High-Order Solutions and Generalized Darboux Transformations of Derivative Nonlinear Schrödinger Equations

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By means of certain limit technique, two kinds of generalized Darboux transformations are constructed for the derivative nonlinear Schrödinger equations (DNLS). These transformations are shown to lead to two solution formulas for DNLS in terms of determinants. As applications, several different types of high-order solutions are calculated for this equation.

1. Introduction

The derivative nonlinear Schrödinger equations (DNLS) [1, 2]

$$\mathrm{i}u_t + u_{xx} + \mathrm{i}(|u|^2u)_x = 0, \tag{1}$$

has many physical applications, especially in space plasma physics and nonlinear optics. It well describes small-amplitude nonlinear Alfvén waves in a low-β plasma, propagating strictly parallel or at a small angle to the ambient magnetic field. It was shown that the DNLS also models large-amplitude magnetohydrodynamic (MHD) waves in a high-β plasma propagating at an arbitrary angle to the ambient magnetic field. In nonlinear optics, the modified nonlinear Schrödinger equations [3], which is gauge equivalent to DNLS, arises in the theory of ultrashort femtosecond nonlinear pulses in optical fibres, when the spectral width of the pulses becomes comparable with the carrier frequency and the effect of self-steepening of the pulse should be taken into account.

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High-order solitons describe the interaction between N solitons of equal amplitude but having a particular chirp [4]. In the terminology of inverse scattering transformation (IST), they correspond to multiple-pole solitons. In the case of the Korteweg-de Vries equation (KdV), the poles must be simple, that is the reason why high-order nonsingular solitons do not exist. Indeed we could obtain multiple-pole solutions by Darboux transformation (DT), such as positon solutions [5]. The high-order solitons for nonlinear Schrödinger equations (NLS) had been studied by many authors [4, 6, 7]. To the best of our knowledge, the high-order solitons of DNLS have never been reported. The aim of this paper is to show that such solutions may be obtained by so-called generalized Darboux transformations (gDT).

Recently, the rogue wave phenomenon [8], which "appears from nowhere and disappears without a trace," has been a subject of extensive study. Those waves, also known as freak, monster, or giant waves, are characterized with large amplitudes and often appear on the sea surface. One of the possible ways to explain the rogue waves is the rogue wave solutions and modulation instability and there is a series of works done by Akhmediev's group [9–11]. Different approaches have been proposed to construct the generalized rogue wave solutions of NLS, for example, the algebro-geometric method is adopted by Dubard et al. [12, 13], Ohta and Yang work in the framework of Hirota bilinear method [14] while the present authors use the gDT as a tool [15]. In the IST terminology, the high-order rogue wave corresponds to multiple-pole solution at the branch points of the spectrum in the non-vanishing background [16]. The first-order rogue wave for DNLS was obtained by Xu and coworkers recently [17]. However, the high-order rogue waves had never been studied. We will tackle this problem by constructing gDT.

The DT [18, 19], which does not need to do the inverse spectral analysis, provides a direct way to solve the Lax pair equations algebraically. However, there is a defect that classical DT cannot be iterated at the same spectral parameter. This defect makes it impossible to construct the high-order rogue wave solutions by DT directly. Thus, we must modify the DT method. In this work, we extend the DT by the limit technique, so that it may be iterated at the same spectral parameter. The modified transformation is referred as gDT.

The inverse scattering method was used to study DNLS with vanishing background (VBC) and non-vanishing background (NVBC) [3, 20–23]. The N-bright soliton formula for DNLS was established by Nakamura and Chen by the Hirota bilinear method [24] and the DT for DNLS was constructed by Imai [25] and Steudel [26] (see also [17]). One key aim of this work is the construction of the gDT and based on it, the high-order soliton solutions and rogue waves are obtained. In addition to above two kinds of solutions, new N-solitons and high-order rational solutions are also found.

The organization of this paper is as follows. In Section 2, we provide a rigorous proof for elementary DT of Kaup-Newell system (KN). Furthermore, based on the elementary DT, we construct for KN the binary DT, which is referred as the DT-II while the elementary DT is referred as DT-I. We also iterate these DT's and work out the N-fold DT's both for DT-I and DT-II, and consider two different kinds of reductions of the DT of the KN to the DNLS. In Section 3, the generalized DT-I and DT-II are constructed in detail by the limit technique. In Section 4, we consider the applications of the gDT and calculate various high-order solutions, which include high-order bright solitons with the VBC, and high-order rogue wave solutions. Final section concludes the paper and offers some discussions.

2. Darboux transformation for DNLS

Let us start with the following system—KN system [27]

$$\begin{aligned} &\mathrm{i}u_t + u_{xx} + \mathrm{i}(u^2 v)_x = 0, \\ &-\mathrm{i}v_t + v_{xx} - \mathrm{i}(uv^2)_x = 0, \end{aligned} \tag{2}$$

which may be written as the compatibility condition

$$U_t - V_x + [U, V] = 0, \tag{3}$$

of the linear system or Lax pair

$$\Phi_x = U\Phi, \tag{4a}$$

$$\Phi_t = V\Phi, \tag{4b}$$

where

$$U = -\frac{\mathrm{i}}{\zeta^2}\sigma_3 + \frac{1}{\zeta}Q, \quad V = -\frac{2\mathrm{i}}{\zeta^4}\sigma_3 + \frac{2}{\zeta^3}Q - \frac{\mathrm{i}}{\zeta^2}Q^2\sigma_3 + \frac{1}{\zeta}Q^3 - \frac{\mathrm{i}}{\zeta}Q_x\sigma_3,$$

with

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & u \\ -v & 0 \end{pmatrix}.$$

These Equations (2) are reduced to the DNLS (1) for $v = u^*$. For convenience, we introduce the following adjoint linear system for (4),

$$-\Psi_x = \Psi U, \tag{5a}$$

$$-\Psi_t = \Psi V. \tag{5b}$$

2.1. DT-I

In the following, we first consider the DT of the unreduced linear system (4). Generally speaking, DT is a special gauge transformation which keeps the form of Lax pair invariant. The explicit steps for constructing DT in $1+1$ dimensional integrable system are as following: first, we consider the gauge transformation

$$D[1] = \zeta D_1 + D_0,$$

where D_1 and D_0 are unknown matrices which do not depend on ζ. Then imposing that $D[1]$ is a DT we have

$$D[1]_x + D[1]U = U[1]D[1], \quad (\det(D[1]))_x = \mathrm{Tr}(U[1]-U)\det(D[1]),$$

where $U[1]$ represents the transformed U matrix. After some analysis, we find the following elementary DT (eDT) for (4):

$$D[1] = \sigma_1 (\zeta + \zeta_1 - 2\zeta_1 P_1), \quad P_1 = \frac{\Phi_1 \Phi_1^T \sigma_1}{\Phi_1^T \sigma_1 \Phi_1}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{6}$$

where $\Phi_1 = (\varphi_1, \phi_1)^T$ is a special solution for linear system (4) at $\zeta = \zeta_1$, and $\Phi_1^T \sigma_1$ is a special solution for adjoint linear system (5) at $\zeta = -\zeta_1$. Here we point out that this DT for DNLS was first derived by Imai [25]. With the help of the DT, Steudel [26] and Xu et al. [17] calculated various solutions for DNLS. Next, we give a rigorous proof that the above transformation does qualify as a DT.

THEOREM 1. *With Φ_1 and $D[1]$ defined above, $\Phi[1] = D[1]\Phi$ solves*

$$\Phi[1]_x = U[1]\Phi[1], \quad \Phi[1]_t = V[1]\Phi[1]$$

where

$$U[1] = -\frac{\mathrm{i}}{\zeta^2}\sigma_3 + \frac{1}{\zeta} Q[1], \quad Q[1] = \sigma_1 Q \sigma_1 - 2\zeta_1 \sigma_1 P_{1,x} \sigma_1,$$

and

$$V[1] = -\frac{2\mathrm{i}}{\zeta^4}\sigma_3 + \frac{2}{\zeta^3} Q[1] - \frac{\mathrm{i}}{\zeta^2} Q[1]^2 \sigma_3 + \frac{1}{\zeta} Q[1]^3 - \frac{\mathrm{i}}{\zeta} Q[1]_x \sigma_3.$$

Namely, $D[1]$ qualifies as a Darboux matrix. Correspondingly, $D[1]^{-1}$ is a Darboux matrix for the adjoint Lax system.

Proof: To begin with, we notice

$$D[1]^{-1} = \frac{1}{\zeta + \zeta_1} \left(I + \frac{2\zeta_1}{\zeta - \zeta_1} P_1 \right) \sigma_1.$$

What we need to do is to verify

$$U[1] = D[1]_{,x} D[1]^{-1} + D[1] U D[1]^{-1}, \tag{7}$$

$$V[1] = D[1]_{,t} D[1]^{-1} + D[1] V D[1]^{-1}. \tag{8}$$

First we consider (7). The residue for function $F_1(\zeta) \equiv D[1]_{,x} D[1]^{-1} + D[1]UD[1]^{-1} - U[1]$ at $\zeta = \zeta_1$ is

$$\begin{aligned}\mathrm{Res}_{\zeta_1}(F_1(\zeta) &= 2\zeta_1\sigma_1[-(I-P_1)P_{1,x} + (I-P_1)U(\zeta_1)P_1]\sigma_1 \\ &= -2\zeta_1\sigma_1(I-P_1)\Phi_1\left[\frac{\sigma_1\Phi_1^T}{\sigma_1\Phi_1^T\Phi_1}\right]_x \sigma_1 = 0,\end{aligned}$$

where we used the relation $D[1]_{,x}D[1]^{-1} = -D[1](D[1]^{-1})_x$. Similarly, the residue of function $F_1(\zeta)$ at $\zeta = -\zeta_1$ is

$$\begin{aligned}\mathrm{Res}_{-\zeta_1}(F_1(\zeta)) &= -2\zeta_1\sigma_1[-P_{1,x}(I-P_1) + P_1U(-\zeta_1)(I-P_1)]\sigma_1 \\ &= 2\zeta_1\sigma_1\left[\frac{\Phi_1}{\sigma_1\Phi_1^T\Phi_1}\right]_x \sigma_1\Phi_1^T(I-P_1)\sigma_1 = 0.\end{aligned}$$

Due to

$$U[1] = -\frac{1}{\zeta^2}\sigma_3 + \frac{1}{\zeta}Q[1],$$

and

$$Q[1] = \sigma_1 Q\sigma_1 - 2\zeta_1\sigma_1 P_{1,x}\sigma_1, \tag{9}$$

the function $F_1(\zeta)D[1]$ is equal to zero at $\zeta = 0$. Thus the function $F_1(\zeta)$ is analytic at $\zeta = 0$. It is easy to see that $F_1(\zeta) \to 0$ at $\zeta \to \infty$. Therefore the equality (7) is valid.

