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Bifurcations and Exact Solutions in a Nonlinear Wave Equation*

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The dynamical model of a nonlinear wave is governed by a partial differential equation which is a special case of the b-family equation. Its traveling system is a singular system with a singular straight line. On this line, there exist two degenerate nodes of the associated regular system. By using the method of dynamical systems and the theory of singular traveling wave systems, in this paper we show that, corresponding to global level curves, this wave equation has global periodic wave solutions and anti-solitary wave solutions. We obtain their exact representations. Specially, we discover some new phenomena. (i) Infinitely many periodic orbits of the traveling wave system pass through the singular straight line. (ii) Inside some homoclinic orbits of the traveling wave system there is not any singular point. (iii) There exist periodic wave bifurcation and double anti-solitary waves bifurcation.

Keywords: Bifurcation; solitary wave; periodic wave; singular system; nonlinear wave equation.

1. Introduction

Many authors have studied the b-family equation.

$$u_t - u_{xxt} + (b+1)uu_x = bu_xu_{xx} + uu_{xxx}. \quad (1)$$

When $b = 2$, Eq. (1) becomes the CH equation formulated by Camassa and Holm [1993], showing that there are peakons in the equation. Boyd [1997] found coshoidal waves in the CH equation. Constantin and Escher [1998] discovered wave breaking phenomenon in the CH equation. Constantin and Strauss [2000] proved the stability of peakons for the CH equation. Reyes [2002] studied the integrability of the CH equation. Johnson [2002] discussed the CH equation and related models for water waves.

When $b = 3$, the b-equation reduces to the DP equation given by Degasperis and Procesi [1999]. Lundmark and Szmigielski [2003, 2005] gave an inverse scattering method for computing the n-peakon solutions of the DP equation and obtained concrete expressions of the 3-peakon solutions. Chen and Tang [2006] confirmed that the DP equation has kink-like waves.

The solutions of the b-equation were investigated numerically for some values of b by Holm and Staley [2003]. For arbitrary $b > 1$, Guo and Liu [2005] proved that Eq. (1) has periodic cusp waves with explicit expressions. Guha [2007] presented an Euler–Poincaré formalism of the DP equation.

To investigate the bifurcation of the peakon waves, Liu and Qian [2001] proposed a generalized CH equation

$$u_t + 2ku_x - u_{xxt} + au^2u_x = 2u_xu_{xx} + uu_{xxx}. \quad (2)$$

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Similarly, to study the change of peakons, Wazwaz [2006, 2007] suggested a generalized DP equation

$$u_t - u_{xxt} + 4u^2 u_x = 3u_x u_{xx} + u u_{xxx} \tag{3}$$

and another type of b-family equation

$$u_t - u_{xxt} + (b+1)u^2 u_x = b u_x u_{xx} + u u_{xxx}. \tag{4}$$

Since the CH and DP equations possess complex structures and properties, many pioneers were interested in their modified forms [Eqs. (2)–(4)]. Tian and Song [2004] presented physical explanation for Eq. (2). Shen and Xu [2005] studied the existence of some traveling waves for Eq. (2). Denoting c as the constant wave speed of traveling waves, for some special values of c, the explicit traveling wave solutions were searched for Eqs. (2) and (3). When $c = 1$, Khuri [2005] obtained a singular wave solution composed of triangular functions for Eq. (2). When $c = 1$ and $c = 2$ respectively, Wazwaz [2007] gave 11 explicit traveling wave solutions consisting of triangular functions or hyperbolic functions for Eq. (2), and Liu and Ouyang [2007] obtained a peakon solution formed of hyperbolic functions for Eq. (2). He *et al.* [2008a] employed the bifurcation theory to get some solutions for Eq. (2). When $a = 3$, Liu and Liang [2011] studied some nonlinear waves and their bifurcations for Eq. (2). When $c = 5/2$, Wazwaz [2007] got nine exact solutions for Eq. (3). Besides, Liu and Ouyang [2007] gave a peakon solution for Eq. (3). Zhang *et al.* [2007] employed the bifurcation method of dynamical systems to show the existence of some nonlinear waves for Eq. (2). Wang and Tang [2008] obtained the exact solutions for Eq. (2) when $c = \frac{1}{3}$ and $c = 3$ respectively, and gave two explicit solutions for Eq. (3) when $c = \frac{1}{4}$ and $c = 4$ respectively. Yomba [2008a, 2008b] presented two methods to find the explicit traveling wave solutions for Eqs. (2) and (3). He *et al.* [2008b] utilized the method of dynamical systems to give some exact solutions for Eq. (3). Liu and Tang [2010] investigated the bifurcations of periodic wave solutions for Eqs. (2) and (3).

When the wave speed $c = \frac{2+b}{2}$, Wazwaz [2007] obtained two soliton solutions for Eq. (4). When $b > 1$, Liu [2010] studied the coexistence of multifarious exact solutions for Eq. (4). When $b \neq 0, -1, -2$, Chen *et al.* [2016] studied the periodic waves and their limit forms for Eq. (4). When $b > 1$, Yang *et al.* [2018] studied the existence and bifurcation of peakons for Eq. (4) of high order.

In this paper, we consider the case of $b = 0$ in Eq. (4), that is, the equation

$$u_t - u_{xxt} + u^2 u_x = u u_{xxx}. \tag{5}$$

Using the qualitative analysis and bifurcation method of dynamical systems (see [Li & Chen, 2007, 2013; Li *et al.*, 2016; Liu & Yan, 2013] for instance), we discovered some interesting properties which are rarely seen in the literature.

This paper is organized as follows. In Sec. 2, we derive the traveling wave system of Eq. (5) and investigate its bifurcations in phase portraits. In Sec. 3, we calculate the exact representations for solitary wave solutions and periodic wave solutions. In Sec. 4, a short conclusion is given.

