International Journal of Bifurcation and Chaos, Vol. 20, No. 10 (2010) 3193–3208

DOI: 10.1142/S0218127410027623

COEXISTENCE OF MULTIFARIOUS EXACT NONLINEAR WAVE SOLUTIONS FOR GENERALIZED *b*-EQUATION*

RUI LIU
Department of Mathematics,
South China University of Technology,
Guangzhou 510640, P. R. China
scliurui@scut.ed.cn

Received December 23, 2009; Revised January 11, 2010

In this paper, we consider the generalized b-equation $u_t - u_{xxt} + (b+1)u^2u_x = bu_xu_{xx} + u_{xxx}$. For a given constant wave speed, we investigate the coexistence of multifarious exact nonlinear wave solutions including smooth solitary wave solution, peakon wave solution, smooth periodic wave solution, single singular wave solution and periodic singular wave solution. Not only is the coexistence shown, but the concrete expressions are given via phase analysis and special integrals. From our work, it can be seen that the types of exact nonlinear wave solutions of the generalized b-equation are more than that of the b-equation. Many previous results are turned to our special cases. Also, some conjectures and questions are presented.

Keywords: Generalized b-equation; phase analysis; special integrals; coexistence of multifarious exact solutions.

1. Introduction

In 2002, Degasperis *et al.* [2002a, 2002b] introduced a type of nonlinear partial differential equation

$$u_t - u_{xxt} + (b+1)uu_x = bu_xu_{xx} + uu_{xxx}, \quad (1)$$

which is called the b-equation. When $b = 2$, the b-equation becomes the CH equation. Camassa and Holm [1993] showed that the CH equation is integrable and has peaked solitons. Cooper and Shepard [1994] derived an approximate solitary wave solution of the CH equation by using some variational functions. Constantin [1997], Constantin and Strauss [2000] gave the mathematical description of the existence of interacting solitary waves and showed that the peakons are stable for the CH equation. Boyd [1997] derived a perturbation series which converges even at the peakon limit, and gave three analytical representations for the spatially periodic generalization of the peakon which is called "the coshoidal wave" in the CH equation. Recently, the CH equation has been studied successively by many authors (e.g. [Johnson, 2002; Lenells, 2002; Liu *et al.* 2004, 2006; Reyes, 2002]).

When $b = 3$, the b-equation becomes the DP equation which was given by Degasperis and Procesi [1999]. Lundmark and Szmigielski [2003, 2005] presented an inverse scattering approach for computing the n-peakon solutions of the DP equation and gave the concrete expressions of the 3-peakon solutions. Chen and Tang [2006] showed that the DP equation has kink-like waves.

The solutions of the b-equation were studied numerically for various values of b by Holm and Staley [2003]. For arbitrary $b > 1$, Guo and Liu [2005] showed that Eq. (1) has periodic cusp waves

*Research is supported by the National Natural Science Foundation of China (No. 10871073).

and constructed their presentations. Guha [2007] proposed an Euler–Poincare formalism of the DP equation.

To study the bifurcation of the peakon waves, Liu and Qian [2001] suggested a generalized CH equation

$$u_t - u_{xxt} + 3u^2 u_x = 2u_x u_{xx} + u u_{xxx}. \tag{2}$$

Similarly, to investigate the change of peakon waves, Wazwaz [2006, 2007] proposed a generalized DP equation

$$u_t - u_{xxt} + 4u^2 u_x = 3u_x u_{xx} + u u_{xxx} \tag{3}$$

and a generalized b-equation

$$u_t - u_{xxt} + (b+1)u^2 u_x = b u_x u_{xx} + u u_{xxx}. \tag{4}$$

Since the CH and the DP equations possess rich structure and properties, many authors were interested in their modified forms, Eqs. (2)–(4). Tian and Song [2004] gave some physical explanation for Eq. (2). Shen and Xu [2005] discussed the existence of smooth and nonsmooth traveling waves for Eq. (2). Letting c denote the constant wave speed of traveling waves, for some special values of c, the exact traveling wave solutions were studied for Eqs. (2) and (3). When $c = 1$, Khuri [2005] obtained a singular wave solution composed of triangle functions for Eq. (2). When $c = 1$ and $c = 2$ respectively, Wazwaz [2007] obtained eleven exact traveling wave solutions composed of triangle functions or hyperbolic functions for Eq. (2), and Liu and Ouyang [2007] got a peakon solution composed of hyperbolic functions for Eq. (2). He *et al.* [2008a] used the integral bifurcation method to obtain some exact solutions for Eq. (2). Liu and Guo [2008] investigated the periodic blow-up solutions and their limit forms for Eq. (2). When $c = 5/2$, Wazwaz [2007] obtained nine exact traveling wave solutions composed of hyperbolic functions for Eq. (3). Besides, Liu and Ouyang [2007] got a peakon solution composed of hyperbolic functions for Eq. (3). Zhang *et al.* [2007] used the bifurcation theory of dynamical systems to show the existence of some traveling waves for Eq. (2). Wang and Tang [2008] obtained two exact solutions for Eq. (2) when $c = 1/3$ and $c = 3$ respectively, and gave two exact solutions for Eq. (3) when $c = 1/4$ and $c = 4$, respectively. Yomba [2008a, 2008b] presented two methods, the sub-ODE method and the generalized auxiliary equation method, to find the exact traveling wave solutions for Eqs. (2) and (3). He *et al.* [2008b] used the bifurcation method of dynamical systems to obtain some exact solutions for Eq. (3). Liu and Pan [2009] studied the coexistence of multifarious explicit nonlinear wave solutions for Eqs. (2) and (3).

When the wave speed $c = (2+b)/2$, Wazwaz [2007] obtained two solitary wave solutions for Eq. (4). When $c = 1/(1+b)$, $(2+b)/2$, $1+b$ respectively, Liu [2010] investigated the solitary wave solution for Eq. (4).

In this paper, we use the phase analysis of planar systems and special integrals to study the coexistence of multifarious exact nonlinear wave solutions for Eq. (4). We will show that when the constant wave speed c satisfies $0 < c < b+1$, multifarious exact nonlinear wave solutions coexist for Eq. (4). We will also give the concrete expressions of these solutions and test their correctness by using the software Mathematica. Some previous results (e.g. [He *et al.*, 2008a, 2008b; Khuri, 2005; Liu, 2010; Liu & Guo, 2008; Liu & Ouyang, 2007; Liu & Pan, 2009; Shen & Xu, 2005; Tian & Song, 2004; Wang & Tang, 2008; Wazwaz, 2006, 2007; Yomba, 2008a, 2008b; Zhang *et al.*, 2007]) become our special cases.

This paper is organized as follows. In Sec. 2, we state our main results. In Sec. 3, we give some preliminaries. In Sec. 4, we give the demonstrations to our main results. A conclusion is given in Sec. 5.

2. Main Results

In this paper, we suppose the parameter $b > 1$. Under this supposition it follows that

$$0 < \frac{1}{1+b} < \frac{4(1+b)}{4+2b+b^2} < 1+b. \tag{5}$$

Thus for the constant wave speed $c \in (0, 1+b)$ we have some results in the following five propositions.

Proposition 1. *If the wave speed c satisfies $0 < c < 1+b$ and $c \neq 1/(1+b)$, then four types of exact nonlinear wave solutions coexist for Eq. (4). These exact solutions are hyperbolic smooth solitary wave solution, hyperbolic peakon wave solution, hyperbolic singular wave solution and trigonometric periodic singular wave solution respectively. Denote*

$$\xi = x - ct, \tag{6}$$

$$p = \sqrt{b(2+b)(1+b-c)c}, \tag{7}$$

$$\alpha = \sqrt{\frac{p}{2b(2+b)}}. \tag{8}$$

Then the explicit expressions of these solutions are as follows:

(i) *The hyperbolic smooth solitary wave solution*

$$u_1(\xi, c) = \frac{1}{b(b+1)}[p - bc - 3p\,\mathrm{sech}^2\,\alpha\xi]. \tag{9}$$

(ii) *The hyperbolic peakon wave solution*

$$u_2(\xi, c) = \frac{1}{b(b+1)}\left[p - bc + \frac{3p(2bc + b^2c - p)}{(\sqrt{3p}\cosh\alpha\xi + \sqrt{2bc + b^2c + 2p}\sinh|\alpha\xi|)^2}\right]. \tag{10}$$

(iii) *The hyperbolic singular wave solution*

$$u_3(\xi, c) = \frac{1}{b(b+1)}(p - bc + 3p\,\mathrm{csch}^2\alpha\xi). \tag{11}$$

(iv) *The trigonometric periodic singular wave solution*

$$u_4(\xi, c) = \frac{1}{b(b+1)}(2p - bc + 3p\tan^2\alpha\xi). \tag{12}$$

For the figures of $u = u_i(\xi, c)$ $(i = 1, 2, 3, 4)$ with $b = 4$ and $c = 1/2$, see Figs. 1(a)–1(d).

Remark 1. Since $u_4(\xi + \pi/2, c)$ is also a solution of Eq. (4) and $1 + \tan^2\xi = \csc^2\xi$, the following functions

$$u_4^{\wedge}(\xi, c) = \frac{1}{b(1+b)}[2p - bc + 3p\cot^2\alpha\xi], \tag{13}$$

$$u_4^{*}(\xi, c) = \frac{1}{b(1+b)}[-p - bc + 3p\csc^2\alpha\xi], \tag{14}$$

and

$$u_4^{\oplus}(\xi, c) = \frac{1}{b(1+b)}[-p - bc + 3p\sec^2\alpha\xi] \tag{15}$$

are the solutions of Eq. (4) too.