Now we turn to the time evolution part (8). We introduce a matrix $\widehat{V[1]} = -\frac{2\mathrm{i}}{\zeta^4}\sigma_3 + \frac{2}{\zeta^3}Q[1] + \frac{\mathrm{i}}{\zeta^2}V_2 + \frac{1}{\zeta}V_1$, so that $F_2(\zeta) \equiv D[1]_{,t}D[1]^{-1} + D[1]VD[1]^{-1} - \widehat{V[1]}$. Proceeding similarly as above, it is found that $F_2(\zeta)$ is analytic at $\zeta = \pm\zeta_1, 0$ and tends to zero as $\zeta \to \infty$, thus $F_2(\zeta) \equiv 0$.

In the following, we show $\widehat{V[1]} = V[1]$. Because of the compatibility condition $(D[1]\Phi)_{xt} = (D[1]\Phi)_{tx}$, we have

$$U[1]_t - \widehat{V[1]}_x + [U[1], \widehat{V[1]}] = 0. \tag{10}$$

Identifying terms of $\mathrm{O}(\zeta)$ in (10), we have

$$[\sigma_3, V_2] = 0, \tag{11}$$

$$[\sigma_3, V_1] = [V_2, Q[1]] + 2Q[1]_x, \tag{12}$$

$$[Q[1], V_1] = V_{2,x}. \tag{13}$$

From (11), we have $V_2^{\mathrm{off}} = 0$, where V_2^{off} denotes the off-diagonal part of V_2. Similarly, we have

$$V_1^{\mathrm{off}} = \mathrm{i}\sigma_3 Q[1]_x - \frac{\mathrm{i}}{2}\sigma_3[Q[1], V_2] \tag{14}$$

through (12). Substituting (14) into (13) and solving it yields

$$V_2 = -\mathrm{i}Q[1]^2\sigma_3 + f(t).$$

Letting $\zeta_1 = 0$, one can readily obtain $V_2 = -\mathrm{i}\sigma_1 Q^2\sigma_1\sigma_3$. Therefore we have $f(t) = 0$. Moreover, $V_1 = Q[1]^3 - \mathrm{i}Q[1]_x\sigma_3$.

Similar argument could show that $D[1]^{-1}$ does qualify as a Darboux matrix for the adjoint Lax system. Thus the proof is completed. ■

Remark 1. In addition to (9) there is a different representation for $Q[1]$

$$Q[1] = \sigma_1(I - 2P_1)Q(I - 2P_1)\sigma_1 + \frac{2\mathrm{i}}{\zeta_1}\sigma_3\sigma_1(I - 2P_1)\sigma_1, \tag{15}$$

which appeared in papers [17, 26]. We will use (9) rather than (15), since the former is more compact.

To derive the N-fold DT for this elementary DT (6), which is referred as DT-I, we rewrite it as

$$D[1] = \begin{pmatrix} -\zeta_1\frac{\cancel{\varphi_1}}{\cancel{\phi_1}} & \zeta \\ \zeta & -\zeta_1\frac{\cancel{\phi_1}}{\cancel{\varphi_1}} \end{pmatrix}.$$

Assuming N different solutions $\Phi_i = (\varphi_i, \phi_i)^T$ of (4) at $\zeta = \zeta_i (i = 1, 2, \ldots, N)$ are given, we may have

PROPOSITION 1. [17, 25, 26] *The N-fold DT for DT-I can be represented as*

$$D_N = D[N]D[N-1]\cdots D[1] = \zeta^N\sigma_1^N + \sum_{k=0}^{N-1}\begin{pmatrix} \alpha_k & 0 \\ 0 & \beta_k \end{pmatrix}\sigma_1^k\zeta^k, \tag{16}$$

where α_i are determined by the following equations

$$\begin{cases} \alpha_0\varphi_i + \alpha_1\zeta_i\phi_i + \cdots + \alpha_{2l-1}\zeta_i^{2l-1}\phi_i + \alpha_{2l}\zeta_i^{2l}\varphi_i = -\zeta_i^{2l+1}\phi_i, \\ \qquad \text{when } N = 2l+1; \\ \alpha_0\varphi_i + \alpha_1\zeta_i\phi_i + \cdots + \alpha_{2l-2}\zeta_i^{2l-2}\varphi_i + \alpha_{2l-1}\zeta_i^{2l-1}\phi_i = -\zeta_i^{2l}\varphi_i, \\ \qquad \text{when } N = 2l, \end{cases}$$

$i = 1, 2, \ldots, N$. And $\beta_i = \alpha_i(\varphi_j \leftrightarrow \phi_j), (i, j = 1, 2, \ldots, N)$.

The transformation between the fields is the following:

(*i*) *When* $N = 2l + 1$

$$u[N] = -v - \left[\frac{\det(B)}{\det(A)}\right]_x, \quad v[N] = -u + \left[\frac{\det(B(\varphi_j \leftrightarrow \phi_j))}{\det(A(\varphi_j \leftrightarrow \phi_j))}\right]_x, \tag{17}$$

where $A=\left(A_1^T,A_2^T,\ldots,A_N^T\right)$, $B=\left(B_1^T,B_2^T,\ldots,B_N^T\right)$,

$$A_i=\left(\varphi_i,\zeta_i\phi_i,\ldots,\zeta_i^{2l-2}\varphi_i,\zeta_i^{2l-1}\phi_i,\zeta_i^{2l}\varphi_i\right),$$

$$B_i=\left(\varphi_i,\zeta_i\phi_i,\ldots,\zeta_i^{2l-2}\varphi_i,\zeta_i^{2l-1}\phi_i,\zeta_i^{2l+1}\phi_i\right).$$

(ii) *When* $N=2l$

$$u[N]=u-\left[\frac{\det(D)}{\det(C)}\right]_x,\quad v[N]=v+\left[\frac{\det(D(\varphi_j\leftrightarrow\phi_j))}{\det(C(\varphi_j\leftrightarrow\phi_j))}\right]_x,\quad(18)$$

where $C=\left(C_1^T,C_2^T,\ldots,C_N^T\right)$, $D=\left(D_1^T,D_2^T,\ldots,D_N^T\right)$,

$$C_i=\left(\varphi_i,\zeta_i\phi_i,\ldots,\zeta_i^{2l-3}\phi_i,\zeta_i^{2l-2}\varphi_i,\zeta_i^{2l-1}\phi_i\right),$$

$$D_i=\left(\varphi_i,\zeta_i\phi_i,\ldots,\zeta_i^{2l-3}\phi_i,\zeta_i^{2l-2}\varphi_i,\zeta_i^{2l}\varphi_i\right).$$

2.2. DT-II

In this section, we will show that the so-called dressing-Bäcklund transformation [28, 29], denoted by DT-II in this paper, may be constructed from above DT-I. For convenience, we rewrite $D[1]$ and $D[1]^{-1}$ as following

$$D[1]=\begin{pmatrix}-\zeta_1\frac{\varphi_1}{\phi_1} & \zeta\\ \zeta & -\zeta_1\frac{\phi_1}{\varphi_1}\end{pmatrix},\quad D[1]^{-1}=\frac{1}{\zeta^2-\zeta_1^2}\begin{pmatrix}\zeta_1\frac{\phi_1}{\varphi_1} & \zeta\\ \zeta & \zeta_1\frac{\varphi_1}{\phi_1}\end{pmatrix}.\quad(19)$$

Suppose another solution $\Psi_1=(\chi_1,\psi_1)$ for the adjoint system (5) at $\zeta=\xi_1$ is given, then $\Psi_1[1]=\Psi_1D[1]^{-1}|_{\zeta=\xi_1}$ is a new solution for adjoint system $(\Psi[1],U[1],V[1])$ at $\zeta=\xi_1$. It is easy to see that $\sigma_1\Psi_1[1]^T$ is a special solution for Lax pair $(\Phi[1],U[1],V[1])$ at $\zeta=-\xi_1$. Therefore, we could construct the second step DT $D[2]$ by the seed solution $\sigma_1\Psi_1[1]^T$. By direct calculations, removing the overall factor $\zeta^2-\xi_1^2$, we have the DT-II

$$T[1]=I+\frac{A}{\zeta-\xi_1}-\frac{\sigma_3A\sigma_3}{\zeta+\xi_1},\quad A=\frac{\xi_1^2-\zeta_1^2}{2}\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}\Phi_1\Psi_1,\quad(20)$$

where

$$\alpha^{-1}=\Psi_1\begin{pmatrix}\xi_1 & 0\\ 0 & \zeta_1\end{pmatrix}\Phi_1,\quad \beta^{-1}=\Psi_1\begin{pmatrix}\zeta_1 & 0\\ 0 & \xi_1\end{pmatrix}\Phi_1.$$

Furthermore, we have

$$T[1]^{-1}=I+\frac{B}{\zeta-\zeta_1}-\frac{\sigma_3B\sigma_3}{\zeta+\zeta_1},\quad B=\frac{\zeta_1^2-\xi_1^2}{2}\Phi_1\Psi_1\begin{pmatrix}\beta & 0\\ 0 & \alpha\end{pmatrix}.\quad(21)$$

The transformation between Q and new potential function $\widehat{Q}$ is

$$\widehat{Q} = Q + (A - \sigma_3 A \sigma_3)_x \,. \tag{22}$$

Above discussion indicates that $T[1]$, namely the DT-II, is indeed a two-fold DT for $D[1]$ in the case of DNLS. We remark that for the two component DNLS the analogy of DT-II exists [30] while the corresponding DT-I has not been constructed.