2. Bifurcations of Phase Portraits for the Traveling Wave System of Eq. (5)

In this section, we describe traveling wave system and study its bifurcations of phase portraits for Eq. (5). Let $\xi = x - ct$. Substituting $u = \varphi(\xi)$ into Eq. (5), and integrating once, we have

$$(\varphi - c)\varphi'' = g - \varphi + \frac{1}{3}\varphi^3 + \frac{1}{2}(\varphi')^2, \tag{6}$$

where g is an integral constant.

Equation (6) is equivalent to the planar dynamical system

$$\frac{d\varphi}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{g - \varphi + \frac{1}{3}\varphi^3 + \frac{1}{2}y^2}{\varphi - c}, \tag{7}$$

with the first integral

$$\frac{y^2}{\varphi - c} - \frac{1}{3}\varphi^2 - \frac{4}{3}c\varphi + 2(1-c)c\ln|\varphi - c| + \frac{2(3g - 3c^2 + c^3)}{3(\varphi - c)} = h. \tag{8}$$

Clearly, system (7) is a singular nonlinear traveling wave system of first class as defined in [Li & Chen, 2007] and [Li, 2013] with a singular straight line $\varphi = c$.

In order to find the exact solutions of system (7), we next assume that $c = 1$. Thus, system (7) becomes

$$\frac{d\varphi}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{g - \varphi + \frac{1}{3}\varphi^3 + \frac{1}{2}y^2}{\varphi - 1}, \tag{9}$$

with the first integral

$$H(\varphi, y) = \frac{y^2 - \frac{1}{3}\varphi^3 - \varphi^2 + \frac{4}{3}\varphi + 2g - \frac{4}{3}}{\varphi - 1} = h. \tag{10}$$

We study the associated regular system of system (9) as

$$\begin{cases} \dfrac{d\varphi}{d\tau} = y(\varphi - 1), \\ \dfrac{dy}{d\tau} = g - \varphi + \dfrac{1}{3}\varphi^3 + \dfrac{1}{2}y^2, \end{cases} \tag{11}$$

where $d\tau = \frac{d\xi}{\varphi - 1}$ for $\varphi \neq 1$.

Notice that both systems (9) and (11) own the same first integral as (10). But, these two systems define different vector fields. On the left side of the straight line $\varphi = 1$, the direction of the vector field defined by (9) is just inverse with the direction of the vector field defined by (11). The straight line $\varphi = 1$ is a solution of the first equation in system (11), but is not a solution of system (9).

To investigate the equilibrium points of system (11), we write that $f(\varphi) = \varphi^3 - 3\varphi + 3g$. If φ_e is a real zero of $f(\varphi)$, then system (11) has the equilibrium point $E_e(\varphi_e, 0)$.

For given parameter g, we will utilize the following notations:

$$\sigma_1 = \frac{1}{2}(12g + 4\sqrt{9g^2 - 4})^{\frac{1}{3}}, \quad \text{when } g \geq \frac{2}{3}, \tag{12}$$

$$\sigma_2 = \frac{\sqrt{4 - 9g^2}}{-3g}, \quad \text{when } |g| \leq \frac{2}{3}, \tag{13}$$

$$\sigma_3 = \frac{1}{2}(-12g + 4\sqrt{9g^2 - 4})^{\frac{1}{3}}, \quad \text{when } g < -\frac{2}{3}, \tag{14}$$

$$\alpha = \begin{cases} 2\cos\left[\dfrac{1}{3}(\pi + \arctan\sigma_2)\right], & \text{when } 0 < g \leq \dfrac{2}{3}, \\ \sqrt{3}, & \text{when } g = 0, \\ 2\cos\left[\dfrac{1}{3}\arctan\sigma_2\right], & \text{when } -\dfrac{2}{3} \leq g < 0, \\ \dfrac{1}{\sigma_3} + \sigma_3, & \text{when } g < -\dfrac{2}{3}, \end{cases} \tag{15}$$

$$\beta = \frac{1}{2}(-\alpha + \sqrt{3(4 - \alpha^2)}), \quad \text{when } |g| < \frac{2}{3}, \tag{16}$$

$$\gamma = -\frac{1}{\sigma_1} - \sigma_1, \quad \text{when } g \geq -\frac{2}{3}, \tag{17}$$

$$\alpha_* = -2\alpha - 3, \quad \text{when } g < \frac{2}{3}, \tag{18}$$

$$\beta_* = -2\beta - 3, \quad \text{when } |g| < \frac{2}{3} \tag{19}$$

and

$$y_0 = \sqrt{2\left(\frac{2}{3} - g\right)}, \quad \text{when } g < \frac{2}{3}. \tag{20}$$

Thus, the following conclusions hold:

(i) When $g < -\frac{2}{3}$, $f(\varphi)$ has only one real zero point $\varphi = \alpha$.
(ii) When $g = -\frac{2}{3}$, $f(\varphi)$ has two real zero points $\varphi = \alpha = 2$, $\beta = \gamma = -1$.
(iii) When $-\frac{2}{3} < g < \frac{2}{3}$, $f(\varphi)$ has three real zero points $\varphi = \alpha, \beta, \gamma$.
(iv) When $g = \frac{2}{3}$, $f(\varphi)$ has two real zero points $\varphi = \alpha = \beta = 1$, $\gamma = -2$.
(v) When $g > \frac{2}{3}$, $f(\varphi)$ has only one real zero point $\varphi = \gamma$. For $6 > g > \frac{2}{3}$, write

$$\sigma_4 = \frac{\sqrt{60g - 36 - 9g^2}}{3g - 10}, \tag{21}$$

$$\alpha_1 = -1 + 4\cos\left[\frac{1}{3}(\pi + \arctan\sigma_4)\right], \tag{22}$$

$$\beta_1 = \frac{1}{2}(-3 - \alpha_1 + \sqrt{45 - 6\alpha_1 - 3\alpha_1^2}) \tag{23}$$