Proposition 2. *If the wave speed c satisfies $0 < c < 4(1+b)/(4+2b+b^2)$ and $c \neq 1/(1+b)$, then six types of exact nonlinear wave solutions coexist for Eq. (4). These exact solutions are hyperbolic smooth solitary wave solution $u_1(\xi, c)$, hyperbolic*

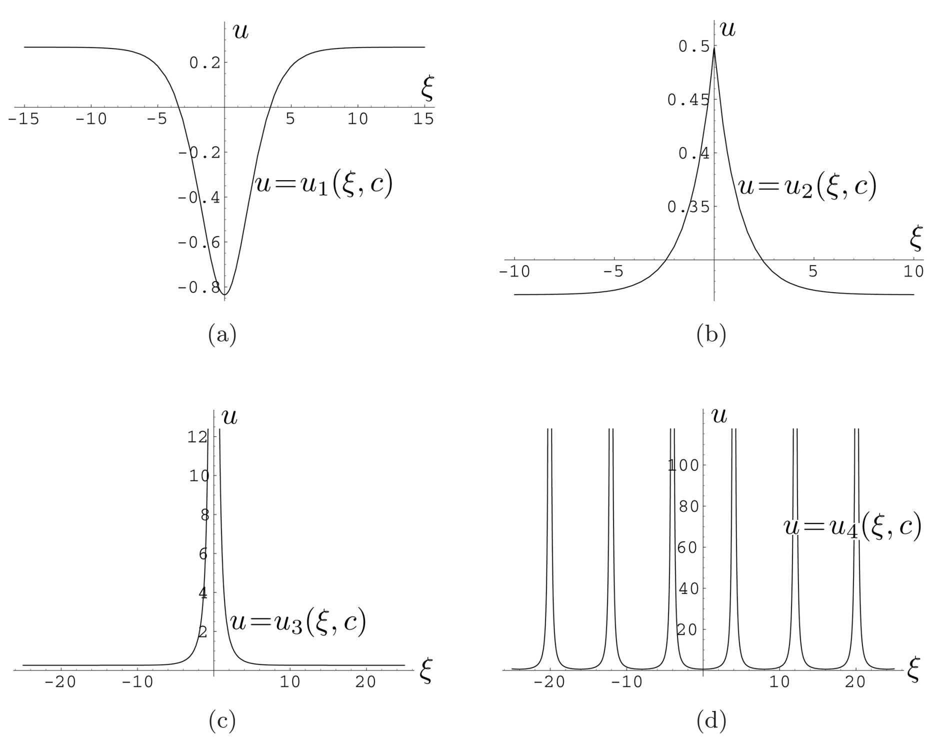


Fig. 1. The figures of $u = u_i(\xi, c)$ $(i = 1, 2, 3, 4)$ with $b = 4$ and $c = 1/2$.

peakon wave solution $u_2(\xi, c)$, *hyperbolic singular wave solution* $u_3(\xi, c)$, *trigonometric periodic singular wave solution* $u_4(\xi, c)$, *elliptic smooth periodic wave solution*

$$u_5(\xi,c) = \begin{cases} a_1 + a_2\,\mathrm{sn}^2(\beta\,\xi,\ k_1) \\ \qquad \text{for } 0 < c < \dfrac{1}{1+b}, \\ a_1 + a_3\,\mathrm{sn}^2(\gamma\,\xi,\ k_2) \\ \qquad \text{for } \dfrac{1}{1+b} < c < \dfrac{4(1+b)}{4+2b+b^2}, \end{cases} \tag{16}$$

and elliptic periodic singular wave solution

$$u_6(\xi,c) = \begin{cases} a_1 + a_3\,\mathrm{sn}^{-2}(\beta\,\xi,\ k_1) \\ \qquad \text{for } 0 < c < \dfrac{1}{1+b}, \\ a_1 + a_2\,\mathrm{sn}^{-2}(\gamma\,\xi,\ k_2) \\ \qquad \text{for } \dfrac{1}{1+b} < c < \dfrac{4(1+b)}{4+2b+b^2}, \end{cases} \tag{17}$$

where

$$q = \sqrt{3bc(2+b)(4+4b-4c-2bc-b^2c)}, \tag{18}$$

$$a_1 = -\frac{bc(4+b)+q}{2b(1+b)}, \tag{19}$$

$$a_2 = \frac{3bc(2+b)+q}{2b(1+b)}, \tag{20}$$

$$a_3 = \frac{q}{b(1+b)}, \tag{21}$$

$$k_1 = \frac{3bc(2+b)+q}{2q}, \tag{22}$$

$$k_2 = \frac{2q}{3bc(2+b)+q}, \tag{23}$$

$$\beta = \sqrt{\frac{q}{6b(2+b)}}, \tag{24}$$

and

$$\gamma = \sqrt{\frac{3bc(2+b)+q}{12b(2+b)}}. \tag{25}$$

Proposition 3. *If the wave speed* $c = 1/(1+b)$, *then three types of exact nonlinear wave solutions coexist for Eq. (4). These solutions are as follows*:

(i) *Smooth solitary wave solution*

$$u_1^\circ(x,t) = \frac{1}{1+b} - \frac{3(2+b)}{(1+b)^2}\,\mathrm{sech}^2\sqrt{\frac{1}{2(1+b)}}\left(x - \frac{t}{1+b}\right). \tag{26}$$

(ii) *Hyperbolic singular wave solution*

$$u_3^\circ(x,t) = \frac{1}{1+b} + \frac{3(2+b)}{(1+b)^2}\,\mathrm{csch}^2\sqrt{\frac{1}{2(1+b)}}\left(x - \frac{t}{1+b}\right). \tag{27}$$

(iii) *Trigonometric periodic singular wave solution*

$$u_4^\circ(x,t) = \frac{1}{(1+b)^2}\left[3+2b+3(2+b)\tan^2\sqrt{\frac{1}{2(1+b)}}\left(x - \frac{t}{1+b}\right)\right]. \tag{28}$$

Remark 2. The solutions $u_1^\circ(x,t)$ and $u_3^\circ(x,t)$ had been obtained in [Liu, 2010], also the following results:

If the wave speed $c = 1+b$, then two types of exact nonlinear wave solutions coexist for Eq. (4). These solutions are peakon wave solution

$$u_7(x,t) = \frac{6(2+b)}{(\sqrt{6}+\sqrt{1+b}\,|x-(1+b)t|)^2} - 1, \tag{29}$$

and singular wave solution

$$u_8(x,t) = \frac{6(2+b)}{(1+b)(x-(1+b)t)^2} - 1. \tag{30}$$

Proposition 4. *For these solutions, there are the following relations or properties.*

(1) *When* c *tends to* $1/(1+b)$, *it follows that*:

(i) *The hyperbolic smooth solitary wave solution* $u_1(\xi, c)$ *becomes* $u_1^\circ(x,t)$, *that is, the hyperbolic smooth solitary wave persists.*

(ii) *The hyperbolic peakon wave solution* $u_2(\xi, c)$ *becomes a constant solution* $u = 1/(1+b)$, *that is, the peakon wave disappears.*

(iii) *The hyperbolic singular wave solution* $u_3(\xi, c)$ *becomes the hyperbolic singular wave solution* $u_3^\circ(x, t)$, *that is, the hyperbolic singular wave persists.*

(iv) *The elliptic smooth periodic wave solution* $u_5(\xi, c)$ *becomes the hyperbolic smooth solitary wave solution* $u_1^\circ(x, t)$. *For the varying figures with* $b = 4$ *and* $c = 0.1$, 0.19, 0.1999, 0.2001, 0.21, 0.3 *respectively, see Figs. 2(a)–2(f).*

(v) *The elliptic periodic singular wave solution* $u_6(\xi, c)$ *becomes the hyperbolic singular wave solution* $u_3^\circ(x, t)$. *For the varying figures with* $b = 4$ *and* $c = 0.1$, 0.19, 0.1999, 0.2001, 0.21, 0.3 *respectively, see Figs. 3(a)–3(f).*

(vi) *The trigonometric periodic singular wave solution* $u_4(\xi, c)$ *becomes the trigonometric periodic singular wave solution* $u_4^\circ(x, t)$, *that is, the trigonometric periodic singular wave persists.*

(2) *When* c *tends to* $4(1+b)/(4+2b+b^2)$, *it follows that*:

(vii) *The smooth periodic wave solution* $u_5(\xi, c)$ *becomes a constant solution* $u = -2(4+b)/(4+2b+b^2)$.

(viii) *The periodic singular wave solution* $u_6(\xi, c)$ *becomes a trigonometric periodic singular wave solution*

$$u_9(x,t) = \frac{2}{\delta}\left[2(1+b) + 3(2+b) \times \tan^2 \sqrt{\frac{1+b}{\delta}}\left(x - \frac{4(1+b)t}{\delta}\right)\right], \tag{31}$$

where

$$\delta = 4 + 2b + b^2. \tag{32}$$

(3) *When* c *tends to* $1+b$, *the four solutions* $u_i(\xi, c)$ $(i = 1, 2, 3, 4)$ *become a constant solution* $u = -1$.