In what follows, we consider the iteration of the DT-II. Assume that we have N distinct solutions $\Phi_i(\mu_i) = (\varphi_i, \phi_i)^T$ of (4) at $\zeta = \mu_i$ and N distinct solutions $\Psi_i(\nu_i) = (\chi_i, \psi_i)$ of (5) at $\zeta = \nu_i$. Similar to DT-I, we work with DT-II and have the following proposition

Proposition 2. *The N-fold DT for the DT-II could be written as the following form*

$$T_N = T[N]T[N-1]\cdots T[1] = I + \sum_{i=1}^{N} \left(\frac{C_i}{\zeta - \nu_i} - \frac{\sigma_3 C_i \sigma_3}{\zeta + \nu_i} \right), \tag{23}$$

and

$$T_N^{-1} = T[1]^{-1} T[2]^{-1} \cdots T[N]^{-1} = I + \sum_{i=1}^{N} \left(\frac{D_i}{\zeta - \mu_i} - \frac{\sigma_3 D_i \sigma_3}{\zeta + \mu_i} \right). \tag{24}$$

Proof: We calculate the residues for both sides of (23)

$$\begin{aligned} &\mathrm{Res}|_{\zeta=\nu_i}(T_N) \\ &= \left(I + \frac{A_N}{\nu_i - \nu_N} - \frac{\sigma_3 A_N \sigma_3}{\nu_i + \nu_N} \right) \cdots A_i \cdots \left(I + \frac{A_1}{\nu_i - \nu_1} - \frac{\sigma_3 A_1 \sigma_3}{\nu_i + \nu_1} \right), \\ &\mathrm{Res}|_{\zeta=-\nu_i}(T_N) \\ &= -\left(I + \frac{A_N}{-\nu_i - \nu_N} - \frac{\sigma_3 A_N \sigma_3}{-\nu_i + \nu_N} \right) \cdots \sigma_3 A_i \sigma_3 \cdots \left(I + \frac{A_1}{-\nu_i - \nu_1} - \frac{\sigma_3 A_1 \sigma_3}{-\nu_i + \nu_1} \right). \end{aligned}$$

Because of $\mathrm{Res}|_{\zeta=\nu_i}(T_N) = -\sigma_3 \mathrm{Res}|_{\zeta=-\nu_i}(T_N)\sigma_3$, Equation (23) is valid. Similarly, (24) can be proved. ■

The N-fold DT-II T_N allows us to find the transformations between the fields u, v and $u[N]$, $v[N]$, which are given below

Theorem 2. *The N-fold DT-II T_N induces the following transformations for the fields*

$$u[N] = u - 2\left(\frac{\det M_1}{\det M} \right)_x, \quad v[N] = v + 2\left(\frac{\det N_1}{\det N} \right)_x, \tag{25}$$

where $M = (M_{ij})_{N\times N}$, $N = (N_{ij})_{N\times N}$, $M_{ij} = \frac{\Psi_i\sigma_3\Phi_j}{\mu_j+\nu_i} - \frac{\Psi_i\Phi_j}{\mu_j-\nu_i}$, $N_{ij} = -[\frac{\Psi_i\sigma_3\Phi_j}{\mu_j+\nu_i} + \frac{\Psi_i\Phi_j}{\mu_j-\nu_i}]$,

$$M_1 = \begin{pmatrix} M_{11} & M_{12} & \cdots & M_{1N} & \psi_1 \\ M_{21} & M_{22} & \cdots & M_{2N} & \psi_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ M_{N1} & M_{N2} & \cdots & M_{NN} & \psi_N \\ \phi_1 & \phi_2 & \cdots & \phi_N & 0 \end{pmatrix}, \quad N_1 = \begin{pmatrix} N_{11} & N_{12} & \cdots & N_{1N} & \chi_1 \\ N_{21} & N_{22} & \cdots & N_{2N} & \chi_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ N_{N1} & N_{N2} & \cdots & N_{NN} & \chi_N \\ \varphi_1 & \varphi_2 & \cdots & \varphi_N & 0 \end{pmatrix}.$$

Proof: Since T_N (23) is the N-fold DT of (4), we have

$$T_{N,x} + T_N U = U[N]T_N.$$

It follows that

$$Q[N] = Q + \sum_{i=1}^{N}[C_i - \sigma_3 C_i \sigma_3]_x.$$

Thus, we need to calculate the explicit forms for C_i. Proposition 2 gives $C_i = \mathrm{Res}|_{\zeta=\mu_i}(T_N)$, and implies that C_i's are the matrices of rank one. Thus we may assume $C_i = |x_i\rangle\langle y_i|$. Similarly we may set $D_i = |w_i\rangle\langle v_i|$.

On the one hand, because of $T_N T_N^{-1} = I$, we have

$$\langle y_l \big| T_N^{-1}\big|_{\zeta=\nu_l} = 0, \tag{26}$$

where the fact that the residue of $T_N T_N^{-1}$ at $\zeta = \nu_l$ equals to zero is taken account of. On the other hand, we have

$$\Psi_l T_N^{-1}\big|_{\zeta=\nu_l} = 0.$$

Noticing that the rank of $T_N^{-1}|_{\zeta=\nu_i}$ is 1, we may obtain

$$\langle y_l| = \Psi_l.$$

Now substituting $\langle y_l|$ into (26) leads to

$$\Phi_l + \sum_{i=1}^{N}\left(\frac{|x_i\rangle\Psi_i\Phi_l}{\mu_l - \nu_i} - \frac{\sigma_3|x_i\rangle\Psi_i\sigma_3\Phi_l}{\mu_l + \nu_i}\right) = 0, \quad (l = 1, 2, \ldots, N). \tag{27}$$

Solving (27) gives us

$$[|x_1\rangle, |x_2\rangle, \ldots, |x_N\rangle]_1 = [\varphi_1, \varphi_2, \ldots, \varphi_N]\, M^{-1},$$
$$[|x_1\rangle, |x_2\rangle, \ldots, |x_N\rangle]_2 = [\phi_1, \phi_2, \ldots, \phi_N]\, N^{-1},$$

where subscripts $_1$ and $_2$ stand the first and second rows, respectively. Finally, the relations between the fields can be represented as

$$u[N]=u+2\left[[\varphi_1,\varphi_2,\ldots,\varphi_N]\,M^{-1}\begin{bmatrix}\psi_1\\ \psi_2\\ \vdots\\ \psi_N\end{bmatrix}\right]_x=u-2\left(\frac{\det M_1}{\det M}\right)_x,$$

$$v[N]=v-2\left[[\phi_1,\phi_2,\ldots,\phi_N]\,N^{-1}\begin{bmatrix}\chi_1\\ \chi_2\\ \vdots\\ \chi_N\end{bmatrix}\right]_x=v+2\left(\frac{\det N_1}{\det N}\right)_x.$$

This completes the proof. ■

2.3. Reductions

So far we have been working with the DTs for the general Lax problem (4) and certain solution formulae have been given for the system (2). However our main task is to construct solutions for DNLS (1), therefore we have to study reduction problem. It is easy to see that two reductions $v=u^*$ and $v=-u^*$ are simply related [26], so we may consider either of them. For the DT-I, let us assume $v=u^*$ or $Q^\dagger=-Q$, where † denotes the complex conjugation and matrix transpose. To implement the reduction, we need to choose the seed solutions properly. Indeed, assuming

$$\zeta_1\in \mathrm{i}\mathbb{R},\quad \text{and}\quad \varphi_1=\phi_1^*, \tag{28}$$

then $Q[1]$, defined by (9), satisfies the reduction relation $Q[1]^\dagger=-Q[1]$. The DT (6) with the reduction condition (28) may be employed to construct bright or dark solitons of DNLS with the nonvanishing background.

Let us now turn to the reduction of the DT-II. Assuming $Q=-Q^\dagger$ and

$$\xi_1=\zeta_1^*,\quad \text{and}\quad (\chi_1,\psi_1)=(\varphi_1^*,\phi_1^*), \tag{29}$$

(22) yields $\widehat{Q}=-\widehat{Q}^\dagger$.

To iterate the reduced DT-I and DT-II, we must verify that they keep the reduction conditions (28) and (29). The latter merely depends on the symmetry of Equation (4), thus it holds automatically. For the former (28) we claim that

Proposition 3. *Both DT $D[1]$ and $T[1]$ keep the reduction condition* (28) *invariant.*

Proof: Direct calculations. ■

Due to above analysis, both DT-I and DT-II may be reduced to find solutions for DNLS. However, the DT-I under (28) is conveniently used to construct the N-dark or bright soliton solutions of DNLS with NVBC, while DT-II with (29) may be properly adopted to represent the N-bright solitons and N-breathers of DNLS.

3. Generalized Darboux transformations

In this section, we construct the corresponding gDT's associated with $D[1]$ and $T[1]$. We will follow the approach proposed for the nonlinear Schrödinger equations in [15]. Indeed, while both DT-I and DT-II considered above are degenerate at $\zeta = \zeta_1$ in the sense that $D[1]|_{\zeta=\zeta_1}\Phi_1 = T[1]|_{\zeta=\zeta_1}\Phi_1 = 0$, we may work with

$$\Phi_1^{[1]} = \lim_{\epsilon\to 0}\frac{(D[1]\Phi_1)|_{\zeta=\zeta_1+\epsilon}}{\epsilon},$$

or

$$\Phi_1^{[1]} = \lim_{\epsilon\to 0}\frac{(T[1]\Phi_1)|_{\zeta=\zeta_1+\epsilon}}{\epsilon},$$

which serves the seed solution for doing the next step transformation.