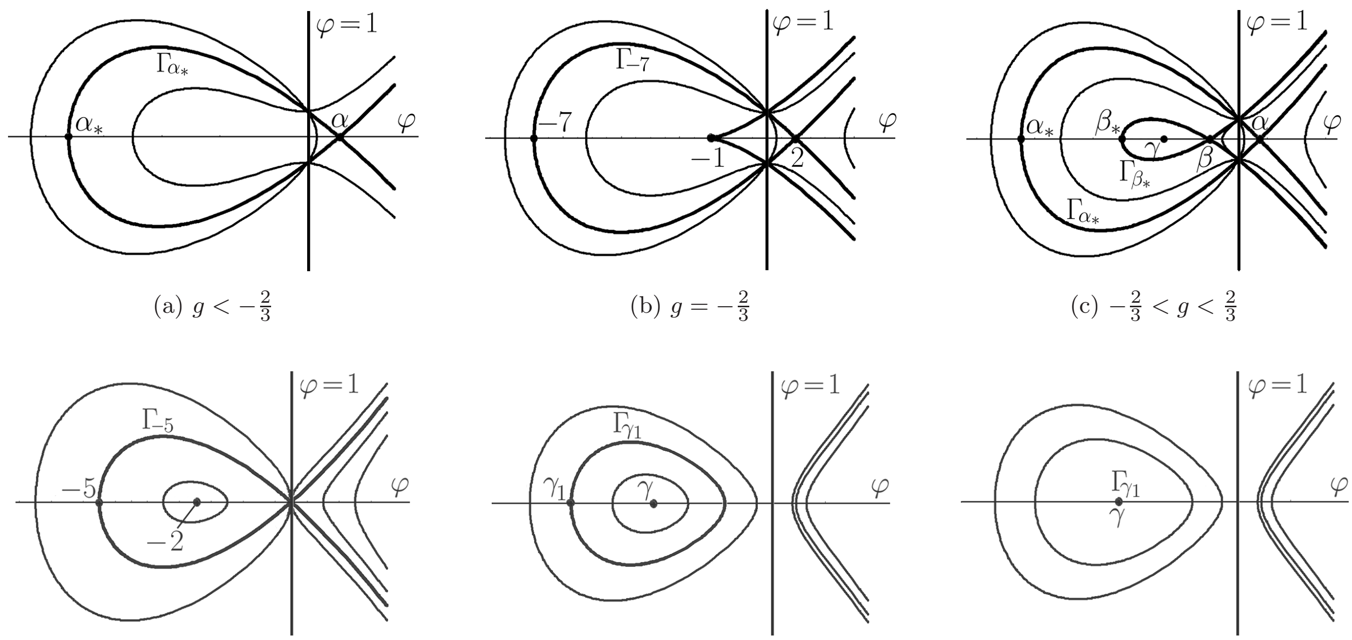
and

$$\gamma_1 = \frac{1}{2}(-3 - \alpha_1 - \sqrt{45 - 6\alpha_1 - 3\alpha_1^2}). \tag{24}$$

Let $M(\varphi_e, y_e)$ be the coefficient matrix of the linearized system of system (11) at an equilibrium point $E_e(\varphi_e, y_e)$ and $J(\varphi_e, y_e) = \det M(\varphi_e, y_e)$. Then we have

$$\begin{gathered} J(\varphi_e, 0) = -(\varphi_e - 1)^2(\varphi_e + 1), \\ J(1, \mp y_0) = y_0^2. \end{gathered} \tag{25}$$

By the theory of planar dynamical systems (for instance, see [Li, 2013]), we know that the distributions and properties of equilibrium points of



(a) $g < -\frac{2}{3}$ (b) $g = -\frac{2}{3}$ (c) $-\frac{2}{3} < g < \frac{2}{3}$

(d) $g = \frac{2}{3}$ (e) $6 > g > \frac{2}{3}$ (f) $g \geq 6$

Fig. 1. Bifurcations of phase portraits of system (11).

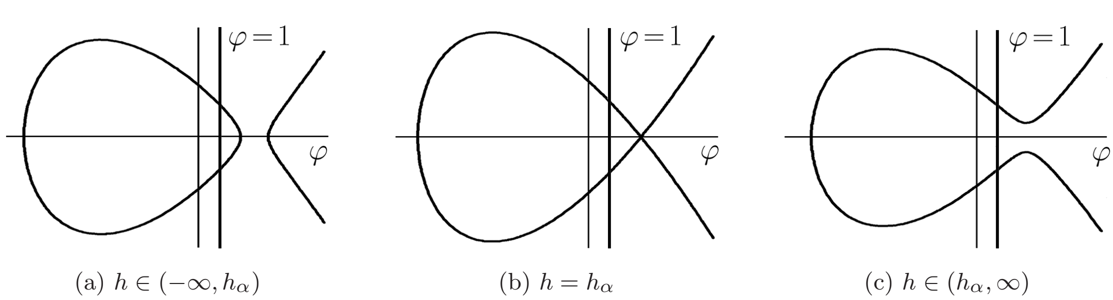
system (11) are as follows:

(1) When $g < -\frac{2}{3}$, system (11) has three equilibrium points $(\alpha, 0)$ and $(1, \mp y_0)$. $(\alpha, 0)$ is a saddle and $(1, -y_0)$ is a stable degenerate node, while $(1, y_0)$ is an unstable degenerate node.
(2) When $g = -\frac{2}{3}$, system (11) has a double equilibrium point $(-1, 0)$ and a simple equilibrium point $(2, 0)$ and $(1, \mp\frac{2\sqrt{6}}{3})$. $(2, 0)$ is a saddle point; $(-1, 0)$ is a cusp and $(1, -\frac{2\sqrt{6}}{3})$ is a stable degenerate node, while $(1, \frac{2\sqrt{6}}{3})$ is an unstable degenerate node.
(3) When $-\frac{2}{3} < g < \frac{2}{3}$, system (11) has five simple equilibrium points $(\gamma, 0)$, $(\beta, 0)$, $(\alpha, 0)$, and $(1, \mp y_0)$. $(\gamma, 0)$ is a center; $(\beta, 0)$ and $(\alpha, 0)$ are two saddles and $(1, -y_0)$ is a stable degenerate node, while $(1, y_0)$ is an unstable degenerate node.
(4) When $g = \frac{2}{3}$, system (11) has a high-order equilibrium point $(1, 0)$ and a center at $(-2, 0)$.
(5) When $g > \frac{2}{3}$, system (11) has a simple singular point $(\gamma, 0)$ which is a center.

From the above discussion, we obtain the bifurcations of phase portraits of system (11) as shown in Figs. 1(a)–1(f).

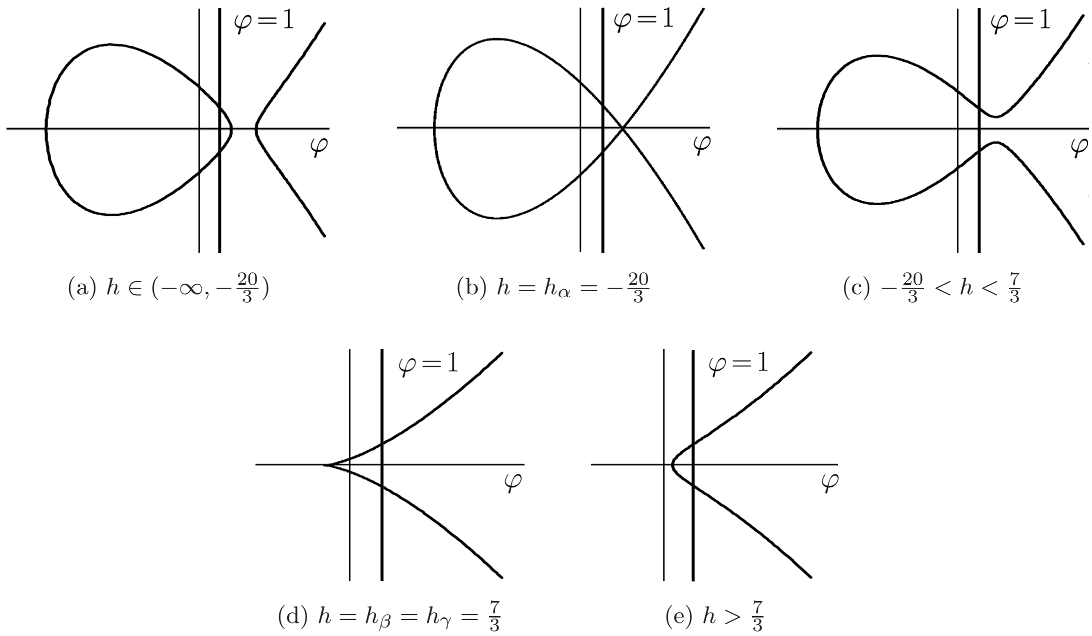
Corresponding to the phase portraits given by Figs. 1(a)–1(f), when h varies from $-\infty$ to ∞, the curves defined by $H(\varphi, y) = h$ in (10) are changed. Figures 2–6 show all possible cases for $-\infty < g < \infty$.

Remark 1. Notice that for the regular system (11), on the left side of the straight line solution $\varphi = 1$, the directions of all orbits are anti-clockwise because $(1, -y_0)$ is a stable degenerate node, $(1, y_0)$ is an unstable degenerate node. But, for the singular



(a) $h \in (-\infty, h_\alpha)$ (b) $h = h_\alpha$ (c) $h \in (h_\alpha, \infty)$

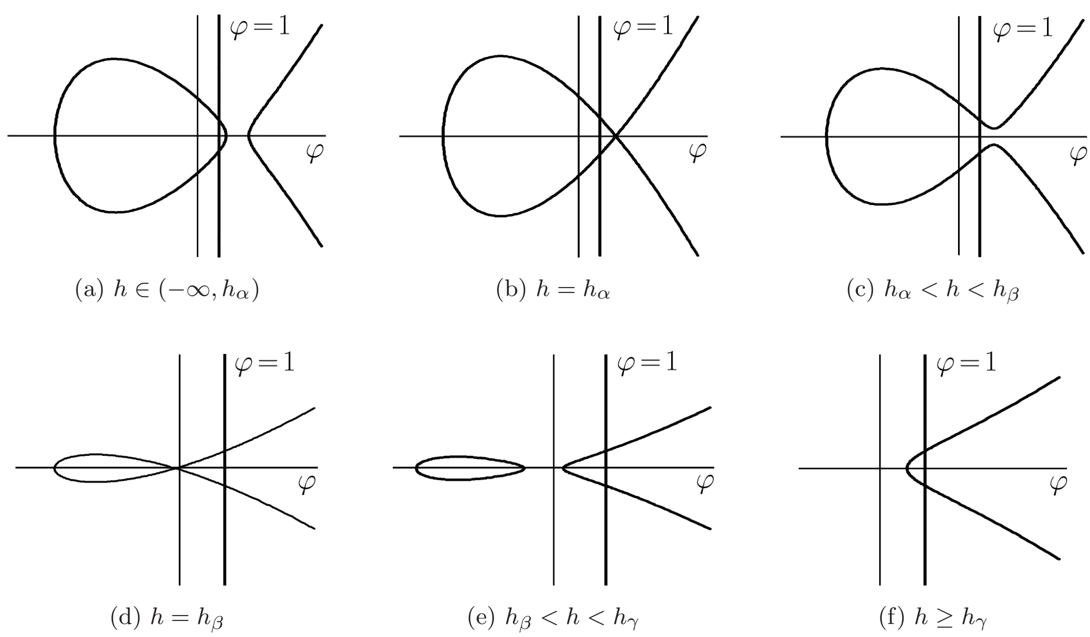
Fig. 2. The changes of the curves defined by $H(\varphi, y) = h$ when $-\infty < g < -\frac{2}{3}$.



