Chapter 2 Matrix Algebra

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§ 2.9  Dimension and Rank
A basis and a spanning set

A spanning set of $\mathbb{R}^3$: $Span\{\overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3}, \overrightarrow{e_4}\}$, where

$$\overrightarrow{e_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \overrightarrow{e_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \overrightarrow{e_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \overrightarrow{e_4} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

A basis of $\mathbb{R}^3$ (standard basis)

$$\overrightarrow{e_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \overrightarrow{e_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \overrightarrow{e_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$
A basis and a spanning set

A spanning set of a subspace $H$ in $\mathbb{R}^n$.

1. It may be linearly dependent.
2. Each vector in $H$ may have many linear combination form.

A basis of a subspace $H$ in $\mathbb{R}^n$.

1. It must be linearly independent.
2. Each vector in $H$ have a unique linear combination form.
Coordinate Systems

The coordinates (坐标) relative to a basis

**Definition (Coordinates)**

Suppose the set $B = \{\vec{b}_1, ..., \vec{b}_p\}$ is a basis for a subspace $H$. For each $\vec{x}$ in $H$, the coordinates of $\vec{x}$ relative to the basis $B$ are the weights $c_1, ..., c_p$ such that $\vec{x} = c_1 \vec{b}_1 + \cdots + c_p \vec{b}_p$, and the vector in $\mathbb{R}^p$

$$[\vec{x}]_B = \begin{pmatrix} c_1 \\ \vdots \\ c_p \end{pmatrix}$$

is called the coordinate vector of $\vec{x}$ (relative to $B$) or the $B$-coordinate vector of $\vec{x}$. 
The coordinates relative to a basis

Example

Let \( \vec{v}_1 = \begin{pmatrix} 3 \\ 6 \\ 2 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \vec{x} = \begin{pmatrix} 3 \\ 12 \\ 7 \end{pmatrix}, \) and \( B = \{\vec{v}_1, \vec{v}_2\}. \)

1. Determine if \( B \) is a basis for \( H = \text{span}\{\vec{v}_1, \vec{v}_2\}. \)
2. Determine if \( \vec{x} \) is in \( H. \)
3. If \( B \) is a basis for \( H, \) find the coordinate vector of \( \vec{x} \) relative to \( B. \)
Solution: Obviously, $B$ is a basis of $H$ since $\vec{v}_1$ and $\vec{v}_2$ are linearly independent.

To determine if $\vec{x}$ is in $H$ is equivalent to that the equation

$$y_1 \vec{v}_1 + y_2 \vec{v}_2 = \vec{x}$$

is consistent, where $y_1, y_2$ are variables.

Applying row operations to the augmented matrix:

$$A = \begin{pmatrix} 3 & -1 & 3 \\ 6 & 0 & 12 \\ 2 & 1 & 7 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}.$$ 

Thus the equation is consistent and $\vec{x}$ is in $H$. 
From above echelon form it follows that $y_1 = 2$, $y_2 = 3$.

Therefore, $[x]_B = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$. 
1. Notice that although points in $H$ are also in $\mathbb{R}^3$, they are completely determined by their coordinate vectors, which belong to $\mathbb{R}^2$.

2. The correspondence $\vec{x} \mapsto [\vec{x}]_B$ is a one-to-one and onto mapping from $H$ to $\mathbb{R}^2$. 
The dimension of a subspace

1. It is clear that if a subspace $H$ has a basis of $n$ vectors, then every basis of $H$ must consist of exactly $n$ vectors.

2. For example, $\mathbb{R}^m$ has $m$ vectors $\vec{e}_1, \ldots, \vec{e}_m$ in the standard basis, thus any basis for $\mathbb{R}^m$ must consist of $m$ vectors.
The dimension of a subspace

Definition (The dimension of a subspace)

1. The dimension of a nonzero subspace $H$, denoted by $\dim H$, is the number of vectors in any basis for $H$.

2. The dimension of the zero subspace $\{\vec{0}\}$ is defined to be zero. The zero subspace has no basis.
Basis for a subspace

Example (Example 1)

The space $\mathbb{R}^n$ has dimension $n$.

Standard basis for $\mathbb{R}^n$:

$$
\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots, \quad \vec{e}_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.
$$
Basis for a subspace

Example (Example 2)

A plane through $\vec{0}$ in $\mathbb{R}^3$ is two-dimensional.
Example (Example 3)

A line through \( \vec{0} \) in \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \) is one-dimensional.

\[ \mathbf{v}_1 \neq \mathbf{0}, \mathbf{v}_2 = k \mathbf{v}_1. \]
Basis for a subspace

Example (Example 4)

For equation $A\vec{x} = \vec{0}$, the dimension of $\text{Null } A$ is exactly the number of free variables.
Questions:

1. What is the dimension of \( \text{Null } A \) of an invertible \( n \times n \) matrix \( A \)?

2. What is the dimension of \( \text{Col } A \) of an invertible \( n \times n \) matrix \( A \)?

3. What is the dimension of \( \text{Row } A \) (the row space) of an invertible \( n \times n \) matrix \( A \)?

4. What is the dimension of \( \text{span} \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\} \)?

5. The dimension of \( \text{span} \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right\} \)?
The rank of a matrix

Definition (Rank of a matrix (矩阵的秩))

The rank of a matrix $A$, denoted by $\text{rank } A$, is the dimension of the column space of $A$.

$$
B = \begin{pmatrix}
1 & 0 & -3 & 5 & 0 \\
0 & 1 & 2 & -1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
$$

The basis of $\text{Col } B$ is $\{ \vec{b}_1, \vec{b}_2, \vec{b}_5 \}$.

$$
\text{rank } B = \text{dim } (\text{Col } B) = 3
$$
The rank of a matrix

Example:

\[ C = \begin{pmatrix}
  3 & -7 & -2 & 2 \\
  0 & -2 & -1 & 2 \\
  0 & 0 & -1 & 1 \\
  0 & 0 & 0 & -1 \\
\end{pmatrix} \]

\[ \text{rank } C = \text{dim } (\text{Col } C) = 4 \]
The rank of a matrix

Example:

For the identity matrix $I_4$:

$$I_4 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}$$

$$\text{rank } I_4 = \dim (\text{Col } I_4) = 4$$
Proposition (Rank of a matrix)

The rank of a matrix $A$, is equivalent to the number of leading entries of the (reduced) echelon form of $A$.

$$A = \begin{pmatrix} 1 & -3 & -4 \\ -4 & 6 & -2 \\ -3 & 7 & 6 \end{pmatrix} \sim \begin{pmatrix} 1 & -3 & -4 \\ 0 & -6 & -18 \\ 0 & 0 & 0 \end{pmatrix}$$

\[ \text{rank } A = 2. \]
The rank of a matrix

Example (Rank of a matrix)

Find the rank of the following matrix $A$, and $\dim \ (Nul \ A)$:

$$A = \begin{pmatrix}
2 & 5 & -3 & -4 & 8 \\
4 & 7 & -4 & -3 & 9 \\
6 & 9 & -5 & 2 & 4 \\
0 & -9 & 6 & 5 & -6
\end{pmatrix}$$
The rank of a matrix

Solution:

\[
A = \begin{pmatrix}
2 & 5 & -3 & -4 & 8 \\
4 & 7 & -4 & -3 & 9 \\
6 & 9 & -5 & 2 & 4 \\
0 & -9 & 6 & 5 & -6 \\
0 & -9 & 6 & 5 & -6
\end{pmatrix}
\sim
\begin{pmatrix}
2 & 5 & -3 & -4 & 8 \\
0 & -3 & 2 & 5 & -7 \\
0 & 0 & 0 & 4 & -6 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
\text{rank } A = \dim(\text{Col } A) = 3, \quad \dim(\text{Nul } A) = 2.
\]
The rank of a matrix

Solution:

\[ A = \begin{pmatrix}
  2 & 5 & -3 & -4 & 8 \\
  4 & 7 & -4 & -3 & 9 \\
  6 & 9 & -5 & 2 & 4 \\
  0 & -9 & 6 & 5 & -6
\end{pmatrix} \sim \begin{pmatrix}
  2 & 5 & -3 & -4 & 8 \\
  0 & -3 & 2 & 5 & -7 \\
  0 & 0 & 0 & 4 & -6 \\
  0 & 0 & 0 & 0 & 0
\end{pmatrix} \]

\[ \text{rank } A = \text{dim } (\text{Col } A) = 3, \quad \text{dim } (\text{Nul } A) = 2. \]
The rank of a matrix

**Proposition (Rank of a matrix)**

*If a matrix $A$ has $n$ columns, then*

$$\text{rank } A + \dim Nul A = n.$$  

*Note: Here $A$ can be an $m \times n$ matrix.*
The rank of a matrix

For an $m \times n$ matrix,

the number of free variables ($\dim \text{Nul} \ A$) 
+ 
the number of leading entries of the echelon form ($\text{rank} \ A$) 
= $n$.

$$A = \begin{pmatrix}
2 & 5 & -3 & -4 & 8 \\
0 & -3 & 2 & 5 & -7 \\
0 & 0 & 0 & 4 & -6 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

The number of free variables $= 5 - 3 = 2$.