3.1. gDT-I

To construct the gDT associated with DT-I, we assume that n solutions $(\varphi_i, \phi_i)^T$ are given for the Lax pair at $\zeta = \zeta_i$ $(i = 1, \ldots, n)$. First, we have the elementary DT

$$D_1^{[0]} = \begin{pmatrix} -\zeta_1\frac{\varphi_1}{\phi_1} & \zeta \\ \zeta & -\zeta_1\frac{\phi_1}{\varphi_1}\end{pmatrix}.$$

As observed earlier, by virtue of the limit process, we find that

$$\begin{aligned}\begin{pmatrix}\varphi_1^{[1]}\\ \phi_1^{[1]}\end{pmatrix} &= \lim\nolimits_{\epsilon\to 0}\frac{D_1^{[0]}\big|_{\zeta=\zeta_1} + \epsilon\sigma_1}{\epsilon}\begin{pmatrix}\varphi_1(\zeta_1+\epsilon)\\ \phi_1(\zeta_1+\epsilon)\end{pmatrix}\\ &= D_1^{[0]}\big|_{\zeta=\zeta_1}\frac{\mathrm{d}}{\mathrm{d}\zeta}\begin{pmatrix}\varphi_1(\zeta)\\ \phi_1(\zeta)\end{pmatrix}_{\zeta=\zeta_1} + \sigma_1\begin{pmatrix}\varphi_1(\zeta_1)\\ \phi_1(\zeta_1)\end{pmatrix}\end{aligned}$$

is a nontrivial solution for Lax pair (4) with $u = u[1]$ and $v = v[1]$ at $\zeta = \zeta_1$, which may lead to the next step transformation

$$D_1^{[1]} = \begin{pmatrix} -\zeta_1\frac{\varphi_1^{[1]}}{\phi_1^{[1]}} & \zeta \\ \zeta & -\zeta_1\frac{\phi_1^{[1]}}{\varphi_1^{[1]}}\end{pmatrix}.$$

This process may be continued and we have the following theorem

THEOREM 3. *Let $(\varphi_i, \phi_i)^T$ be the solutions of Lax pair at $\zeta = \zeta_i$ $(i = 1, \ldots, n)$ and assume that DT-I possesses m_i order zeros at $\zeta = \zeta_i$. Then we have the following gDT-I:*

$$D_N = D_n^{[m_n-1]} \cdots D_n^{[1]} D_n^{[0]} \cdots D_1^{[m_1-1]} \cdots D_1^{[1]} D_1^{[0]}, \tag{30}$$

where

$$N = \sum_{i=1}^{n} m_i,$$

and

$$D_i^{[j]} = \begin{pmatrix} -\zeta_i \frac{\varphi_i^{[j-1]}}{\phi_i^{[j-1]}} & \zeta \\ \zeta & -\zeta_i \frac{\phi_i^{[j-1]}}{\varphi_i^{[j-1]}} \end{pmatrix},$$

$$\begin{pmatrix} \varphi_i^{[j-1]} \\ \phi_i^{[j-1]} \end{pmatrix} = \sum_{l=1}^{j-1} \frac{\Omega_{j-1-l}}{l!} \frac{\mathrm{d}}{d\zeta^l} \begin{pmatrix} \varphi_i \\ \phi_i \end{pmatrix} \Big|_{\zeta=\zeta_i}, \quad \begin{pmatrix} \varphi_i^{[0]} \\ \phi_i^{[0]} \end{pmatrix} = \begin{pmatrix} \varphi_i \\ \phi_i \end{pmatrix}, \tag{31}$$

and

$$\Omega_l = \sum_{\sum \delta_i^k = l} M_i^{[j-2]} \cdots M_i^{[0]} \cdots M_1^{[m_1-1]} \cdots M_1^{[0]}, \quad M_i^{[k]} = \begin{cases} \sigma_1, & \textit{if}\ \delta_i^k = 1; \\ D_i^{[k]}\big|_{\zeta=\zeta_i}, & \textit{if}\ \delta_i^k = 0. \end{cases}$$

Proof: To construct the gDT-I, we start with the elementary DT

$$D_1^{[0]} = \begin{pmatrix} -\zeta_1 \frac{\varphi_1}{\phi_1} & \zeta \\ \zeta & -\zeta_1 \frac{\phi_1}{\varphi_1} \end{pmatrix}.$$

By means of the nontrivial solutions $(\varphi_1[1], \phi_1[1])$, we may do the next step of transformation $D_1^{[1]}$. Taking account of the given seeds $(\varphi_i, \phi_i)^T$, we perform the following limit

$$\begin{pmatrix} \varphi_i^{[j]} \\ \phi_i^{[j]} \end{pmatrix} = \lim_{\epsilon\to 0} \frac{\left[D_i^{[j-1]} \cdots D_i^{[1]} D_i^{[0]} \cdots D_1^{[m_1-1]} \cdots D_1^{[1]} D_1^{[0]} \right] \Big|_{\zeta=\zeta_i+\epsilon}}{\epsilon^j} \begin{pmatrix} \varphi_i(\zeta_i+\epsilon) \\ \phi_i(\zeta_i+\epsilon) \end{pmatrix},$$

which yields the formulae presented in above theorem. This completes the proof. ■

To have a compact determinant representation for the gDT-I, we may take the limit directly on the N-fold DT-I (16). It follows from (17) and (18) that the transformations between the fields are:

(i) When $N = 2l + 1$

$$u[N] = -v - \left[\frac{\det(B)}{\det(A)}\right]_x, \quad v[N] = -u + \left[\frac{\det(B(\varphi_j \leftrightarrow \phi_j))}{\det(A(\varphi_j \leftrightarrow \phi_j))}\right]_x, \tag{32}$$

where

$$A=\left(A_1^T, \frac{\mathrm{d}}{\mathrm{d}\zeta}A_1^T, \ldots, \frac{d^{m_1-1}}{(m_1-1)!\mathrm{d}\zeta^{m_1-1}}A_1^T, \ldots, A_n^T, \right.$$
$$\left. \frac{\mathrm{d}}{\mathrm{d}\zeta}A_n^T, \ldots, \frac{d^{m_n-1}}{(m_n-1)!\mathrm{d}\zeta^{m_n-1}}A_n^T\right),$$

$$B=\left(B_1^T, \frac{\mathrm{d}}{\mathrm{d}\zeta}B_1^T, \ldots, \frac{d^{m_1-1}}{(m_1-1)!\mathrm{d}\zeta^{m_1-1}}B_1^T, \ldots, B_n^T, \right.$$
$$\left. \frac{\mathrm{d}}{\mathrm{d}\zeta}B_n^T, \ldots, \frac{d^{m_n-1}}{(m_n-1)!\mathrm{d}\zeta^{m_n-1}}B_n^T\right),$$

and A_i, B_i are the same as (17).

(ii) When $N = 2l$

$$u[N] = u - \left[\frac{\det(D)}{\det(C)}\right]_x, \quad v[N] = v + \left[\frac{\det(D(\varphi_j \leftrightarrow \phi_j))}{\det(C(\varphi_j \leftrightarrow \phi_j))}\right]_x, \tag{33}$$

where

$$C = \left(C_1^T, \frac{\mathrm{d}}{\mathrm{d}\zeta}C_1^T, \ldots, \frac{d^{m_n-1}}{(m_n-1)!\mathrm{d}\zeta^{m_n-1}}C_1^T, \ldots, C_n^T, \right.$$
$$\left. \frac{\mathrm{d}}{\mathrm{d}\zeta}C_n^T, \ldots, \frac{d^{m_n-1}}{(m_n-1)!\mathrm{d}\zeta^{m_n-1}}C_n^T\right),$$

$$D = \left(D_1^T, \frac{\mathrm{d}}{\mathrm{d}\zeta}D_1^T, \ldots, \frac{d^{m_n-1}}{(m_n-1)!\mathrm{d}\zeta^{m_n-1}}D_1^T, \ldots, D_n^T, \right.$$
$$\left. \frac{\mathrm{d}}{\mathrm{d}\zeta}D_n^T, \ldots, \frac{d^{m_n-1}}{(m_n-1)!\mathrm{d}\zeta^{m_n-1}}D_n^T\right),$$

and C_i, D_i are the same as (18).

Thus, we complete the construction of gDT-I for (4), which could be considered as a generalization for DT studied in [17, 25, 26].

3.2. gDT-II

In this subsection, we consider the generalization for DT-II. To this end, we assume that n solutions $\Phi_i(\zeta = \mu) = (\varphi_i, \phi_i)^T$ are given for the Lax pair (4) at

$\mu = \mu_i$ and n solutions $\Psi_i(\zeta = \nu) = (\chi_i, \psi_i)$ are given for the adjoint Lax pair (5) at $\nu = \nu_i$ $(i = 1, \ldots, n)$.

THEOREM 4. *Let Φ_i be the solutions of Lax pair (4) at$\zeta = \mu_i$ and Ψ_i be the solutions of adjoint Lax pair (5) at $\zeta = \nu_i$ $(i = 1, \ldots, n)$,*

$$\sum_{i=1}^{r} m_i = N,$$

assume that DT-II possesses m_i order zeros at $\zeta = \pm\mu_i$ and inverse of DT-II possesses m_i order zeros at$\zeta = \pm\nu_i$. Then we have the following gDT-II

$$\begin{aligned} T_N &= T_n^{[m_i-1]} \cdots T_n^{[0]} \cdots T_1^{[m_1-1]} \cdots T_1^{[0]}, \\ T_N^{-1} &= \left(T_1^{[0]}\right)^{-1} \cdots \left(T_1^{[m_1-1]}\right)^{-1} \cdots \left(T_n^{[0]}\right)^{-1} \cdots \left(T_n^{[m_i-1]}\right)^{-1} \end{aligned} \tag{34}$$

where

$$T_i^{[j]} = I + \frac{A_i^{[j]}}{\zeta - \nu_i} - \frac{\sigma_3 A_i^{[j]} \sigma_3}{\zeta + \nu_i}, \quad \left(T_i^{[j]}\right)^{-1} = I + \frac{B_i^{[j]}}{\zeta - \mu_i} - \frac{\sigma_3 B_i^{[j]} \sigma_3}{\zeta + \mu_i},$$

$$A_i^{[j]} = \frac{\nu_i^2 - \mu_i^2}{2} \begin{pmatrix} \alpha_i^{[j]} & 0 \\ 0 & \beta_i^{[j]} \end{pmatrix} \Phi_i^{[j]} \Psi_i^{[j]}, \quad B_i^{[j]} = \frac{\mu_i^2 - \nu_i^2}{2} \Phi_i^{[j]} \Psi_i^{[j]} \begin{pmatrix} \beta_i^{[j]} & 0 \\ 0 & \alpha_i^{[j]} \end{pmatrix},$$