(a) $h \in (-\infty, -\frac{20}{3})$ (b) $h = h_\alpha = -\frac{20}{3}$ (c) $-\frac{20}{3} < h < \frac{7}{3}$

(d) $h = h_\beta = h_\gamma = \frac{7}{3}$ (e) $h > \frac{7}{3}$

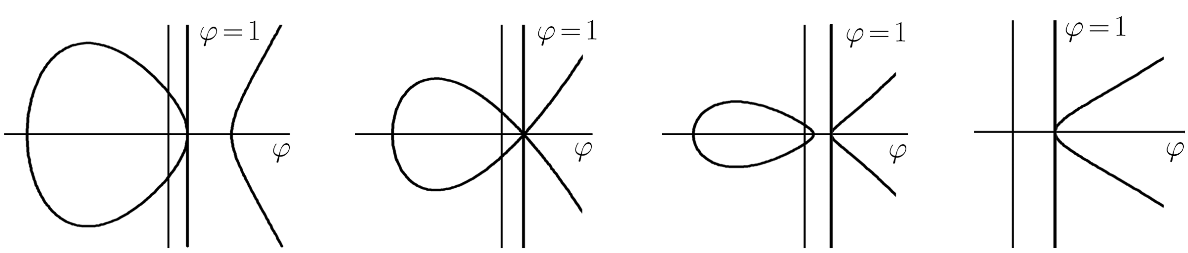
Fig. 3. The changes of the curves defined by $H(\varphi, y) = h$ when $g = -\frac{2}{3}$.



(a) $h \in (-\infty, h_\alpha)$ (b) $h = h_\alpha$ (c) $h_\alpha < h < h_\beta$

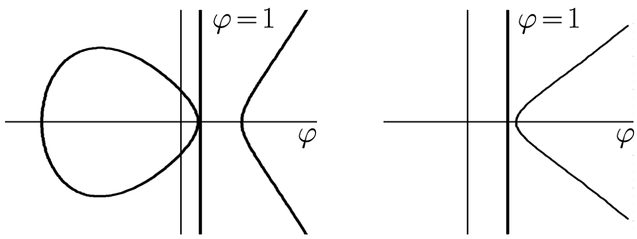
(d) $h = h_\beta$ (e) $h_\beta < h < h_\gamma$ (f) $h \geq h_\gamma$

Fig. 4. The changes of the curves defined by $H(\varphi, y) = h$ when $-\frac{2}{3} < g < \frac{2}{3}$.



(a) $h \in (-\infty, -\frac{5}{3})$ (b) $h = h_\alpha = -\frac{5}{3}$ (c) $-\frac{5}{3} < h < h_\gamma = \frac{4}{3}$ (d) $h \geq \frac{4}{3}$

Fig. 5. The changes of the curves defined by $H(\varphi, y) = h$ when $g = \frac{2}{3}$.



(a) $h \in (-\infty, h_\gamma)$ (b) $h \in [h_\gamma, \infty)$

Fig. 6. The changes of the curves defined by $H(\varphi, y) = h$ when $\frac{2}{3} < g < \infty$.

system (9), on the left side of the straight line solution $\varphi = 1$, the directions of all orbits are clockwise by the vector fields defined by system (9).

Therefore, we emphasize that in Figs. 2(a), 2(b)–5(a), 5(b), the direction of vector fields for the closed curves passing through the singular straight line $\varphi = 1$ is clockwise. Thus, these curves in Figs. 2(a)–5(a) give rise to global periodic orbits of system (9). In Figs. 2(b)–5(b), the closed curves passing through the singular straight line $\varphi = 1$ are homoclinic orbits of system (9). Of course, in Fig. 4(d) the closed curve is a homoclinic orbit of system (9) too.

3. Exact Solutions of Eq. (5)

In this section, we will derive the exact solutions of Eq. (5). From (10) we see that

$$y^2 = h(\varphi - 1) + \frac{1}{3}\varphi^3 + \varphi^2 - \frac{4}{3}\varphi - 2g + \frac{4}{3}. \tag{26}$$

Thus, by using the first equation of system (9), we obtain

$$\int_{\varphi_0}^{\varphi} \frac{\mathrm{d}s}{\sqrt{s^3 + 3s^2 + (3h-4)s - 6g + 4 - 3h}} = \frac{1}{\sqrt{3}}\xi. \tag{27}$$

By calculating (27), we can obtain all exact solutions.