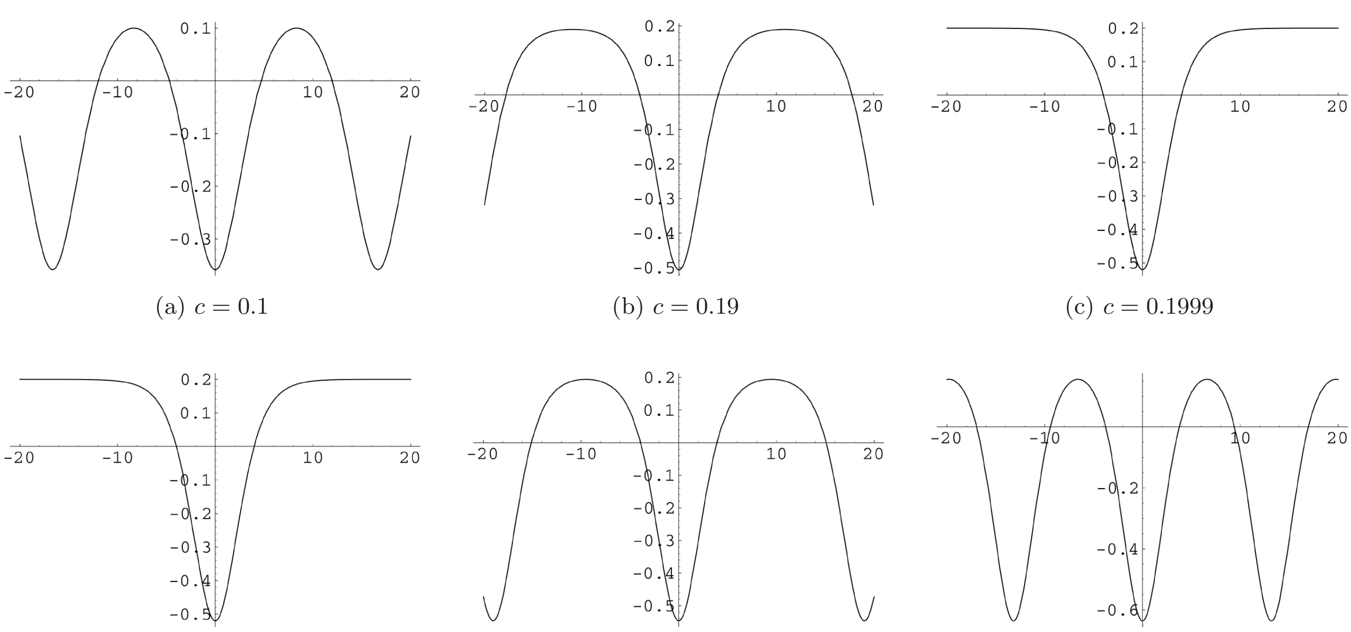
Remark 3. Using the software Mathematica, we have tested the correctness of these solutions. For instance, testing $u_1(\xi, c)$ with $b = 4$, the orders are as follows (testing other solutions, the orders are the same):

$$b = 4$$

$$p = \sqrt{b(2+b)(1+b-c)c}$$

$$q = \frac{1}{b(1+b)}$$

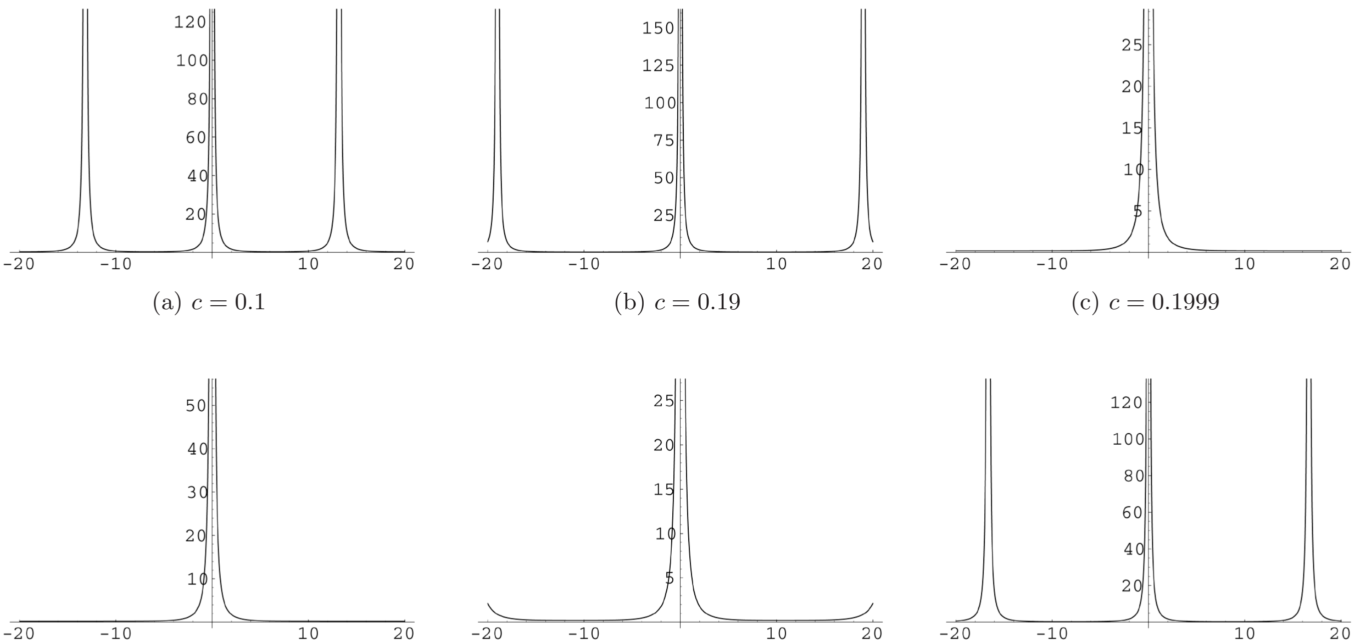
$$z = \sqrt{\frac{p}{2b(2+b)}}(x - c\,t)$$



(a) $c = 0.1$ (b) $c = 0.19$ (c) $c = 0.1999$

(d) $c = 0.2001$ (e) $c = 0.21$ (f) $c = 0.3$

Fig. 2. The varying figures of $u_5(\xi, c)$ when $b = 4$ and c tends to $1/(1+b) = 0.2$.



(a) $c = 0.1$ (b) $c = 0.19$ (c) $c = 0.1999$

(d) $c = 0.2001$ (e) $c = 0.21$ (f) $c = 0.3$

Fig. 3. The varying figures of $u_6(\xi, c)$ when $b = 4$ and c tends to $1/(1+b) = 0.2$.

$$u1 = q(p - bc - 3\,p\,\mathrm{Sech}\,[z]^2)$$

$$u = u1$$

Simplify $[D[u,t] - D[D[u,t],\{x,2\}] + (b+1)*u*u*D[u,x] - b*D[u,x]*D[u,\{x,2\}] - u*D[u,\{x,3\}]]$.

3. Preliminaries

For a given constant $c > 0$, substituting $u = \varphi(\xi)$ into Eq. (4) with $\xi = x - ct$, it follows that

$$-c\varphi' + c\varphi''' + (1+b)\varphi^2\varphi' = b\varphi'\varphi'' + \varphi\varphi'''. \quad (33)$$

Integrating (33) once, we have

$$(\varphi - c)\varphi'' = g - c\varphi + \frac{1+b}{3}\varphi^3 - \frac{b-1}{2}(\varphi')^2, \quad (34)$$

where g is the constant of integration.

Letting $y = \varphi'$, it yields the following planar system

$$\begin{cases} \dfrac{d\varphi}{d\xi} = y, \\[2ex] \dfrac{dy}{d\xi} = \dfrac{g - c\varphi + \dfrac{1+b}{3}\varphi^3 - \dfrac{b-1}{2}y^2}{\varphi - c}. \end{cases} \quad (35)$$

By using the transformation $d\tau = d\xi/(\varphi - c)$, (35) can be written as the planar system

$$\begin{cases} \dfrac{d\varphi}{d\tau} = (\varphi - c)y, \\[2ex] \dfrac{dy}{d\tau} = g - c\varphi + \dfrac{1+b}{3}\varphi^3 - \dfrac{b-1}{2}y^2. \end{cases} \quad (36)$$

Let

$$a_0 = \frac{6g - 6c^2 + 2(1+b)c^3}{3(b-1)}, \quad (37)$$

$$a_1 = \frac{2(1+b)c^2 - 2c}{b}, \quad (38)$$

$$a_2 = 2c, \quad (39)$$

$$a_3 = \frac{2(1+b)}{3(2+b)}, \quad (40)$$

and

$$H(\varphi, y) = (\varphi - c)^{b-1}[a_0 + a_1(\varphi - c) + a_2(\varphi - c)^2 + a_3(\varphi - c)^3 - y^2]. \quad (41)$$

It is easy to check that

$$H(\varphi, y) = h \quad (42)$$

is the first integration for both systems (35) and (36). Therefore, both systems (35) and (36) have the same topological phase portraits except the line $\varphi = c$. This implies that one can study the phase portraits of system (35) from that of system (36).