$$\left(\alpha_i^{[j]}\right)^{-1} = \Psi_i^{[j]} \begin{pmatrix} \nu_i & 0 \\ 0 & \mu_i \end{pmatrix} \Phi_i^{[j]}, \quad \left(\beta_i^{[j]}\right)^{-1} = \Psi_i^{[j]} \begin{pmatrix} \mu_i & 0 \\ 0 & \nu_i \end{pmatrix} \Phi_i^{[j]},$$

and

$$\Phi_i^{[j]} = \sum_{l=0}^{j} \frac{\Omega_l}{(j-l)!} \frac{\mathrm{d}^{j-l}}{\mathrm{d}\mu^{j-l}} \Phi_i|_{\mu=\mu_i}, \quad \Omega_l = \sum^{\sum \delta_i^j = l} M_i^{[j-1]} \cdots M_i^{[0]} \cdots M_1^{[m_1-1]} \cdots M_1^{[0]},$$

$$\Psi_i^{[j]} = \sum_{l=0}^{j} \frac{\mathrm{d}^{j-l}}{\mathrm{d}\mu^{j-l}} \Psi_i|_{\nu=\nu_i} \frac{\Lambda_l}{(j-l)!}, \quad \Lambda_l = \sum^{\sum \delta_i^j = l} N_i^{[j-1]} \cdots N_i^{[0]} \cdots N_1^{[m_1-1]} \cdots N_1^{[0]},$$

$$M_i^{[j]} = \begin{cases} \frac{1}{\mu_i^2 - \nu_i^2}, & \text{if } \delta_i^j = 2; \\ \frac{2\mu_i + A_i^{[j]} - \sigma_3 A_i^{[j]} \sigma_3}{\mu_i^2 - \nu_i^2}, & \text{if } \delta_i^j = 1; \\ T_i^{[j]}\big|_{\zeta=\mu_i}, & \text{if } \delta_i^j = 0. \end{cases}, \quad N_i^{[j]} = \begin{cases} \frac{1}{\nu_i^2 - \mu_i^2}, & \text{if } \delta_i^j = 2; \\ \frac{2\nu_i + B_i^{[j]} - \sigma_3 B_i^{[j]} \sigma_3}{\nu_i^2 - \mu_i^2}, & \text{if } \delta_i^j = 1; \\ \left(T_i^{[j]}\right)^{-1}\big|_{\zeta=\nu_i}, & \text{if } \delta_i^j = 0. \end{cases}$$

Proof: Noting that the DT-II is given (20) and using the limit technique, we could obtain special solutions for new Lax pair (4) ($\Phi[1], U[1], V[1]$) at $\zeta = \mu_1$ and adjoint Lax pair (5) ($\Psi[1], U[1], V[1]$) at $\zeta = \nu_1$, that is,

$$\Phi_1^{[1]} = \lim_{\delta\to 0} \frac{T_1^{[0]}\big|_{\zeta=\mu_1+\delta}\Phi_1(\mu_1+\delta)}{\delta} = T_1^{[0]}\big|_{\zeta=\mu_1}\frac{\mathrm{d}}{\mathrm{d}\mu}\Phi_1|_{\mu=\mu_1} + S_1\Phi_1(\mu_1),$$

$$\Psi_1^{[1]} = \lim_{\delta\to 0} \frac{\Psi_1(\nu_1+\delta)T_1^{[0]-1}\big|_{\zeta=\nu_1+\delta}}{\delta} = \frac{\mathrm{d}}{\mathrm{d}\nu}\Psi_1|_{\nu=\nu_1}T_1^{[0]-1}\big|_{\zeta=\nu_1} + \Psi_1(\nu_1)R_1,$$

where

$$S_1 = \frac{2\mu_1 + A_1 - \sigma_3 A_1\sigma_3}{\mu_1^2-\nu_1^2}, \qquad R_1 = \frac{2\nu_1 + B_1 - \sigma_3 B_1\sigma_3}{\nu_1^2-\mu_1^2}.$$

Therefore, we may continue to construct for the new system the DT-II $T[2]$

$$T_1^{[1]} = I + \frac{A_1^{[1]}}{\zeta-\nu_1} - \frac{\sigma_3 A_1^{[1]}\sigma_3}{\zeta+\nu_1}, \qquad \left(T_1^{[1]}\right)^{-1} = I + \frac{B_1^{[1]}}{\zeta-\mu_1} - \frac{\sigma_3 B_1^{[1]}\sigma_3}{\zeta+\mu_1}.$$

Generally, taking account of the given seeds Φ_i and Ψ_i, we perform the following limit

$$\Phi_i^{[j]} = \lim_{\delta\to 0} \frac{\left(T_i^{[j-1]}\cdots T_i^{[0]}\cdots T_1^{[m_1-1]}\cdots T_1^{[0]}\right)\big|_{\zeta=\mu_i+\delta}}{\delta^j}\Phi_i(\mu_i+\delta),$$

$$\Psi_i^{[j]} = \lim_{\delta\to 0} \Psi_i(\nu_i+\delta)\frac{\left(T_1^{[0]-1}\cdots\left(T_1^{[m_1-1]}\right)^{-1}\cdots\left(T_i^{[0]}\right)^{-1}\cdots\left(T_i^{[j-1]}\right)^{-1}\right)\big|_{\zeta=\nu_i+\delta}}{\delta^j},$$

and mathematical induction leads to gDT-II (34). This completes the proof. ■

Due to above proposition, the transformations between the fields are

$$u[N] = u + 2\sum_{i=1}^{n}\sum_{j=0}^{m_i-1}\left[\left(A_i^{[j]}\right)_{12}\right]_x, \tag{35}$$

$$v[N] = v - 2\sum_{i=1}^{n}\sum_{j=0}^{m_i-1}\left[\left(A_i^{[j]}\right)_{21}\right]_x. \tag{36}$$

As before, the formulas (35) and (36) could be rewritten in terms of determinants, that is,

$$u[N] = u - 2\left(\frac{\det(P_1)}{\det(P)}\right)_x, \quad v[N] = v + 2\left(\frac{\det(Q_1)}{\det(Q)}\right)_x \tag{37}$$

where

$$P_1 = \begin{pmatrix} P^{[11]} & P^{[12]} & \cdots & P^{[1r]} & \widehat{\psi_1} \\ P^{[21]} & P^{[22]} & \cdots & P^{[2r]} & \widehat{\psi_2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ P^{[r1]} & P^{[r2]} & \cdots & P^{[rr]} & \widehat{\psi_r} \\ \widehat{\varphi_1} & \widehat{\varphi_2} & \cdots & \widehat{\varphi_r} & 0 \end{pmatrix}, \quad P = \begin{pmatrix} P^{[11]} & P^{[12]} & \cdots & P^{[1r]} \\ P^{[21]} & P^{[22]} & \cdots & P^{[2r]} \\ \vdots & \vdots & \ddots & \vdots \\ P^{[r1]} & P^{[r2]} & \cdots & P^{[rr]} \end{pmatrix},$$

with

$$\widehat{\psi_i} = \left(\psi_i, \frac{\partial}{\partial \nu}\psi_i, \dots, \frac{1}{(m_i-1)!}\frac{\partial^{m_i-1}}{\partial \nu^{m_i-1}}\psi_i\right)^T \Bigg|_{\nu=\nu_j},$$

$$\widehat{\varphi_j} = \left(\varphi_j, \frac{\partial}{\partial \mu}\varphi_j, \dots, \frac{1}{(m_j-1)!}\frac{\partial^{m_j-1}}{\partial \mu^{m_j-1}}\varphi_j\right)\Bigg|_{\mu=\mu_i},$$

$$P^{[ij]} = \left(P^{[ij]}_{kl}\right)_{m_k, m_l},$$

$$P^{[ij]}_{kl} = \frac{1}{(k-1)!(l-1)!}\frac{\partial^{k+l-2}}{\partial \nu^{k-1}\partial \mu^{l-1}}\left(\frac{\Psi_i(\nu)\sigma_3\Phi_j(\mu)}{\mu+\nu} - \frac{\Psi_i(\nu)\Phi_j(\mu)}{\mu-\nu}\right)\Bigg|_{\mu=\mu_i, \nu=\nu_j},$$

and

$$Q_1 = \begin{pmatrix} Q^{[11]} & Q^{[12]} & \cdots & Q^{[1r]} & \widehat{\chi_1} \\ Q^{[21]} & Q^{[22]} & \cdots & Q^{[2r]} & \widehat{\chi_2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ Q^{[r1]} & Q^{[r2]} & \cdots & Q^{[rr]} & \widehat{\chi_r} \\ \widehat{\phi_1} & \widehat{\phi_2} & \cdots & \widehat{\phi_r} & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} Q^{[11]} & Q^{[12]} & \cdots & Q^{[1r]} \\ Q^{[21]} & Q^{[22]} & \cdots & Q^{[2r]} \\ \vdots & \vdots & \ddots & \vdots \\ Q^{[r1]} & Q^{[r2]} & \cdots & Q^{[rr]} \end{pmatrix},$$

with

$$\widehat{\chi_i} = \left(\chi_i, \frac{\partial}{\partial \nu}\chi_i, \dots, \frac{1}{(m_i-1)!}\frac{\partial^{m_i-1}}{\partial \nu^{m_i-1}}\chi_i\right)^T \Bigg|_{\nu=\nu_i},$$

$$\widehat{\phi_j} = \left(\phi_j, \frac{\partial}{\partial \mu}\phi_j, \dots, \frac{1}{(m_j-1)!}\frac{\partial^{m_j-1}}{\partial \mu^{m_j-1}}\phi_j\right)\Bigg|_{\mu=\mu_j},$$

$$Q^{[ij]} = \left(Q^{[ij]}_{kl}\right)_{m_k, m_l},$$

$$Q^{[ij]}_{kl} = -\frac{1}{(k-1)!(l-1)!}\frac{\partial^{k+l-2}}{\partial \nu^{k-1}\partial \mu^{l-1}}\left(\frac{\Psi_i(\nu)\sigma_3\Phi_j(\mu)}{\mu+\nu} + \frac{\Psi_i(\nu)\Phi_j(\mu)}{\mu-\nu}\right)\Bigg|_{\mu=\mu_i, \nu=\nu_j}.$$

According to above theorems, it is not difficult to see that the reductions (28) and (29) are still valid for gDT-I and gDT-II. For the gDT-II, to reduce system (4) to DNLS, we must set $\mu_i = \nu_i^*$.