3.1. *Periodic wave solutions and bifurcation*

For given $g \in (-\infty, \infty)$, we use the notations of α, β and γ as (15)–(17), and let

$$h_\alpha = H(\alpha, 0), \quad h_\beta = H(\beta, 0), \quad h_\gamma = H(\gamma, 0). \tag{28}$$

From Figs. 2–6, we see the following facts:

(i) If we write

$$h_\rho = \begin{cases} h_\gamma, & \text{when } g > \dfrac{2}{3}, \\ -\dfrac{5}{3}, & \text{when } g = \dfrac{2}{3}, \\ h_\alpha, & \text{when } g < \dfrac{2}{3}, \end{cases} \tag{29}$$

and let $h \in (-\infty, h_\rho)$, then in the curves defined by $H(\varphi, y) = h$ there exist global families of periodic orbits of system (9).

(ii) If $-\frac{2}{3} < g < \frac{2}{3}$ and $h_\beta < h < h_\gamma$, or $g = \frac{2}{3}$ and $-\frac{5}{3} < h < \frac{4}{3}$, then in the curves defined by $H(\varphi, y) = h$ there exist local families of periodic orbits of system (9).

On the φ–y plane, the periodic orbits possess expression

$$y^2 = \frac{1}{3}(r_1 - \varphi)(r_2 - \varphi)(\varphi - r_3), \tag{30}$$

where

$$r_3 \le \varphi < r_2, \tag{31}$$

$$r_1 = -1 - \frac{\sqrt[3]{2}\delta_2}{3\sqrt[3]{\delta_*}} + \frac{\sqrt[3]{\delta_*}}{3 \times \sqrt[3]{2}}, \tag{32}$$

$$r_2 = -1 + \frac{(1 - \mathrm{i}\sqrt{3})\delta_2}{3 \times \sqrt[3]{2^2}\sqrt[3]{\delta_*}} - \frac{(1 + \mathrm{i}\sqrt{3})\sqrt[3]{\delta_*}}{6 \times \sqrt[3]{2}}, \tag{33}$$

$$r_3 = -1 + \frac{(1 + \mathrm{i}\sqrt{3})\delta_2}{3 \times \sqrt[3]{2^2}\sqrt[3]{\delta_*}} + \frac{(-1 + \mathrm{i}\sqrt{3})\sqrt[3]{\delta_*}}{6 \times \sqrt[3]{2}}, \tag{34}$$

$$\delta_0 = \sqrt{729(-5 + 3g + 3h)^2 + (-21 + 9h)^3}, \tag{35}$$

$$\delta_1 = -270 + 162g + 162h, \tag{36}$$

$$\delta_2 = -21 + 9h \tag{37}$$

and

$$\delta_* = 2\delta_0 + \delta_1. \tag{38}$$

Thus along the periodic orbits, the integral equation (27) changes into

$$\int_{\gamma_3}^{\varphi} \frac{\mathrm{d}s}{\sqrt{(r_1 - s)(r_2 - s)(s - r_3)}} = \frac{1}{\sqrt{3}}\xi. \tag{39}$$

Completing the integration and solving φ in the integral equation (39), we obtain the periodic wave solution

$$u(x,t,g,h) = r_3 + (r_2 - r_3)\mathrm{sn}^2(\Omega\xi, k), \quad (40)$$

where

$$\Omega = \frac{\sqrt{r_1 - r_3}}{2\sqrt{3}}, \quad (41)$$

$$\xi = x - t \quad (42)$$

and

$$k^2 = \frac{r_2 - r_3}{r_1 - r_3}. \quad (43)$$

Remark 2. When $\frac{2}{3} < g < 6$ and $h = 0$, it follows that

$$r_1 = \alpha_1, \quad r_2 = \beta_1 \quad \text{and} \quad r_3 = \gamma_1. \quad (44)$$

Thus we have

$$u(\xi, g, 0) = \gamma_1 + (\beta_1 - \gamma_1)\mathrm{sn}^2(\eta\xi, k_0), \quad (45)$$

where

$$\eta = \sqrt{\frac{\alpha_1 - \gamma_1}{12}} \quad (46)$$

and

$$k_0^2 = \frac{\beta_1 - \gamma_1}{\alpha_1 - \gamma_1}. \quad (47)$$

From (45) we have

$$\lim_{g \to \frac{2}{3}+0} u(\xi, g, 0) = u_1(\xi), \quad \text{[see (51)]}. \quad (48)$$

This implies that an anti-solitary wave is a bifurcation from a periodic wave when $g \to \frac{2}{3} + 0$. The evolution of wave profiles of $u(\xi, g, 0)$ is given in Fig. 7.

3.2. *Anti-solitary wave solutions and bifurcations*

Now, we derive anti-solitary wave solutions as follows.

(1) When $g = \frac{2}{3}$, on φ–y plane, the homoclinic orbit [see Fig. 5(b)] owns expression

$$y^2 = \frac{1}{3}(1-\varphi)^2(\varphi+5) \quad \text{for } -5 \le \varphi < 1. \quad (49)$$

Along the homoclinic orbit, Eq. (27) changes into

$$\int_{-5}^{\varphi} \frac{1}{(1-s)\sqrt{s+5}}\mathrm{d}s = \frac{1}{\sqrt{3}}|\xi|. \quad (50)$$

Solving (50) for φ, we obtain the anti-solitary wave solution

$$u_1(\xi) = \varphi = 1 - 6\,\mathrm{sech}^2\left(\frac{\xi}{\sqrt{2}}\right). \quad (51)$$