Now we begin to study the bifurcation phase portraits of system (36). Let

$$z = f(\varphi), \tag{43}$$

where

$$f(\varphi) = g - c\varphi + \frac{1+b}{3}\varphi^3. \tag{44}$$

We have

$$f'(\varphi) = (1+b)\varphi^2 - c. \tag{45}$$

When $g = 0$, it follows

$$f\left(\pm\sqrt{\frac{c}{1+b}}\right) = \mp\frac{2c}{3}\sqrt{\frac{c}{1+b}}. \tag{46}$$

Let

$$\varphi^* = \sqrt{\frac{3c}{1+b}}, \tag{47}$$

$$\varphi_0 = \sqrt{\frac{c}{1+b}}, \tag{48}$$

and

$$g_0 = \frac{2|c|}{3}\sqrt{\frac{c}{1+b}}. \tag{49}$$

We draw the graph of $z = f(\varphi)$ as in Fig. 4.

On the other hand, it is seen that $(\tilde{\varphi}, 0)$ is a singular point of system (36) if and only if $f(\tilde{\varphi}) = 0$. At the singular point $(\tilde{\varphi}, 0)$, it is easy to see that the linearized system of (36) has the eigenvalues

$$\lambda_{\pm}(\tilde{\varphi}, 0) = \pm\sqrt{(\tilde{\varphi} - c)f'(\tilde{\varphi})}. \tag{50}$$

From (42) and (50) we see that the singular point $(\tilde{\varphi}, 0)$ is of the following properties:

(i) If $(\tilde{\varphi} - c)f'(\tilde{\varphi}) > 0$, then $(\tilde{\varphi}, 0)$ is a saddle point of system (36).

(ii) If $(\tilde{\varphi} - c)f'(\tilde{\varphi}) = 0$, then $(\tilde{\varphi}, 0)$ is a degenerate saddle point of system (36).

(iii) If $(\tilde{\varphi} - c)f'(\tilde{\varphi}) < 0$, then $(\tilde{\varphi}, 0)$ is a center point of system (36).

Let

$$y_0 = \frac{2(1+b)c^3 - 6c^2 + 6g}{3(b-1)}. \tag{51}$$

Similarly, it can be seen that if $y_0 > 0$, then $(c, -\sqrt{y_0})$ and $(c, \sqrt{y_0})$ are two saddle points of system (36). According to the analysis above and the values of $H(\varphi, y)$ at the singular points, we obtain five bifurcation curves:

$$g_1(c) = -\frac{2c\sqrt{c}}{3\sqrt{1+b}}, \tag{52}$$

$$g_2(c) = \frac{2c(bc(3+3b-3c-bc) - (-1+c-bc+b^2)\Omega)}{3b^2(1+b)^2}, \tag{53}$$

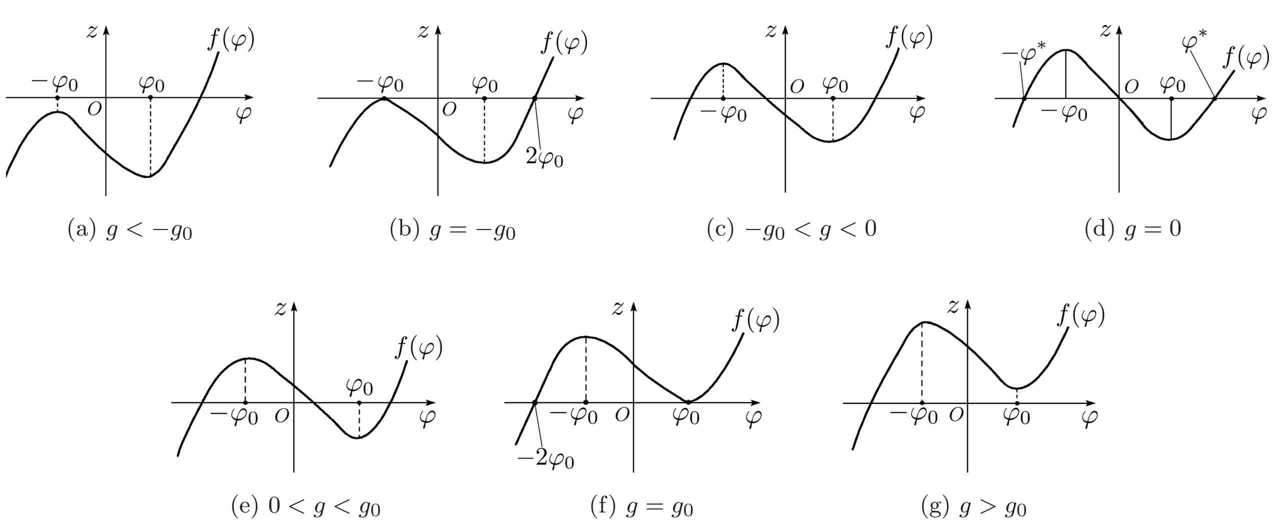


Fig. 4. The graph of $z = f(\varphi)$.

$$g_3(c) = c^2\left(1 - \frac{1+b}{3}c\right), \tag{54}$$

$$g_4(c) = \frac{2c(bc(3+3b-3c-bc) + (-1+c-bc+b^2)\Omega)}{3b^2(1+b)^2}, \tag{55}$$

and

$$g_5(c) = \frac{2c\sqrt{c}}{3\sqrt{1+b}}, \tag{56}$$

where

$$\Omega = \sqrt{b(2+b)(1+b-c)c}. \tag{57}$$

On a c–g plane, it is easy to test the following properties:

(i) The five curves $g = g_i(c)$ $(i = 1-5)$ intersect at $(0, 0)$.
(ii) The three curves $g = g_i(c)$ $(i = 3, 4, 5)$ have another intersection $(1/(1+b), 2/(3(1+b)^2))$.
(iii) The two curves $g = g_2(c)$ and $g = g_3(c)$ have another intersection $4(1+b)/(4+2b+b^2), 16(1+b)^2(8-2b-b^2)/(3(4+2b+b^2)^3)$.
(iv) The two curves $g = g_1(c)$ and $g = g_3(c)$ have another intersection $(4/(1+b), -(16/(3(1+b)^2)))$.
(v) The three curves $g = g_1(c)$, $g = g_2(c)$ and $g = g_4(c)$ have another intersection $(1+b, -2(1+b)/3)$.
(vi) When $0 < c < 4(1+b)/(4+2b+b^2)$ and $c \neq (1/(1+b))$, it follows that

$$g_5(c) > g_4(c) > g_3(c) > g_2(c) > g_1(c). \tag{58}$$

(vii) When $4(1+b)/(4+2b+b^2) < c < 4/(1+b)$, it follows that

$$g_5(c) > g_4(c) > g_2(c) > g_3(c) > g_1(c). \tag{59}$$

(viii) When $4/(1+b) < c < 1+b$, it follows that

$$g_5(c) > g_4(c) > g_2(c) > g_1(c) > g_3(c). \tag{60}$$

(ix) When $1+b < c < +\infty$, it follows that

$$g_5(c) > g_1(c) > g_3(c), \tag{61}$$

and $g_2(c)$, $g_4(c)$ have no definition.

From the discussion above, we draw the figures of $g = g_i(c)$ $(i = 1-5)$ as in Fig. 5.

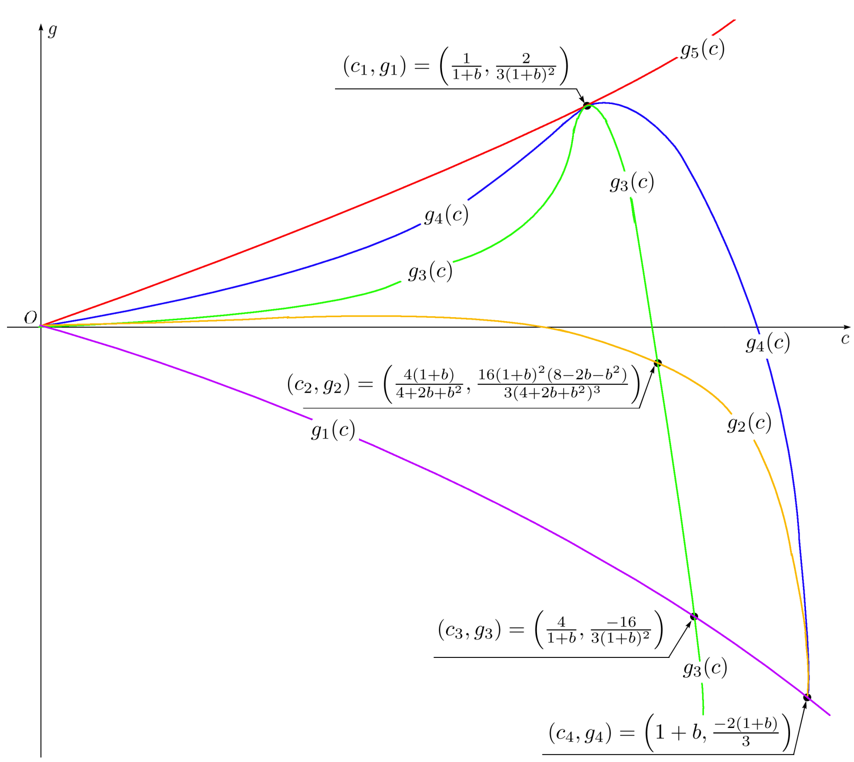


Fig. 5. The bifurcation curves of systems (35) and (36).

Next, we will study the phase portraits of system (35) on the curves $g = g_j(c)$ $(j = 2, 3, 4)$ and use some special orbits of the phase portraits to derive our main results.