4. High-order solutions for DNLS

Integrable nonlinear partial differential equations are well known for their richness of solutions. To construct those solutions, a number of approaches have been proposed including IST, Dressing method, Hirota's bilinear theory and Darboux (Bäcklund) method, etc.

While classical DT is known to be a convenient tool to construct N-soliton solutions, it may not be directly used to obtain the high-order solutions, which correspond to multiple poles of the reflection coefficient in the IST terminology (see [6] and the references there). We will show in this section that the gDT derived above can be applied to obtain various solutions for DNLS. Indeed, apart from the high-order solutions, a kind of new N-soliton solutions will also appear. In Section 4.1, we consider the solutions with the VBC. In particular, N-rational solitons, high-order rational solitons and high-order solitons are worked out. In Section 4.2, we construct the high-order solutions with NVBC which include the high-order rational solutions with NVBC and high-order rogue wave solutions.

4.1. Solutions with VBC

Applying DT-I to vacuum, we may obtain three kinds of solutions, namely plane wave solutions, N-phase solutions (periodic solutions) and N-soliton solutions (see [17]). Additionally, if we take limit of the soliton solutions, we can find rational solutions [17, 27]. In this section, we consider the N-rational solutions first. As we pointed out, the rational solitons are the limit cases to the soliton solutions. The different behaviours of the high-order rational solitons and high-order solitons are indicated.

In the first two cases, gDT-I will be used, while for the case 3, it is more convenient to use the gDT-II since the spectral parameters need to be conjugated with each other.

Case 1: N-rational solutions

We first consider the rational solitons and their higher order analogies. For the seed solution $u = 0$, the special solution for Lax pair (4) with the reduction $u = v^*$ is

$$\begin{pmatrix} \varphi \\ \phi \end{pmatrix} = \begin{pmatrix} e^{-i\zeta^{-2}(x+2\zeta^{-2}t+c)} \\ e^{i\zeta^{-2}(x+2\zeta^{-2}t+c)} \end{pmatrix}, \tag{38}$$

where c is a complex constant, which will be taken as a polynomial function of ζ so that the high-order solutions with free parameters may be constructed.

To obtain the N-rational solitons, we introduce vectors

$$y = (\varphi, \zeta\phi, \ldots, \zeta^{2N-2}\varphi, \zeta^{2N}\varphi), \quad z = (\varphi, \zeta\phi, \ldots, \zeta^{2N-2}\varphi, \zeta^{2N-1}\phi),$$

and define the matrices

$$Y = \begin{pmatrix} y_1 \\ y_1^{(1)} \\ \vdots \\ y_N \\ y_N^{(1)} \end{pmatrix}, \quad Z = \begin{pmatrix} z_1 \\ z_1^{(1)} \\ \vdots \\ z_N \\ z_N^{(1)} \end{pmatrix},$$

where $y_i = y|_{\zeta=\zeta_i, c=c_i}$ and $z_i = z|_{\zeta=\zeta_i, c=c_i}$, the superscript $^{(1)}$ represents the first-order derivative to ζ. Then the N-rational soliton can be represented as

$$u[N] = -\left(\frac{\det(Y)}{\det(Z)}\right)_x. \tag{39}$$

Taking $\zeta_1 = \mathrm{i}a$, we have

$$u[1] = \frac{4a^3[4\mathrm{i}(a^2x - 4t + a^2c) - a^4]e^{\frac{2\mathrm{i}(a^2x-2t+a^2c)}{a^4}}}{[4\mathrm{i}(a^2x - 4t + a^2c) + a^4]^2},$$

which appeared already in [26]. The velocity for this rational soliton is $a^2/4$ and the center is along the line $a^2x - 4t + a^2c = 0$. The altitude for $|u[1]|^2$ is $16/a^2$. A simple analysis shows that this two-rational soliton does not possess phase shift when $t \to \pm\infty$, which is different from the two-soliton of NLS. This phenomenon is illustrated by Figure 1.

Case 2: High-order rational solitons

Next we consider the high-order rational solitons. Set the matrices

$$Y_1 = \begin{pmatrix} y_1 \\ y_1^{(1)} \\ \vdots \\ y_1^{(2N-1)} \end{pmatrix}, \quad Z_1 = \begin{pmatrix} z_1 \\ z_1^{(1)} \\ \vdots \\ z_1^{(2N-1)} \end{pmatrix},$$

where the superscript $^{(i)}$ represents the ith derivative with respect to ζ. It follows that the N-order rational soliton for DNLS with VBC can be formulated as

$$u[N] = -\left(\frac{\det(Y_1)}{\det(Z_1)}\right)_x. \tag{40}$$

By choosing appropriate parameters, we have the first and second order rational solitons with VBC, which are plotted in Figure 2.

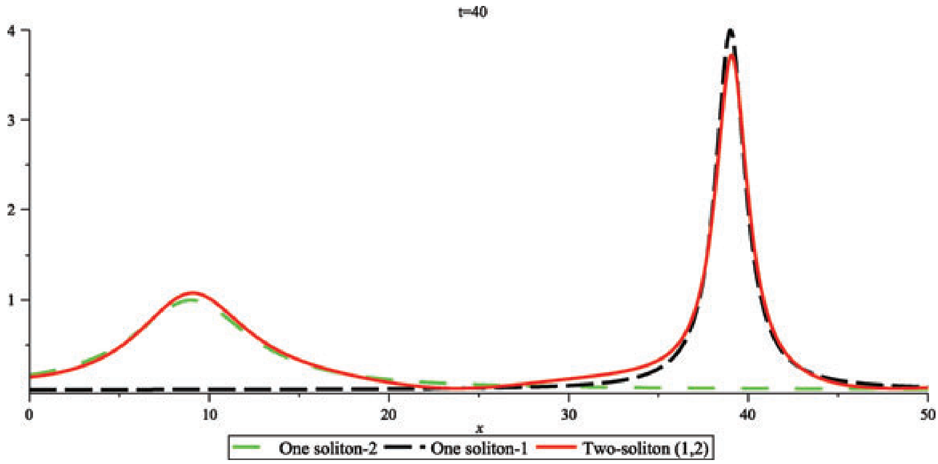

Figure 1. Two-soliton (1,2) with the parameters $\zeta_1 = 2\mathrm{i}$, $\zeta_2 = 4\mathrm{i}$ and $c_1 = c_2 = 1$; One soliton-1 with parameters $\zeta_1 = 4\mathrm{i}$ and $c_1 = 1$; One soliton-2 with parameters $\zeta_1 = 2\mathrm{i}$ and $c_1 = 1$.

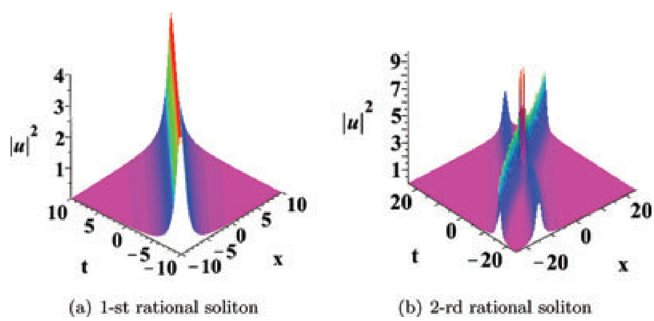

Figure 2. High-order rational solution with VBC: The parameters $\zeta_1 = 2\mathrm{i}$ and $c_1 = 0$.

Case 3: High-order solitons

To obtain the high-order soliton solutions, we start with the seed solution $u = 0$. The special solution for Lax pair (4) with the reduction $u = v^*$ at $u = 0$ is

$$\begin{pmatrix} \varphi \\ \phi \end{pmatrix} = \begin{pmatrix} e^{-\mathrm{i}\mu^{-2}(x+2\mu^{-2}t+c)} \\ e^{\mathrm{i}\mu^{-2}(x+2\mu^{-2}t+d)} \end{pmatrix},$$

and the special solution for adjoint Lax pair (5) with the reduction $u = v^*$ at $u = 0$ reads as

$$(\chi, \psi) = \left(e^{\mathrm{i}\nu^{-2}(x+2\nu^{-2}t+c^*)}, \quad e^{-\mathrm{i}\nu^{-2}(x+2\nu^{-2}t+d^*)}\right),$$

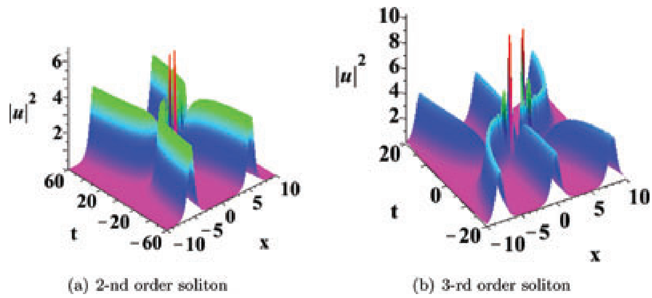

Figure 3. High-order soliton with VBC: The parameters $\mu = 1 + \mathrm{i}$ and $c = d = 0$.

where c, d are constants which may depend on the spectral parameters. Then, the Nth order soliton solution for DNLS is given by

$$u[N] = -2\left(\frac{\det(M_1)}{\det(M)}\right)_x, \quad M = (M_{ij})_{N\times N} \tag{41}$$

where

$$M_{ij} = \frac{\mathrm{d}^{i+j-2}}{\mathrm{d}\nu^{i-1}\mathrm{d}\mu^{j-1}} \frac{2[\nu e^{\mathrm{i}(\nu^{-2}-\mu^{-2})[x+2(\nu^{-2}+\mu^{-2})t]} + \mu e^{-\mathrm{i}(\nu^{-2}-\mu^{-2})[x+2(\nu^{-2}+\mu^{-2})t]}]}{\nu^2-\mu^2}\Bigg|_{\nu=\mu^*},$$

$$M_1 = \begin{pmatrix} M & Y^T \\ X & 0 \end{pmatrix}, \quad X = \left(\varphi, \frac{\mathrm{d}}{\mathrm{d}\mu}\varphi, \ldots, \frac{\mathrm{d}^{N-1}}{\mathrm{d}\mu^{N-1}}\varphi\right), \quad Y = \left(\psi, \frac{\mathrm{d}}{\mathrm{d}\nu}\psi, \ldots, \frac{\mathrm{d}^{N-1}}{\mathrm{d}\nu^{N-1}}\psi\right).$$

The second order and third order soliton solutions are shown in Figure 3. The high-order soliton with more free parameters may be obtained if c or d are taken as polynomial functions of μ.