(2) When $|g| < \frac{2}{3}$, on the φ–y plane, the two homoclinic orbits [see Figs. 4(b) and 4(d)] have expressions

$$y^2 = \frac{1}{3}(\alpha-\varphi)^2(\varphi-\alpha_*) \quad \text{for } \alpha_* \le \varphi < \alpha \quad (52)$$

and

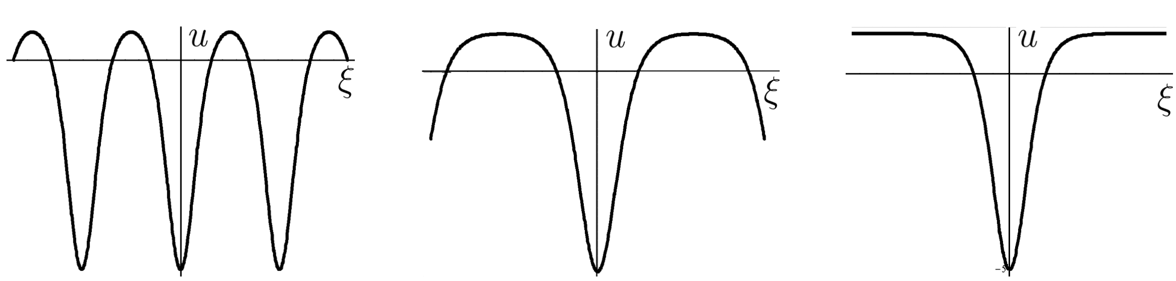
$$y^2 = \frac{1}{3}(\beta-\varphi)^2(\varphi-\beta_*) \quad \text{for } \beta_* \le \varphi < \beta. \quad (53)$$

According to (52), Eq. (27) reduces to

$$\int_{\alpha_*}^{\varphi} \frac{1}{(\alpha-s)\sqrt{s-\alpha_*}}\mathrm{d}s = \frac{1}{\sqrt{3}}|\xi|. \quad (54)$$

Solving (54) for φ, we get anti-solitary wave solution

$$u_\alpha(\xi, g) = \varphi = \alpha - 3(\alpha+1)\mathrm{sech}^2\left(\frac{1}{2}\sqrt{\alpha+1}\,\xi\right). \quad (55)$$



(a) $g = \frac{2}{3} + 10^{-1}$ (b) $g = \frac{2}{3} + 10^{-3}$ (c) $g = \frac{2}{3} + 10^{-5}$

Fig. 7. The evolution of wave profiles of $u(\xi, g, 0)$ when $g \to \frac{2}{3} + 0$.

Via (53), Eq. (27) becomes

$$\int_{\beta_*}^{\varphi} \frac{1}{(\beta - s)\sqrt{s - \beta_*}} \mathrm{d}s = \frac{1}{\sqrt{3}}|\xi|. \tag{56}$$

Solving (56) for φ, we get another anti-solitary wave solution

$$u_\beta(\xi, g) = \varphi = \beta - 3(\beta + 1)\mathrm{sech}^2\left(\frac{1}{2}\sqrt{\beta + 1}\xi\right). \tag{57}$$

(3) When $g = -\frac{2}{3}$, on the φ–y plane, the homoclinic orbit [see Fig. 3(b)] owns the expression

$$y^2 = \frac{1}{3}(2 - \varphi)^2(\varphi + 7) \quad \text{for } -7 \leq \varphi < 2. \tag{58}$$

Along the homoclinic orbit, Eq. (27) changes into

$$\int_{-7}^{\varphi} \frac{1}{(2 - s)\sqrt{s + 7}} \mathrm{d}s = \frac{1}{\sqrt{3}}|\xi|. \tag{59}$$

Solving (59) for φ, we get the anti-solitary wave solution

$$u_2(\xi) = \varphi = 2 - 9\,\mathrm{sech}^2\left(\frac{\sqrt{3}}{2}\xi\right). \tag{60}$$

(4) When $g < -\frac{2}{3}$, on the φ–y plane, the expression of homoclinic orbit [see Fig. 2(b)] is the same as for (52). Therefore, the expression of anti-solitary wave solution is the same as for (55).

From the expressions of anti-solitary waves, we have the following limits:

(1) When $g \to -\frac{2}{3} - 0$, we have

$$\lim_{g \to -\frac{2}{3} - 0} u_\alpha(\xi, g) = u_2(\xi). \tag{61}$$

(2) When $g \to -\frac{2}{3} + 0$, we have

$$\lim_{g \to -\frac{2}{3} + 0} u_\alpha(\xi, g) = u_2(\xi) \tag{62}$$

and

$$\lim_{g \to -\frac{2}{3} + 0} u_\beta(\xi, g) = -1. \tag{63}$$

When $g \to -\frac{2}{3} + 0$, the evolution of wave profiles of $u_\alpha(\xi, g)$ and $u_\beta(\xi, g)$ is given in Fig. 8.

(3) When $g \to \frac{2}{3} - 0$, we have

$$\lim_{g \to \frac{2}{3} - 0} u_\alpha(\xi, g) = u_1(\xi) \tag{64}$$

and

$$\lim_{g \to \frac{2}{3} - 0} u_\beta(\xi, g) = u_1(\xi). \tag{65}$$

When $g \to \frac{2}{3} - 0$, the evolution of wave profiles of $u_\alpha(\xi, g)$ and $u_\beta(\xi, g)$ is given in Fig. 9.