4. The Derivations for the Main Results

In this section, we will derive our main results on three bifurcation curves $g_2(c)$, $g_3(c)$ and $g_4(c)$. Our derivation will be given under the following three cases:

Case 1. $0 < c < 1 + b$ and $g = g_2(c)$.

Case 2. $0 < c < 4(1 + b)/(4 + 2b + b^2)$ and $g = g_3(c)$.

Case 3. $0 < c < 1 + b$ and $g = g_4(c)$.

4.1. *The derivations under Case 1*

For Case 1, that is, $0 < c < 1+b$ and $g = g_2(c)$, system (35) has three singular points $(\varphi_1^\circ, 0)$, $(\varphi_1^-, 0)$ and $(\varphi_1^+, 0)$ where

$$\varphi_1^\circ = \frac{-bc - \Omega}{b(1 + b)}, \tag{62}$$

$$\varphi_1^- = \frac{bc + \Omega - \sqrt{3bc(b + 3b^2 + 2c - 2 - 2\Omega)}}{2b(1 + b)}, \tag{63}$$

$$\varphi_1^+ = \frac{bc + \Omega + \sqrt{3bc(b + 3b^2 + 2c - 2 - 2\Omega)}}{2b(1 + b)}. \tag{64}$$

In (42) letting $\varphi = c$, it follows that $h = 0$. Thus substituting $g = g_2(c)$ into (37) and solving equation

$$a_3(\varphi - c) + a_2(\varphi - c)^2 + a_1(\varphi - c) + a_0 = 0, \tag{65}$$

we get three roots c, φ_1° and φ_2°, where

$$\varphi_2^\circ = \frac{-bc + 2\Omega}{b(1 + b)}. \tag{66}$$

It is easy to see that $\varphi_1^\pm$, φ_1° and φ_2° satisfy the following inequalities:

$$\varphi_1^\circ < \varphi_1^- < c < \varphi_1^+ < \varphi_2^\circ \quad \text{for } 0 < c < \frac{4(1 + b)}{4 + 2b + b^2}, \tag{67}$$

$$\varphi_1^\circ < \varphi_1^- < \varphi_1^+ = \varphi_2^\circ = c \quad \text{for } c = \frac{4(1 + b)}{4 + 2b + b^2}, \tag{68}$$

and

$$\varphi_1^\circ < \varphi_1^- < \varphi_2^\circ < \varphi_1^+ < c \quad \text{for } \frac{4(1 + b)}{4 + 2b + b^2} < c < 1 + b. \tag{69}$$

Let $\Gamma(\varphi_2^\circ)$ denote the orbit passing through the point $(\varphi_2^\circ, 0)$. From Fig. 4, (50) and (67)–(69), we see that $(\varphi_1^\circ, 0)$ is a center, $(\varphi_1^-, 0)$ is a saddle, and the properties of $(\varphi_1^+, 0)$ are as follows:

(i) When $0 < c < 4(1 + b)/(4 + 2b + b^2)$, $(\varphi_1^+, 0)$ is a saddle. The phase portrait and the location of the orbit $\Gamma(\varphi_2^\circ)$ are shown in Fig. 6(a).
(ii) When $c = 4(1 + b)/(4 + 2b + b^2)$, $(\varphi_1^+, 0)$ is a degenerate singular point. The phase portrait and the location of the orbit $\Gamma(\varphi_2^\circ)$ are shown in Fig. 6(b).
(iii) When $4(1 + b)/(4 + 2b + b^2) < c < 1 + b$, $(\varphi_1^+, 0)$ is a center. The phase portrait and the location of the orbit $\Gamma(\varphi_2^\circ)$ are shown in Fig. 6(c).

From (42) and the analysis above, on φ–y plane the orbit $\Gamma(\varphi_2^\circ)$ has expression

$$y = \pm\sqrt{\frac{2(1 + b)}{3(2 + b)}}(\varphi - \varphi_1^\circ)\sqrt{\varphi - \varphi_2^\circ}. \tag{70}$$

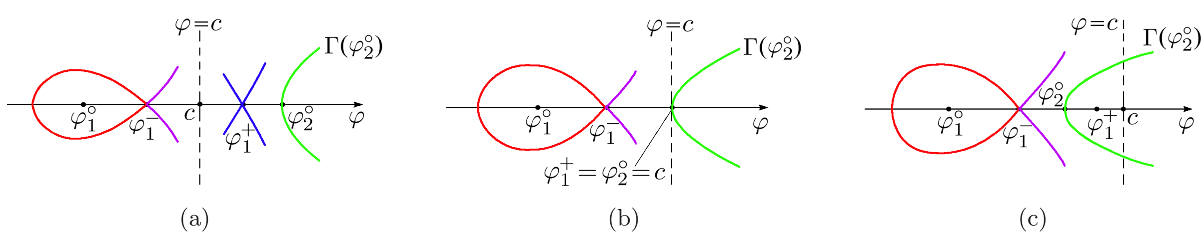


Fig. 6. The location of the orbit $\Gamma(\varphi_2^\circ)$ when $0 < c < 1 + b$ and $g = g_2(c)$, (a) for $0 < c < 4(1 + b)/(4 + 2b + b^2)$, (b) for $c = 4(1 + b)/(4 + 2b + b^2)$, (c) for $4(1 + b)/(4 + 2b + b^2) < c < 1 + b$.

Substituting (70) into the first equation of system (35), it follows that

$$\frac{\mathrm{d}\varphi}{(\varphi-\varphi_1^\circ)\sqrt{\varphi-\varphi_2^\circ}} = \pm\sqrt{\frac{2(1+b)}{3(2+b)}}\,\mathrm{d}\xi. \qquad (71)$$

Suppose $\varphi(0)=+\infty$. From (71) we have

$$\int_\varphi^{+\infty}\frac{\mathrm{d}s}{(s-\varphi_1^\circ)\sqrt{s-\varphi_2^\circ}} = \sqrt{\frac{2(1+b)}{3(2+b)}}\,|\xi|. \qquad (72)$$

In (72) completing the integral and solving the equation for φ, and noting that $u=\varphi(\xi)$, we obtain the trigonometric periodic singular wave solution $u_4(\xi,c)$ as (12). Note that our derivation is available for $c=1/(1+b)$. Thus substituting $c=1/(1+b)$ into (12), we get $u_4^\circ(x,t)$ as (28).

4.2. *The derivations under Case 2*

For Case 2, that is, $0<c<4(1+b)/(4+2b+b^2)$ and $g=g_3(c)$, system (35) has three singular points $(c,0)$, $(\varphi_2^+,0)$ and $(\varphi_2^-,0)$, where

$$\varphi_2^+ = \frac{-(1+b)c+\sqrt{3(1+b)(4-c-bc)c}}{2(1+b)}, \qquad (73)$$

and

$$\varphi_2^- = \frac{-(1+b)c-\sqrt{3(1+b)(4-c-bc)c}}{2(1+b)}. \qquad (74)$$

Clearly, there are the following inequalities

$$\varphi_2^- < c < \varphi_2^+ \quad \text{for } 0<c<\frac{1}{1+b}, \qquad (75)$$

$$\varphi_2^- < c = \varphi_2^+ \quad \text{for } c=\frac{1}{1+b}, \qquad (76)$$

and

$$\varphi_2^- < \varphi_2^+ < c \quad \text{for } \frac{1}{1+b}<c<\frac{4(1+b)}{4+2b+b^2}. \qquad (77)$$

Note that $\varphi=c$ is the singular line. Therefore, $(c,0)$ is a degenerate singular point. $(\varphi_2^-,0)$ is a center, and $(\varphi_2^+,0)$ is a saddle when $c\neq 1/(1+b)$. When $c=1/(1+b)$, the saddle $(\varphi_2^+,0)$ and the degenerate singular point $(c,0)$ coincide. In (42) when $\varphi=c$, we have $h=0$. Therefore, substituting $h=0$ and $g=g_3(c)$ into (42) and solving the equation $H(\varphi,0)=0$, we get three roots c, φ_3^+ and φ_3^-, where

$$\varphi_3^+ = \frac{-bc(4+b)+\sqrt{3bc(2+b)(4+4b-4c-2bc-b^2c)}}{2b(1+b)}, \qquad (78)$$

and

$$\varphi_3^- = \frac{-bc(4+b)-\sqrt{3bc(2+b)(4+4b-4c-2bc-b^2c)}}{2b(1+b)}. \qquad (79)$$

It is easy to check the following inequalities:

$$\varphi_3^- < \varphi_2^- < c < \varphi_2^+ < \varphi_3^+ \quad \text{for } 0<c<\frac{1}{1+b}, \qquad (80)$$

$$\varphi_3^- < \varphi_2^- < c = \varphi_2^+ = \varphi_3^+ \quad \text{for } c=\frac{1}{1+b}, \qquad (81)$$

and

$$\varphi_3^- < \varphi_2^- < \varphi_3^+ < \varphi_2^+ < c \quad \text{for } \frac{1}{1+b}<c<\frac{4(1+b)}{4+2b+b^2}. \qquad (82)$$

These imply the following information:

(i) When $0<c<1/(1+b)$, there is a special closed orbit denoted by Γ_1 which passes through $(\varphi_3^-,0)$ and connects with the degenerate singular point $(c,0)$. And there is another special orbit denoted by Γ_2 which passes through $(\varphi_3^+,0)$ [see Fig. 7(a)].