4.2. Solution with NVBC

The solutions with NVBC may be obtained by applying DT to zero solution. As illustrated in [26], one fold DT-I could be used to yield the plane wave solution. Thus we will consider the high-order rational soliton solutions resulted from vacuum first. To find more general solutions with NVBC, we may apply DT to the general plane wave solution. This will be considered in case 2 and the genuine rational solutions and their higher order analogies with NVBC will be calculated. In case 3, we construct high-order rogue wave solutions. As above, gDT-I will be employed in first two cases and gDT-II will be adopted in the last case.

To the high-order rational solution with NVBC and high-order rogue waves, because they all locate at the branch points of the spectrum, we must use some tricks to deal with this problem. Besides the high-order rogue wave solutions, the high-order breather solutions and periodic solutions can be readily to obtain similarly. Because they are nothing but using the formula (37) directly like the case 3 of above section. Thus we omit them in our work.

Case 1: **High-order rational solitons with NVBC from vacuum**

To obtain high-order rational solutions with NVBC, the order of determinants should be odd. Define the matrices

$$\widehat{Y_1} = \begin{pmatrix} \widehat{y_1} \\ \widehat{y_1}^{(1)} \\ \vdots \\ \widehat{y_1}^{(2N)} \end{pmatrix}, \quad \widehat{Z_1} = \begin{pmatrix} \widehat{z_1} \\ \widehat{z_1}^{(1)} \\ \vdots \\ \widehat{z_1}^{(2N)} \end{pmatrix},$$

where

$$\widehat{y_1} = \left(\varphi, \zeta_1\phi, \ldots, \zeta_1^{2N-1}\phi, \zeta_1^{2N+1}\phi\right), \quad \widehat{z_1} = \left(\varphi, \zeta_1\phi, \ldots, \zeta_1^{2N-1}\phi, \zeta_1^{2N}\varphi\right),$$

and ϕ, ψ are given by (38). Then the high-order rational solitons with NVBC can be represented as

$$u[N] = -\left(\frac{\det(\widehat{Y_1})}{\det(\widehat{Z_1})}\right)_x. \tag{42}$$

Taking the parameters $\zeta_1 = \mathrm{i}a$ (a is a real constant) and $c = 0$, we have the first-order rational solutions in NVBC

$$u[1] = \frac{2L_2L_1^*}{aL_1^2}e^{-\frac{2\mathrm{i}(a^2x-2t)}{a^4}}, \tag{43}$$

where

$$L_1 = 16\xi^2 + a^8 + 8\mathrm{i}a^4(\xi - 4t), \quad L_2 = 16\xi^2 - 3a^8 - 8\mathrm{i}a^4(\xi + 4t)$$

and $\xi = a^2x - 4t$. The norm of $u[1]$ attains the maximum value $\frac{6}{|a|}$ at origin, and vanishes at $(x, t) = (0, \pm\frac{\sqrt{3}}{16}a^4)$. The "ridge" of the solution (43) lays approximately on the line $a^2x - 4t = 0$, and decays to $\frac{2}{a}$ slowly.

The first and second order rational solitons in NVBC are plotted in Fig. 4.

Case 2: **High-order rational solitons with NVBC from plane wave solution**

To construct these solutions, we take the seed solution as

$$u = A\exp(2\mathrm{i}\theta_2),$$

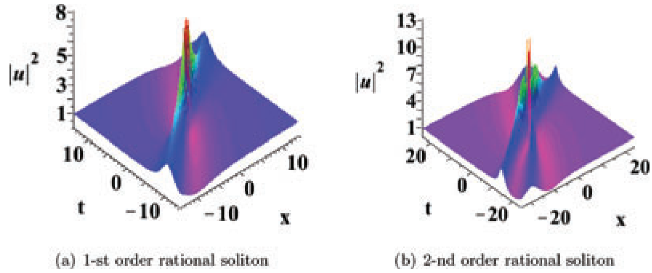

Figure 4. Rational solitons with NVBC: The parameters $\zeta_1 = 2\mathrm{i}$ and $c_1 = 0$.

where $\theta_2 = \frac{1}{2}[ax - (A^2a + a^2)t + c]$, $c \in \mathbb{R}$. The corresponding fundamental solution for Lax pair (4) is

$$\Phi = \begin{pmatrix} e^{\mathrm{i}(\theta_1+\theta_2)+\phi} & e^{-\mathrm{i}(\theta_1-\theta_2)-\phi} \\ e^{-\mathrm{i}(\theta_1+\theta_2)+\phi} & e^{\mathrm{i}(\theta_1-\theta_2)-\phi} \end{pmatrix}, \tag{44}$$

where

$$\theta_1 = \frac{1}{2}\arccos\left(-\frac{2 + a\zeta^2}{2\mathrm{i}A\zeta}\right),$$

and

$$\phi = \frac{1}{2}\sqrt{-(2\zeta^{-2} + a)^2 - 4A^2\zeta^{-2}}[x - (a + A^2 - 2\zeta^{-2})t + d].$$

To resolve the reduction (28), we assume $-4A^2\zeta^2 - (2 + a\zeta^2)^2 > 0$, $d \in \mathbb{R}$, $\zeta \in \mathrm{i}\mathbb{R}$. The corresponding dark solitons and bright solitons were studied and analyzed in [17, 26] in details. The author of [26] also illustrated certain limit cases, but he did not give the explicit expression for those solutions.

In the case $-4A^2\zeta^2 - (2 + a\zeta^2)^2 = 0$, Φ given by the formula (44), which does not qualify as the fundamental solution, is in fact a constant. Thus the gDT could not generate interesting solutions. To obtain meaningful solutions, we must find another solution for (4). To this end, we turn to the limit technique.

For convenience, we consider the special case $a = c = 0$, $A = 1$ which leads to the genuine rational solutions. We will expand the solution for Lax pair at $\zeta = \mathrm{i}$. With the special solution

$$\Phi_1 = \frac{\Phi|_{\zeta=\mathrm{i}(1+f)}C}{f^{1/2}}, \quad C = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

at $f = 0$, which satisfies the reduction (28), we have

$$\Phi_1 = Y_0 + Y_1 f + \cdots + Y_n f^n + \cdots,$$

$$Y_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \lim_{f=0} \frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}f^n}\Phi_1(f).$$

The high-order genuine rational solution in NVBC is represented as the following:

(i) When $N = 2l - 1$, we have

$$u[2l-1] = -1 - \left[\frac{\det(B)}{\det(A)}\right]_x, \tag{45}$$

where

$$A_{i,2j-1} = B_{i,2j-1} = \mathrm{i}^{2j-1} \sum_{k=0}^{\min(i-1,2j-2)} C_{2j-1}^{k} x_k, \quad (j = 1, 2, \ldots, l-1),$$

$$B_{i,2l-1} = \mathrm{i}^{2l} \sum_{k=0}^{i-1} C_{2l}^{k} x_k,$$

$$A_{i,2j} = B_{i,2j} = \mathrm{i}^{2j} \sum_{k=0}^{\min(i-1,2j-1)} C_{2j}^{k} y_k, \quad C_m^n = \frac{m!}{n!(m-n)!}.$$

(ii) When $N = 2l$, we have

$$u[2l] = 1 - \left[\frac{\det(D)}{\det(C)}\right]_x, \tag{46}$$

where

$$C_{i,2j-1} = D_{i,2j-1} = \mathrm{i}^{2j-1} \sum_{k=0}^{\min(i-1,2j-2)} C_{2j-1}^{k} x_k,$$

$$C_{i,2j} = \mathrm{i}^{2j} \sum_{k=0}^{\min(i-1,2j-1)} C_{2j}^{k} y_k, \quad (j = 1, 2, \ldots, l)$$

$$D_{i,2k} = C_{i,2k}, \quad (k = 1, 2, \ldots, l-1), \quad D_{i,2l} = \mathrm{i}^{2l+1} \sum_{k=0}^{i-1} C_{2l+1}^{k} y_k.$$

In particular, taking $d = ef$, where e is real number, we have

$$x_0 = \sqrt{2}(2x - 6t - \mathrm{i}), \quad y_0 = \sqrt{2}(2x - 6t + \mathrm{i}),$$

$$x_1 = \sqrt{2}\left[\frac{2}{3}x^3 - 6x^2 t + 18xt^2 - 18t^3 - 4x + 20t + 2e + \mathrm{i}\left(\frac{1}{2} - x^2 + 6xt - 9t^2\right)\right],$$

$$y_1 = \sqrt{2}\left[\frac{2}{3}x^3 - 6x^2 t + 18xt^2 - 18t^3 - 4x + 20t + 2e + \mathrm{i}\left(x^2 - \frac{1}{2} - 6xt + 9t^2\right)\right].$$

Thus by means of above formula (45), the first order genuine rational soliton solution reads

$$u[1] = -\frac{(-2x+6t-\mathrm{i})(-2x+6t+3\mathrm{i})}{(-2x+6t+\mathrm{i})^2}, \tag{47}$$

which is nothing but the rational traveling wave solution with NVBC. Similarly, (46) provides us the following second order genuine rational soliton with NVBC

$$u[2] = \frac{L_1^* L_2}{L_1^2}, \tag{48}$$

where

$$\begin{aligned} L_1 &= 8\eta^3 + 18\eta + 48t + 12e + \mathrm{i}(12\eta^2 + 3), \\ L_2 &= 8\eta^3 - 30\eta + 48t + 12e + \mathrm{i}(36\eta^2 - 15), \end{aligned}$$

and $\eta = 3t - x$ and e is an arbitrary real number. The norm of solution (48) attains the maximum value five which locates at $(x, t) = (-\frac{3}{4}e, -\frac{1}{4}e)$, and vanishes (48) at $(\frac{7\tau-4\tau^3-6e}{8}, \frac{15\tau-4\tau^3-6e}{24})$, where $\tau = \pm\sqrt{\frac{5}{12}}$. The "ridge" of this soliton (48) approximately lays on the line $x = 3t$. When $t \to \pm\infty$, above $u[2]$ approaches to $u[1]$ represented by (47) along its "ridge." It is should be emphasized that there exists no rational dark soliton.

