Acta Mathematicae Applicatae Sinica, English Series
Vol. 32, No. 2 (2016) 513–528
DOI: 10.1007/s10255-016-0581-x
http://www.ApplMath.com.cn & www.SpringerLink.com

Acta Mathematicae Applicatae Sinica, English Series

The Explicit Periodic Wave Solutions and Their Limit Forms for a Generalized b-equation

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Abstract In this paper, for $b \in (-\infty, \infty)$ and $b \neq -1, -2$, we investigate the explicit periodic wave solutions for the generalized b-equation

$$u_t + 2ku_x - u_{xxt} + (1+b)u^2 u_x = bu_x u_{xx} + uu_{xxx},$$

which contains the generalized Camassa-Holm equation and the generalized Degasperis-Procesi equation. Firstly, via the methods of dynamical system and elliptic integral we obtain two types of explicit periodic wave solutions with a parametric variable α. One of them is made of two elliptic smooth periodic wave solutions. The other is composed of four elliptic periodic blow-up solutions. Secondly we show that there exist four special values for α. When α tends to these special values, these above solutions have limits. From the limit forms we get other three types of nonlinear wave solutions, hyperbolic smooth solitary wave solution, hyperbolic single blow-up solution, trigonometric periodic blow-up solution. Some previous results are extended. For $b = -1$ or $b = -2$, we guess that the equation does not have any one of above solutions.

Keywords generalized b-equation; elliptic integral method; explicit periodic wave solutions; limit formes
2000 MR Subject Classification 34A20; 34C35; 35B65; 58F05

1 Introduction

Degasperis, Holm and Hone[5,6] deduced a nonlinear equation

$$u_t + 2ku_x - u_{xxt} + (1+b)uu_x = bu_x u_{xx} + uu_{xxx}, \tag{1.1}$$

which is called b-equation.

When $b = 2, 3$ respectively, Eq.(1.1) changes into the CH equation

$$u_t + 2ku_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx} \tag{1.2}$$

and DP equation

$$u_t + 2ku_x - u_{xxt} + 4uu_x = 3u_x u_{xx} + uu_{xxx}. \tag{1.3}$$

For various values of b, Holm and Staley[13] studied the solutions of Eq.(1.1) numerically. Guo and Liu[10] investigated the periodic cusp waves and single-solitons of Eq.(1.1). Escher and Seiler[8] discussed the relation between Eq.(1.1) and Euler equation. Vitanov et al.[26] obtained some exact traveling wave solutions for Eq.(1.1).

When $k = 0$, Camassa and Holm[1] showed that Eq.(1.2) is integrable and has peakon solution. Constantin[3], Constantin and Strauss[4] presented the mathematical description of

Manuscript received January 28, 2014. Revised April 8, 2014.
Supported by the National Natural Science Foundation of China (No.11401222), Natural Science Foundation of Guangdong Province (No.S2012040007959) and The Fundamental Research Funds for the Central Universities (No.2014ZZ0064) and Pearl River Science and Technology Nova Program of Guangzhou.
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the existence of interacting solitary waves and demonstrated that the peakons are stable for Eq.(1.2) with $k = 0$. When $k \neq 0$, Liu and Qian[20] certified that Eq.(1.2) has peakons whose stability was showed by Ouyang et al.[23].

Degasperis and Procesi[7] showed that Eq.(1.3) is integrable and has peakons. Lundmark and Szmigielski[21,22] gave an inverse scattering approach for computing the n-peakon solutions of Eq.(1.3) and presented the concrete expressions of the 3-peakon solutions. Chen and Tang[2] studied the kink-like waves for Eq.(1.3). Guha[9] gave an Euler-poincare formalism to Eq.(1.3).

Based on Eq.(1.2), Liu and Qian[19] proposed the generalized CH equation

$$u_t + 2ku_x - u_{xxt} + 3u^2u_x - 2u_xu_{xx} - uu_{xxx} = 0. \tag{1.4}$$

Similarly, Wazwaz[28,29] suggested the generalized DP equation

$$u_t + 2ku_x - u_{xxt} + 4u^2u_x - 3u_xu_{xx} - uu_{xxx} = 0, \tag{1.5}$$

and generalized b-equation

$$u_t + 2ku_x - u_{xxt} + (b+1)u^2u_x = bu_xu_{xx} + uu_{xxx}. \tag{1.6}$$

Tian and Song[25] gave physical explanations to Eq.(1.4). Shen and Xu[24] studied the existence on both smooth and non-smooth traveling waves for Eq.(1.4). Khuri[14] investigated the singular wave solution composed of trigonometric functions for Eq.(1.4). Liu and Ouyang[18] obtained a peakon solution for Eq.(1.4). He, et al.[11] used the integral bifurcation method to study the exact solutions for Eq.(1.4). Zhang et al.[32] researched bifurcation of smooth and nonsmooth traveling wave solutions for Eq.(1.4). Yomba[30,31] presented two methods to look for the exact traveling wave solutions for Eq.(1.4) and Eq.(1.5). Wang and Tang[27] studied the explicit periodic wave solutions and their bifurcations for Eq.(1.4). He et al.[12] obtained some exact solutions for Eq.(1.5). When $k = 0$ Liu and Pan[17] investigated multifarious explicit nonlinear wave solutions for Eq.(1.4) and Eq.(1.5). When $k = 0$, Liu[15] confirmed the coexistence of multifarious exact solutions for Eq.(1.6). When $k \neq 0$ and $b > 1$, Liu[16] obtained five types of solutions, hyperbolic smooth solitary wave solution, hyperbolic peakon wave solution, hyperbolic blow-up solution, fractional peakon wave solution and fractional blow-up solution for Eq.(1.6).

In this paper, we employ the methods of dynamical systems and elliptic integral to study the explicit periodic wave solutions and their limit forms for Eq.(1.6). For $b \in (-\infty, \infty)$ and $b \neq -1, -2$, we obtain six explicit expressions of smooth periodic wave and periodic blow-up waves with a parametric variable α. When α tends to some fixed values, we get smooth solitary wave solution and single blow-up solutions. Our work extends some previous results.

This pager is arranged as follows. Our main results are listed in Section 2. In Section 3 we give derivations to our main results. In Section 4 we make a short conclusion.

2 Main Results

In this section, we expound the expressions of explicit periodic wave solutions and their limit forms for Eq.(1.6). Our main results are made of the following two lemmas and three propositions.

Lemma 1. *For given real coefficients b, k, and constant wave speed c in Eq.(1.6), let*

$$c_1 = \frac{1 + b - \sqrt{(1+b)(1+b-8k)}}{2} \tag{2.1}$$