(ii) When $c=1/(1+b)$, there is a special closed orbit denoted by Γ_3 which passes through $(\varphi_3^-,0)$ and connects with $(c,0)$. And there are other two orbits denoted by Γ_4^- and Γ_4^+ which connect with $(c,0)$, too [see Fig. 7(b)].

(iii) When $1/(1+b)<c<4(1+b)/(4+2b+b^2)$, there is a special closed orbit denoted by Γ_5 which passes through $(\varphi_3^-,0)$ and $(\varphi_3^+,0)$. And there are two special orbits denoted by $\Gamma_6^\pm$ which connect with $(c,0)$ [see Fig. 7(c)].

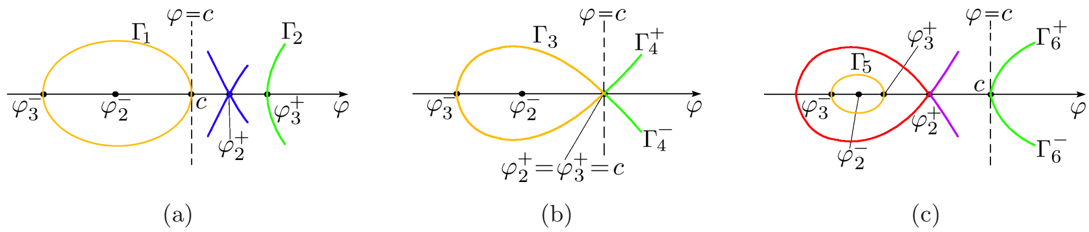


Fig. 7. The graphs of the special orbits when $0 < c < 4(1+b)/(4+2b+b^2)$ and $g = g_3(c)$, (a) for $0 < c < 1/(1+b)$, (b) for $c = 1/(1+b)$, (c) for $1/(1+b) < c < 4(1+b)/(4+2b+b^2)$.

On φ–y plane, these special orbits possess the following expressions:

$$\Gamma_1\colon\ y^2 = \frac{2(1+b)}{3(2+b)}(\varphi_3^+ - \varphi)(c-\varphi)(\varphi - \varphi_3^-),\quad \text{where } \varphi_3^- \le \varphi < c < \varphi_3^+. \tag{83}$$

$$\Gamma_2\colon\ y^2 = \frac{2(1+b)}{3(2+b)}(\varphi - \varphi_3^+)(\varphi - c)(\varphi - \varphi_3^-),\quad \text{where } \varphi_3^- < c < \varphi_3^+ < \varphi. \tag{84}$$

$$\Gamma_3\colon\ y^2 = \frac{2(1+b)}{3(2+b)}(c-\varphi)^2(\varphi - \varphi_3^-),\quad \text{where } \varphi_3^- \le \varphi < c. \tag{85}$$

$$\Gamma_4^{\pm}\colon\ y = \pm\sqrt{\frac{2(1+b)}{3(2+b)}}(\varphi - c)\sqrt{\varphi - \varphi_3^-},\quad \text{where } \varphi_3^- < c < \varphi. \tag{86}$$

$$\Gamma_5\colon\ y^2 = \frac{2(1+b)}{3(2+b)}(c-\varphi)(\varphi_3^+ - \varphi)(\varphi - \varphi_3^-),\quad \text{where } \varphi_3^- < \varphi < \varphi_3^+ < c. \tag{87}$$

$$\Gamma_6^{\pm}\colon\ y = \pm\sqrt{\frac{2(1+b)}{3(2+b)}}\sqrt{(\varphi - c)(\varphi - \varphi_3^+)(\varphi - \varphi_3^-)},\quad \text{where } \varphi_3^- < \varphi_3^+ < c < \varphi. \tag{88}$$

Substituting these expressions into the first equation of system (35) and integrating them respectively, it yields that

$$\int_{\varphi_3^-}^{\varphi} \frac{ds}{\sqrt{(\varphi_3^+ - s)(c-s)(s-\varphi_3^-)}} = \sqrt{\frac{2(1+b)}{3(2+b)}}\,|\xi| \quad (\text{along the orbit } \Gamma_1), \tag{89}$$

where $0 < c < 1/(1+b)$ and $\varphi_3^- < \varphi \le c < \varphi_3^+$.

$$\int_{\varphi}^{\infty} \frac{ds}{\sqrt{(s-\varphi_3^+)(s-c)(s-\varphi_3^-)}} = \sqrt{\frac{2(1+b)}{3(2+b)}}\,|\xi| \quad (\text{along the orbit } \Gamma_2), \tag{90}$$

where $0 < c < 1/(1+b)$ and $\varphi_3^- < c < \varphi_3^+ \le \varphi < \infty$.

$$\int_{\varphi_3^-}^{\varphi} \frac{ds}{(c-s)\sqrt{(s-\varphi_3^-)}} = \sqrt{\frac{2(1+b)}{3(2+b)}}\,|\xi| \quad (\text{along the orbit } \Gamma_3), \tag{91}$$

and

$$\int_{\varphi}^{\infty} \frac{ds}{(s-c)\sqrt{(s-\varphi_3^-)}} = \sqrt{\frac{2(1+b)}{3(2+b)}}\,|\xi| \quad (\text{along the orbit } \Gamma_4^{\pm}), \tag{92}$$

where $c = 1/(1+b)$.

$$\int_{\varphi_3^-}^{\varphi} \frac{ds}{\sqrt{(c-s)(\varphi_3^+ - s)(s-\varphi_3^-)}} = \sqrt{\frac{2(1+b)}{3(2+b)}}\,|\xi| \quad (\text{along the orbit } \Gamma_5), \tag{93}$$

where $1/(1+b) < c < 4(1+b)/(4+2b+b^2)$ and $\varphi_3^- < \varphi \le \varphi_3^+ < c$.

$$\int_{\varphi}^{\infty} \frac{ds}{\sqrt{(s-c)(s-\varphi_3^+)(s-\varphi_3^-)}} = \sqrt{\frac{2(1+b)}{3(2+b)}}\,|\xi| \quad (\text{along the orbit } \Gamma_6^{\pm}), \tag{94}$$

where $1/(1+b) < c < 4(1+b)/(4+2b+b^2)$ and $\varphi_3^- < \varphi_3^+ < c \leq \varphi < \infty$.

In (89) and (93) completing the integrals and solving the equations for φ, and noting that $u = \varphi(\xi)$, we get the elliptic smooth periodic wave solution $u_5(\xi, c)$ given as (16).

Similarly, via (90) and (94), we obtain the elliptic periodic singular wave solution $u_6(\xi, c)$ given as (17). Via (91) we get the smooth solitary wave solution $u_1^\circ(x,t)$ given as (26). Via (92) we get the hyperbolic singular wave solution $u_3^\circ(x,t)$ given as (27).

4.3. *The derivations under Case 3*

Case 3, that is, $0 < c < 1+b$, and $g = g_4(c)$, system (35) has three singular points $(\varphi_4^+, 0)$, $(\varphi_3^\circ, 0)$ and $(\varphi_4^-, 0)$, where

$$\varphi_4^\pm = \frac{bc - \Omega \pm \sqrt{3bc(b+3b^2-2+2c+2\Omega)}}{2b(1+b)}$$

and

$$\varphi_3^\circ = \frac{\Omega - 3bc}{b(1+b)}.$$

It is easy to check that φ_4^+, φ_3° and φ_4^- satisfy the following inequalities

$$\varphi_4^- < c < \varphi_4^+ < \varphi_3^\circ \quad \text{for } 0 < c < \frac{1}{1+b}, \tag{95}$$

$$\varphi_4^- < \varphi_3^\circ = \varphi_4^+ = c \quad \text{for } c = \frac{1}{1+b}, \tag{96}$$

and

$$\varphi_4^- < \varphi_3^\circ < \varphi_4^+ < c \quad \text{for } \frac{1}{1+b} < c < 1+b. \tag{97}$$

From Fig. 4, (50) and (95)–(97) we see the following facts:

(i) When $0 < c < 1+b$ and $c \neq 1/(1+b)$, $(\varphi_4^-, 0)$ and $(\varphi_4^+, 0)$ are two centers. $(\varphi_3^\circ, 0)$ is a saddle.
(ii) When $c = 1/(1+b)$, the center $(\varphi_4^-, 0)$ and the saddle $(\varphi_3^\circ, 0)$ persist. The center $(\varphi_4^+, 0)$ disappears. Let

$$\varphi_4^\circ = -\frac{bc + 2\Omega}{b(1+b)}, \tag{98}$$

and

$$y_0^* = \frac{\sqrt{2c[(2+b)(-c-3b+3c+2bc+b^2c)bc + 2(1+b-c)\Omega]}}{\sqrt{3}b(1+b)}. \tag{99}$$

Assume that $(\varphi_0, 0)$ is an initial point of system (35). We discuss some special orbits as follows.