To sum up, we have proved the following two theorems:

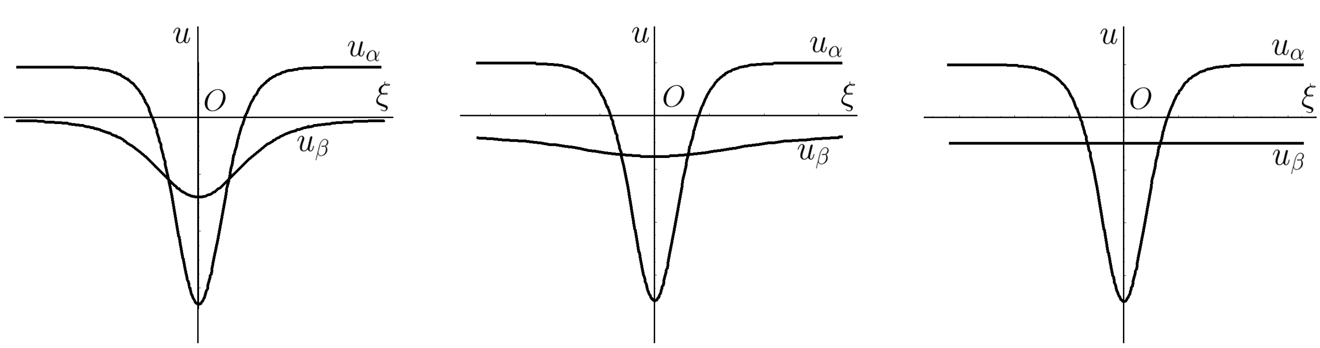
Theorem 1. *Consider the following three parametric conditions*:

(1) *For given* $g \in (-\infty, \infty)$, $h \in (-\infty, h_\rho)$;
(2) *For given* $g \in (-\frac{2}{3}, \frac{2}{3})$, $h \in (h_\beta, h_\gamma)$;
(3) *For given* $g = \frac{2}{3}$, $h \in (-\frac{5}{3}, \frac{4}{3})$.

If one of the three conditions holds, then Eq. (5) *has the family of periodic wave solutions as* (40).

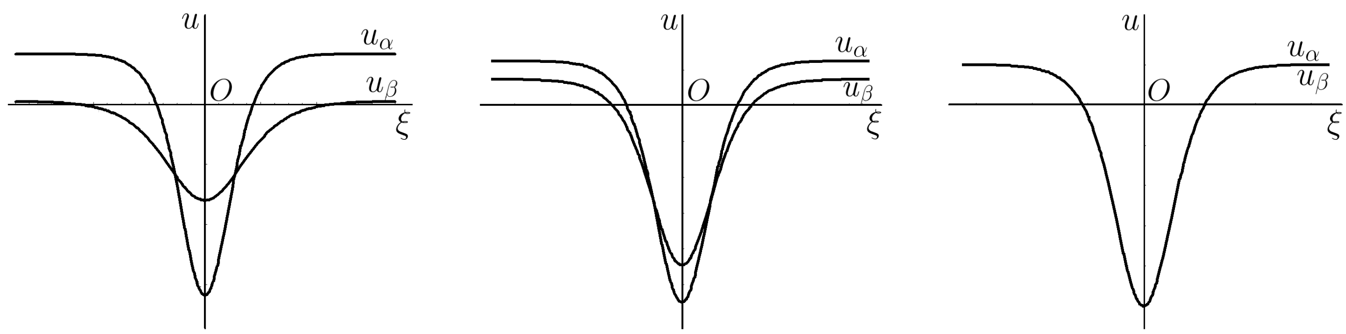
(4) *When* $g \to \frac{2}{3} + 0$, *an anti-solitary wave is bifurcated from a periodic wave* [*see* (48) *and Fig.* 7].

Theorem 2. *For given* g *and* $\xi = x - t$, *Eq.* (5) *has the following anti-solitary wave solutions and*



(a) $g = -\frac{2}{3} + 10^{-1}$ (b) $g = -\frac{2}{3} + 10^{-3}$ (c) $g = -\frac{2}{3} + 10^{-5}$

Fig. 8. The evolution of wave profiles of $u_\alpha(\xi, g)$ and $u_\beta(\xi, g)$ when $g \to -\frac{2}{3} + 0$.



(a) $g = \frac{2}{3} - 10^{-1}$ (b) $g = \frac{2}{3} - 10^{-3}$ (c) $g = \frac{2}{3} - 10^{-5}$

Fig. 9. The evolution of wave profiles of $u_\alpha(\xi, g)$ and $u_\beta(\xi, g)$ when $g \to \frac{2}{3} - 0$.

bifurcations:

(1) *If* $g = \frac{2}{3}$, *then Eq. (5) has only one anti-solitary wave solution* $u = u_1(\xi)$ *as in (51).*
(2) *If* $|g| < \frac{2}{3}$, *then Eq. (5) has two anti-solitary wave solutions* $u = u_\alpha(\xi, g)$ *and* $u = u_\beta(\xi, g)$ *as in (55) and (57).*
(3) *If* $g = -\frac{2}{3}$, *then Eq. (5) has only one anti-solitary wave solution* $u = u_2(\xi)$ *as in (60).*
(4) *If* $g < -\frac{2}{3}$, *then Eq. (5) has a unique anti-solitary wave solution whose expression is the same as with* $u_\alpha(\xi, g)$ *in (55).*
(5) $g = -\frac{2}{3}$ *is a parametric value of double anti-solitary waves bifurcation [see (62) and (63) and Fig. 8].*
(6) $g = \frac{2}{3}$ *is not only the parametric value of periodic wave bifurcation, but also that of double anti-solitary waves bifurcation [see (48), (64), (65) and Figs. 7 and 9].*

4. Conclusion

In this paper, we have investigated the expressions and bifurcations of anti-solitary traveling waves and periodic traveling waves [Eq. (5)], when wave speed equals to 1, because the corresponding traveling system (9) of Eq. (5) is a singular system with the singular straight line $\varphi = 1$ and on this line there exist two nodes of the associated regular system (11). Therefore, there exist no peakon and compacton solutions of Eq. (5). There exist smooth anti-solitary wave solutions and smooth periodic wave solutions of Eq. (5). Specially, we discover some new phenomena. (i) Infinitely many periodic orbits of the traveling wave system pass through the singular straight line. (ii) Inside some homoclinic orbits of the traveling wave system there is no singular point. (iii) There exist periodic wave bifurcation and double anti-solitary waves bifurcation.

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