The number of leading entries of the echelon form $= 3$. 
In an reduced echelon form of $n \times n$ matrix $A$, the number of zero rows ($\dim \text{Nul } A$) plus the number of nonzero rows ($\text{rank } A$) equals $n$. 

$$A = \begin{pmatrix}
1 & 0 & -2 & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}$$
The dimension of a subspace

Example (Basis for $\text{Col } A$ and $\text{Nul } A$)

If the matrix $A$ has the following echelon form. Find bases for $\text{Col } A$ and $\text{Nul } A$, and give the dimensions of these subspaces.

\[
A = \begin{bmatrix}
1 & 3 & 2 & -6 \\
3 & 9 & 1 & 5 \\
2 & 6 & -1 & 9 \\
5 & 15 & 0 & 14
\end{bmatrix} \sim \begin{bmatrix}
1 & 3 & 3 & 2 \\
0 & 0 & 5 & -7 \\
0 & 0 & 0 & 5 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]
The dimension of a subspace

Solution:

\[
A = \begin{bmatrix}
1 & -3 & 2 & -4 \\
-3 & 9 & -1 & 5 \\
2 & -6 & 4 & -3 \\
-4 & 12 & 2 & 7 \\
\end{bmatrix}
\sim
\begin{bmatrix}
1 & -3 & 2 & -4 \\
0 & 0 & \textcircled{5} & -7 \\
0 & 0 & 0 & \textcircled{5} \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

The leading entries are respectively in column 1, 3 and 4. So the bases for \textit{Col} \ A are

\[
\begin{bmatrix}
1 \\
-3 \\
2 \\
-4 \\
\end{bmatrix}, \begin{bmatrix}
2 \\
-1 \\
4 \\
2 \\
\end{bmatrix}, \begin{bmatrix}
-4 \\
5 \\
-3 \\
7 \\
\end{bmatrix}
\]
The dimension of a subspace

In order to find a basis of $\text{Nul } A$, use the augmented matrix of $A\vec{x} = \vec{0}$:

$$
\begin{bmatrix}
1 & -3 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}.
$$

\[ x_1 - 3x_2 = 0 \]
\[ x_3 = 0 \]
\[ x_4 = 0. \]

$x_2$ is the free variable.

$$
x = 
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix}
= 
\begin{bmatrix}
3x_2 \\
x_2 \\
0 \\
0
\end{bmatrix}
= x_2 
\begin{bmatrix}
3 \\
1 \\
0 \\
0
\end{bmatrix}.
$$
The dimension of a subspace

So the vector

\[
\begin{pmatrix}
3 \\
1 \\
0 \\
0
\end{pmatrix}
\]

is the basis for \(\text{Nul } A\).

From the above calculation, we have that

\[
\dim \text{Col } A = 3
\]

and

\[
\dim \text{Nul } A = 1.
\]
The dimension of a subspace

Theorem (The basis theorem)

Let $H$ be a $p$-dimensional subspace of $\mathbb{R}^n$.

1. Any linearly independent set of exactly $p$ vectors in $H$ is automatically a basis for $H$.

2. Also, any set of $p$ vectors of $H$ that spans $H$ is automatically a basis for $H$.

Vector set $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$ is a basis for $\mathbb{R}^2$. 
Rank and its propositions about the invertible matrices:

**Theorem (The invertible matrix theorem)**

Let $A$ be an $n \times n$ square matrix. Then the following statements are equivalent:

1. $A$ is an invertible matrix.
2. The columns of $A$ form a basis of $\mathbb{R}^n$.
3. $\text{Col } A = \text{Span}\{\vec{a}_1, \vec{a}_2, ..., \vec{a}_n\} = \mathbb{R}^n$.
4. $\dim \text{Col } A = n$, \hspace{1cm} (rank $A = n$).
5. $\text{Nul } A = \{\vec{0}\}$.
6. $\dim \text{Nul } A = 0$. 

Subspaces of $\mathbb{R}^n$

**Exercise:** If $A$ is an $m \times n$ matrix:

a. If $\mathcal{B} = \{v_1, \ldots, v_p\}$ is a basis for a subspace $H$ and if $x = c_1v_1 + \cdots + c_pv_p$, then $c_1, \ldots, c_p$ are the coordinates of $x$ relative to the basis $\mathcal{B}$.

b. Each line in $\mathbb{R}^n$ is a one-dimensional subspace of $\mathbb{R}^n$.

c. The dimension of $\text{Col } A$ is the number of pivot columns in $A$.

d. The dimensions of $\text{Col } A$ and $\text{Nul } A$ add up to the number of columns in $A$.

e. If a set of $p$ vectors spans a $p$-dimensional subspace $H$ of $\mathbb{R}^n$, then these vectors form a basis for $H$. 

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Subspaces of $\mathbb{R}^n$

Exercise:

a. True
b. False
c. True
d. True
e. True
Subspaces of $\mathbb{R}^n$

Exercise:

a. If $\mathcal{B}$ is a basis for a subspace $H$, then each vector in $H$ can be written in only one way as a linear combination of the vectors in $\mathcal{B}$.

b. The dimension of Nul $A$ is the number of variables in the equation $Ax = 0$.

c. The dimension of the column space of $A$ is rank $A$.

e. If $H$ is a $p$-dimensional subspace of $\mathbb{R}^n$, then a linearly independent set of $p$ vectors in $H$ is a basis for $H$. 
Subspaces of $\mathbb{R}^n$

Exercise:

a. True
b. False
c. True
e. True
Homework:

Section 2.9

p. 165: 5, 6;

p. 166: 10, 12, 13;