(1) When the wave speed c satisfies $0 < c < 1/(1+b)$, we have:

(i) If $\varphi_4^- < \varphi_0 < c$, then the orbit passing $(\varphi_0, 0)$ is a periodic orbit. As $\varphi_0 \to c - 0$, the periodic orbit tends to a closed orbit denoted by Γ_7 which is composed of a line segment $\varphi = c$ and an arc connecting the three points $(\varphi_4^\circ, 0)$, $(c, -y_0^*)$ and (c, y_0^*) [see Fig. 8(a)].
(ii) If $c < \varphi_0 < \varphi_4^+$, then the orbit passing $(\varphi_0, 0)$ is a periodic orbit. As $\varphi_0 \to c + 0$, the periodic orbit tends to a closed orbit denoted by Γ_8 which is composed of a line segment $\varphi = c$ and an arc connecting the three points $(\varphi_3^\circ, 0)$, $(0, y_0^*)$ and $(0, -y_0^*)$ [see Fig. 8(b)].
(iii) If $\varphi_0 = \varphi_3^\circ$, then there are three orbits connecting to the point $(\varphi_3^\circ, 0)$. These orbits are

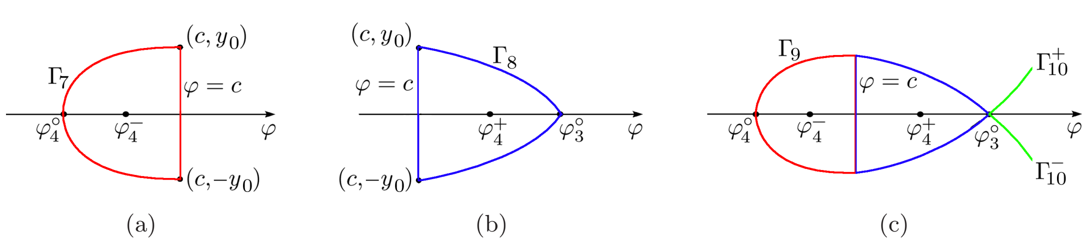


Fig. 8. The graphs of the orbits Γ_i $(i = 7-9)$, Γ_{10}^+, and Γ_{10}^- when $0 < c < 1/(1+b)$ and $g = g_4(c)$. (a) The limit orbit Γ_7 of the periodic orbit passing $(\varphi_0, 0)$ when $\varphi_4^- < \varphi_0 < c$ and $\varphi_0 \to c - 0$. (b) The limit orbit Γ_8 of the periodic orbit passing through $(\varphi_0, 0)$ when $c < \varphi_0 < \varphi_4^+$ and $\varphi_0 \to c + 0$. (c) The orbits Γ_9, Γ_{10}^+ and Γ_{10}^- connecting to $(\varphi_3^\circ, 0)$.

the homoclinic orbit denoted by Γ_9, the heteroclinic orbits denoted by Γ_{10}^+ and Γ_{10}^- respectively [see Fig. 8(c)].

(2) When the wave speed c satisfies $1/(1+b) < c < 1+b$, we have:

(i) If $\varphi_4^- < \varphi_0 < \varphi_3^\circ$, then the orbit passing through $(\varphi_0, 0)$ is a periodic orbit. When $\varphi_0 \to \varphi_3^\circ - 0$, the periodic orbit tends to a closed orbit denoted by Γ_{11} which connects with two points $(\varphi_3^\circ, 0)$ and $(\varphi_4^\circ, 0)$ [see Fig. 9(a)].

(ii) If $\varphi_3^\circ < \varphi_0 < \varphi_4^+$, then the orbit passing $(\varphi_0, 0)$ is a periodic orbit. When $\varphi_0 \to \varphi_3^\circ + 0$, the periodic orbit tends to a closed orbit denoted by Γ_{12} which is composed of a line segment $\varphi = c$ and an arc connecting the three points $(\varphi_3^\circ, 0)$, (c, y_0^*) and $(c, -y_0^*)$ [see Fig. 9(b)].

(iii) If $\varphi_0 = \varphi_3^\circ$, then there are four orbits connecting to the point $(\varphi_3^\circ, 0)$. These orbits are the homoclinic orbits Γ_{11} and Γ_{12}, the heteroclinic orbits denoted by Γ_{13}^+ and Γ_{13}^- respectively [see Fig. 9(c)].

From Figs. 8 and 9 we see that these orbits possess the same expression

$$y^2 = \frac{2(1+b)}{3(2+b)}(\varphi - \varphi_3^\circ)^2(\varphi - \varphi_4^\circ) \tag{100}$$

except the definition intervals.

Substituting (100) into the first equation of system (35) and respectively integrating it along these orbits above, it follows that

$$\int_{\varphi_4^\circ}^{\varphi} \frac{\mathrm{d}s}{(\varphi_3^\circ - s)\sqrt{s - \varphi_4^\circ}} = \sqrt{\frac{2(1+b)}{3(2+b)}}\,|\xi| \quad \text{(along the orbits } \Gamma_9 \text{ or } \Gamma_{11}), \tag{101}$$

$$\int_{\varphi}^{c} \frac{\mathrm{d}s}{(s - \varphi_3^\circ)\sqrt{s - \varphi_4^\circ}} = \sqrt{\frac{2(1+b)}{3(2+b)}}\,|\xi| \quad \text{(along the orbits } \Gamma_8 \text{ or } \Gamma_{12}), \tag{102}$$

and

$$\int_{\varphi}^{+\infty} \frac{\mathrm{d}s}{(s - \varphi_3^\circ)\sqrt{s - \varphi_4^\circ}} = \sqrt{\frac{2(1+b)}{3(2+b)}}\,|\xi| \quad \text{(along the orbits } \Gamma_{10}^\pm \text{ or } \Gamma_{13}^\pm). \tag{103}$$

From (101) we get the hyperbolic smooth solitary wave solution $u_1(\xi, c)$ given as (9). Via (102) we obtain the hyperbolic peakon wave solution $u_2(\xi, c)$ as (10). Via (103) we have the hyperbolic singular wave solution $u_3(\xi, c)$ as (11).

In Secs. 4.1–4.3, we have given derivations for the solutions $u_i(\xi, c)$ $(i = 1-6)$ and $u_1^\circ(x,t)$, $u_2^\circ(x,t)$, $u_4^\circ(x,t)$. Also we have tested their correctness by using the software Mathematica (see Remark 3). Hence, we have completed the proof for Propositions 1–3. Next we will prove Proposition 4.

4.4. *The proof of Proposition 4*

Now we prove Proposition 4. Firstly, when $c \to 1/(1+b)$, it follows that

$$p \to \frac{b(2+b)}{1+b}, \tag{104}$$

$$\alpha \to \frac{1}{\sqrt{2(1+b)}}, \tag{105}$$

$$2bc + b^2c - p \to 0, \tag{106}$$

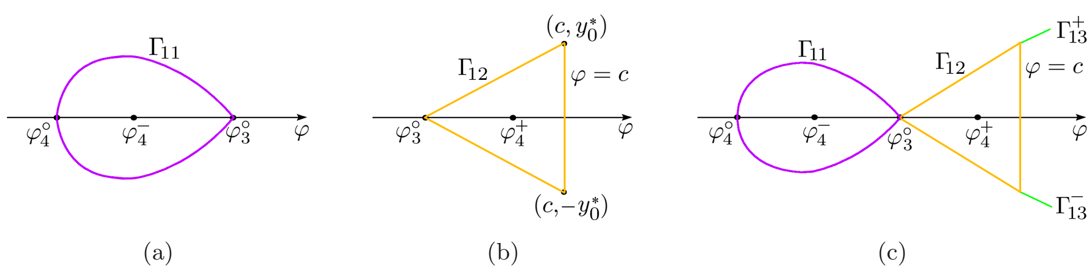


Fig. 9. The graphs of the orbits Γ_i $(i = 11, 12)$, and Γ_{13}^+, Γ_{13}^- when $1/(1+b) < c < 1+b$ and $g = g_4(c)$. (a) The limit orbit Γ_{11} of the periodic orbit passing $(\varphi_0, 0)$ when $\varphi_4^- < \varphi_0 < \varphi_3^\circ$ and $\varphi_0 \to \varphi_3^\circ - 0$. (b) The limit orbit Γ_{12} of the periodic orbit passing through $(\varphi_0, 0)$ when $\varphi_3^\circ < \varphi_0 < \varphi_4^+$ and $\varphi_0 \to \varphi_3^\circ + 0$. (c) The orbits Γ_{11}, Γ_{12}, Γ_{13}^+ and Γ_{13}^- connecting to $(\varphi_3^\circ, 0)$.

$$a_1 \to -\frac{5+2b}{(1+b)^2}, \tag{107}$$

$$a_2 \to \frac{3(2+b)}{(1+b)^2}, \tag{108}$$

$$a_3 \to \frac{3(2+b)}{(1+b)^2}, \tag{109}$$

$$\beta \to \frac{1}{\sqrt{2(1+b)}}, \tag{110}$$

$$\gamma \to \frac{1}{\sqrt{2(1+b)}}, \tag{111}$$

$$k_i \to 1 \quad (i=1,2), \tag{112}$$

$$\mathrm{sn}(v,1) = \tanh v, \tag{113}$$

$$1 - \tanh^2 v = \mathrm{sech}^2 v, \tag{114}$$

and

$$\tanh^{-2} v - 1 = \mathrm{csch}^2 v. \tag{115}$$

Therefore, in (9)–(12) and (16)–(17) letting $c \to 1/(1+b)$ and using the limit values, from the equations above, we get Properties (i)–(vi) in Proposition 4.