Case 3: High-order rogue wave solutions

The rogue wave solutions for DNLS were first derived in [17] via DT, to the best of our knowledge. However, the classical DT can not be used directly to obtain high-order rogue wave solutions. According to above, we can see that the gDT is a very efficient way to obtain high-order solutions.

To get the high-order rogue wave solutions, for simplicity, we consider the seed solution $u[0] = \exp(-\mathrm{i}x)$. The corresponding fundamental-matrix solution for Lax pair is

$$\Phi = E\begin{pmatrix} \alpha & \alpha^{-1} \\ -\alpha^{-1} & -\alpha \end{pmatrix}\begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix}, \quad E = \begin{pmatrix} \exp\left(-\frac{1}{2}\mathrm{i}x\right) & 0 \\ 0 & \exp\left(\frac{1}{2}\mathrm{i}x\right) \end{pmatrix}$$

where

$$\alpha = [(2\zeta)^{-1}(\lambda - 2\mathrm{i} + \mathrm{i}\zeta^2)]^{1/2}, \; \beta = \exp\left[\frac{1}{2}\lambda\zeta^{-2}(x + 2\zeta^{-2}t + F(\zeta))\right],$$

$$\lambda = (-4 - \zeta^4)^{1/2}$$

and $F(\zeta)$ is a polynomial function for ζ. As above, by means of the limit technique, we expand the special solution at $\zeta = 1 + \mathrm{i}$

$$\Phi_1 = \frac{\Phi|_{\zeta=(1+\mathrm{i})(1+f)}C}{f^{1/2}}, \quad C = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

at $f = 0$, and find

$$\Phi_1 = Y_0 + Y_1 f + \cdots + Y_n f^n + \cdots,$$

where

$$Y_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \lim_{f=0} \frac{1}{n!} \frac{\mathrm{d}}{\mathrm{d} f^n} \Phi_1(f).$$

Explicitly, we have

$$x_0 = \exp\left[-\frac{1}{2}\mathrm{i}x\right](2x - 2\mathrm{i}t - 1 - \mathrm{i}),$$

$$y_0 = \exp\left[\frac{1}{2}\mathrm{i}x\right](2x - 2\mathrm{i}t + 1 + \mathrm{i}),$$

$$x_1 = \exp\left[-\frac{1}{2}\mathrm{i}x\right]\left[-\frac{1}{3}x^3 + \mathrm{i}x^2 t + xt^2 - \frac{1}{3}\mathrm{i}t^3 + \frac{1+\mathrm{i}}{2}x^2 + (1-\mathrm{i})xt - \frac{1+\mathrm{i}}{2}t^2 \right.$$
$$\left. - \frac{1}{2}\mathrm{i}x - \frac{5}{2}x + \frac{13}{2}\mathrm{i}t - \frac{1}{2}t + 2e + \frac{1}{2} + 2\mathrm{i}g\right],$$

$$y_1 = \exp\left[\frac{1}{2}\mathrm{i}x\right]\left[-\frac{1}{3}x^3 + \mathrm{i}x^2 t + xt^2 - \frac{1}{3}\mathrm{i}t^3 - \frac{1+\mathrm{i}}{2}x^2 - (1-\mathrm{i})xt + \frac{1+\mathrm{i}}{2}t^2 \right.$$
$$\left. - \frac{1}{2}\mathrm{i}x - \frac{5}{2}x + \frac{13}{2}\mathrm{i}t - \frac{1}{2}t + 2e - \frac{1}{2} + 2\mathrm{i}g\right].$$

Therefore, the Nth order rogue wave can be represented as following

$$u[N] = \exp[-\mathrm{i}x] - 2\left(\frac{\det(M_1)}{\det(M)}\right)_x, \tag{49}$$

where $M = \left(M_{ij}\right)_{N\times N}$,

$$M_1 = \begin{pmatrix} M_{11} & M_{12} & \cdots & M_{1N} & y_0 \\ M_{21} & M_{22} & \cdots & M_{2N} & y_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ M_{N1} & M_{N2} & \cdots & M_{NN} & y_{N-1} \\ x_0 & x_1 & \cdots & x_{N-1} & 0 \end{pmatrix},$$

$$M_{ij} = \sum_{k=0,l=0}^{i-1,j-1} -\left(-\frac{1}{2}\right)^{i+j-(k+l+1)} C_{i+j-(k+l+2)}^{i-k-1}$$
$$\times\left[Y_k^{\dagger}\sigma_3 Y_l (1-\mathrm{i})^{i-k-1}(1+\mathrm{i})^{j-l-1} + \mathrm{i}Y_k^{\dagger} Y_l (1+\mathrm{i})^{i-k-1}(1-\mathrm{i})^{j-l-1}\right].$$

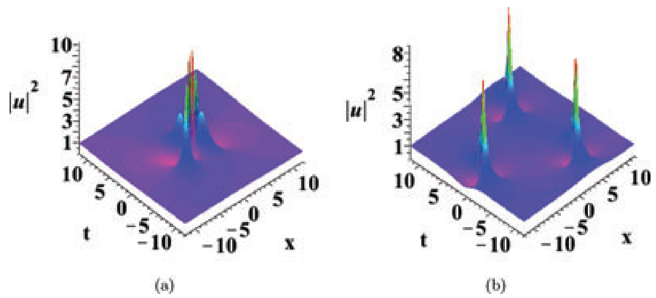

Figure 5. Second order rogue wave with the parameters (a) $e = g = 0$; (b) $e = 0$ and $g = 100$.

In particular, the first order and the second order rogue wave solutions are given by

$$u[1] = -\frac{[2t^2 + 2x^2 - 3 - 2\mathrm{i}(x + 3t)][2x^2 + 2t^2 - 2\mathrm{i}(x - t)]}{[2x^2 + 2t^2 + 1 + 2\mathrm{i}(x - t)]^2} \exp[-\mathrm{i}x],$$

and

$$u[2] = \frac{L_1^* L_2}{L_1^2} \exp[-\mathrm{i}x],$$

where

$$\begin{aligned} L_1 = {} & 72[e^2 + g^2] + [48x^3 - 144xt^2 + 72\mathrm{i}x^2 + 144\mathrm{i}xt - 72\mathrm{i}t^2 - 144x - 72t \\ & + 36\mathrm{i}]e + [-144x^2t + 48t^3 + 72\mathrm{i}x^2 - 144\mathrm{i}xt - 72\mathrm{i}t^2 - 72x + 432t \\ & - 36\mathrm{i}]g + 8x^6 + 24x^4t^2 + 24x^2t^4 + 8t^6 + 24\mathrm{i}x^5 - 24\mathrm{i}x^4t + 48\mathrm{i}x^3t^2 \\ & - 48\mathrm{i}x^2t^3 + 24\mathrm{i}xt^4 - 24\mathrm{i}t^5 - 12x^4 + 48x^3t - 216x^2t^2 + 48xt^3 + 180t^4 \\ & + 48\mathrm{i}x^3 - 288\mathrm{i}xt^2 - 336\mathrm{i}t^3 + 90x^2 - 72xt + 666t^2 + 54\mathrm{i}x - 198\mathrm{i}t + 9, \\ L_2 = {} & 72[e^2 + g^2] + [48x^3 - 144xt^2 - 72\mathrm{i}x^2 + 432\mathrm{i}xt + 72\mathrm{i}t^2 + 144x + 216t \\ & - 180\mathrm{i}]e + [-144x^2t + 48t^3 + 216\mathrm{i}x^2 + 144\mathrm{i}xt - 216\mathrm{i}t^2 + 216x + 144t \\ & + 36\mathrm{i}]g + 8x^6 + 24x^4t^2 + 24x^2t^4 + 8t^6 - 24\mathrm{i}x^5 - 72\mathrm{i}x^4t - 48\mathrm{i}x^3t^2 \\ & - 144\mathrm{i}x^2t^3 - 24\mathrm{i}xt^4 - 72\mathrm{i}t^5 - 60x^4 - 144x^3t - 504x^2t^2 - 144xt^3 \\ & - 60t^4 + 48\mathrm{i}x^3 + 288\mathrm{i}x^2t + 576\mathrm{i}xt^2 - 528\mathrm{i}t^3 - 198x^2 + 504xt - 486t^2 \\ & + 90\mathrm{i}x + 414\mathrm{i}t + 45, \end{aligned}$$

e and f are arbitrary real number. These solutions with different parameters are plotted in Fig. 5.

5. Conclusion and Discussion

The theory of DT is developed and two generalized DTs, gDT-I, and gDT-II, are constructed for DNLS. With the help of them, two generalized determinant solution formulae are obtained for this physically relevant equation. Moreover, high-order solitons, high-order rogue waves, and rational solutions are given explicitly. We remark that the gDT-II is still valid for N-component DNLS system. In addition, the above formula can be easily modified and applied to so-called Fokas-Lenells Equation [28].

As shown in Figure 5, the second-order rogue waves exhibit dynamics which varies according to the different values of the parameters. It is interesting to study the dynamics of the general high-order solutions. Also, it is interesting to calculate the energy of the high-order solutions, which could be done directly. However, the calculations involved in is very tedious. These questions may be answered efficiently by a detailed analysis of DNLS in the framework of inverse scattering method.

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