_1)^{-1}e^{-ic^2t} \end{bmatrix} e^{X_1}, \; X_1 = -\mu_1(x + \lambda_1 t) + \frac{i}{2}c^2 t.$$

Thus, this transformation can be used to add the discrete spectrum of the above spectral problem. Detailed research on this transformation applied to inverse scattering transformation will be undertaken in the future.

Second, the direct and simple application of this transformation is to derive the dark and multi-dark soliton solutions, which are significant for many different physical systems. Besides, the method in our paper can be generalised to look for some other types of nonlinear localised wave solutions. We would like to explore them in the future as well.

Acknowledgments

This work is supported by the National Natural Science Foundation of China (Contact No. 11271052) and Fundamental Research Funds for the Central Universities (Contact No. 2014ZB0034).

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