Secondly, when $c \to 4(1+b)/(4+2b+b^2)$, it follows that

$$a_1 \to -\frac{2(1+b)}{4+2b+b^2}, \tag{116}$$

$$a_2 \to -\frac{6(2+b)}{4+2b+b^2}, \tag{117}$$

$$a_3 \to 0, \tag{118}$$

$$r \to \sqrt{\frac{1+b}{4+2b+b^2}}, \tag{119}$$

$$k_2 \to 0, \tag{120}$$

$$\mathrm{sn}(v,0) = \sin v. \tag{121}$$

Thus in (16) and (17) letting $c \to 4(1+b)/(4+2b+b^2)$, we get Properties (vii) and (viii).

Finally, note that when $c \to 1+b$, it follows that $p \to 0$ and $\alpha \to 0$. Therefore in (9)–(22) letting $c \to 1+b$, we see that $u_i(\xi, c) \to -1$ $(i = 1,2,3,4)$. Hereto, we have completed the proof for our main results.

5. Conclusion

In this paper, for parameter $b > 1$ and constant wave speed c, we have studied the coexistence of multifarious exact nonlinear wave solutions for Eq. (4). Note the following facts:

(i) In [He *et al.*, 2008a, 2008b; Khuri, 2005; Liu & Guo, 2008; Liu & Ouyang, 2007; Liu & Pan, 2009; Liu & Qian, 2001; Shen & Xu, 2005; Tian & Song, 2004; Wang & Tang, 2008; Yomba, 2008a, 2008b; Zhang *et al.*, 2007], the authors investigated the nonlinear wave solutions when $b = 2$ or $b = 3$.

(ii) In [Wazwaz, 2007], the solitary wave solutions were studied for a special wave speed $c = (2+b)/2$.

(iii) In [Liu, 2010], the study was based on three special wave speeds $c = 1/(1+b)$, or $c = (2+b)/2$, or $c = 1+b$.

Therefore, our work includes many previous results.

In Sec. 4, we obtained the bifurcation phase portraits on three bifurcation curves $g = g_i(c)$ $(i = 2,3,4)$. Using a similar method, we obtained all bifurcation phase portraits on c–g plane for $c > 0$ (for $c \le 0$, system (35) has a unique singular point) as in Fig. 10.

From Fig. 10 and previous results, we have the following conjectures:

Conjectures 1. *When $b > 1$ and the wave speed $c = 1/(1+b)$ or $c > 1+b$, Eq. (4) has no peakon wave solution.*

Conjectures 2. *When the wave speed $c \ge 1+b$, Eq. (4) has no explicit smooth solitary wave solution.*

Conjectures 3. *When the wave speed $c \ne 1+b$, Eq. (4) has no fractional solution.*

Some questions also remain as follows.

Questions 1. *Our derivations were based on $b > 1$. However, substituting the expressions of the solutions into Eq. (4) directly, we see the following facts:* (i) *From (6)–(25), for $b > 0$, $u_i(\xi, c)$ $(i = 1-6)$ are still the real solutions.* (ii) *Via (26)–(31), for $b > -1$, $u_1^\circ(x,t)$, $u_3^\circ(x,t)$, $u_4^\circ(x,t)$, $u_7(x,t)$, $u_8(x,t)$, and $u_9(x,t)$ are still the real solutions.* (iii) *$u_8(x,t)$ is still a real solution for $b \ne -1$. But for $b \le 1$, we do not know how to derive these expressions mentioned above.*

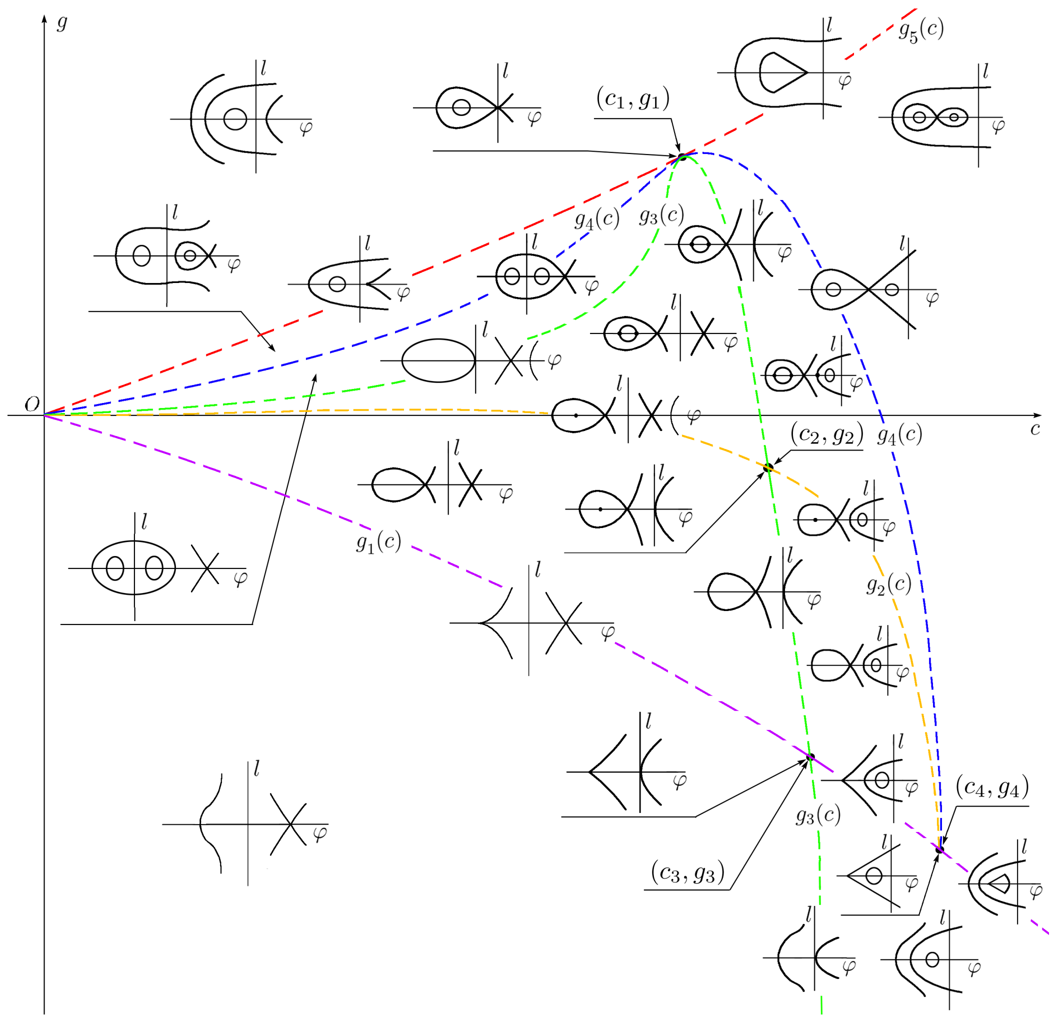


Fig. 10. The bifurcation phase portraits of systems (35) and (36), where the two systems have the same topological phase portraits except the line $\varphi = c$.

Questions 2. *From Liu [2010] we know that for $b > -2$ and $b \neq -1$,*

$$u_2^*(x,t) = \frac{3(2+b)(2+3b+b^2)}{2(1+b)\left[\sqrt{3(2+b)}\cosh\frac{1}{2}\left(x - \frac{2+b}{2}t\right) + \sqrt{8+6b+b^2}\sinh\frac{1}{2}\left|x - \frac{2+b}{2}t\right|\right]^2}$$

is a real solution of Eq. (4). And for $b \neq -1$, Eq. (4) has three real solutions

$$u_1^*(x,t) = -\frac{3(2+b)}{2(1+b)}\operatorname{sech}^2\frac{1}{2}\left(x - \frac{2+b}{2}t\right),$$

$$u_3^*(x,t) = \frac{3(2+b)}{2(1+b)}\operatorname{csch}^2\frac{1}{2}\left(x - \frac{2+b}{2}t\right),$$

and $u_8(x,t)$ given as (30).

For $b < -1$, we do not know whether there is any other explicit real solution except $u_i^*(x,t)$, $(i = 1, 2, 3)$ and $u_8(x,t)$.

References

Boyd, J. P. [1997] "Peakons and coshoidal waves: Travelling wave solutions of the Camassa–Holm equation," *Appl. Math. Comput.* **81**, 173–187.

Camassa, R. & Holm, D. D. [1993] "An integrable shallow water equation with peaked solitons," *Phys. Rev. Lett.* **71**, 1661–1664.

Chen, C. & Tang, M. Y. [2006] "A new type of bounded waves for Degasperis–Procesi equations," *Chaos Solit. Fract.* **27**, 698–704.

Constantin, A. [1997] "Soliton interactions for the Camassa–Holm equation," *Exp. Math.* **15**, 251–264.

Constantin, A. & Strauss, W. A. [2000] "Stability of peakons," *Commun. Pure Appl. Math.* **53**, 603–610.

Cooper, F. & Shepard, H. [1994] "Solitons in the Camassa–Holm shallow water equation," *Phys. Lett. A* **194**, 246–250.

Degasperis, A. & Procesi, M. [1999] "Asymptotic integrability," *Symmetry and Perturbation Theory*, eds. Degasperis, A. & Gaeta, G. (World Scientific, Singapore), pp. 23–37.

Degasperis, A., Holm, D. D. & Hone, A. N. W. [2002a] "A new integrable equation with peakon solutions," *Theor. Math. Phys.* **133**, 170–183.

Degasperis, A., Holm, D. D. & Hone, A. N. W. [2002b] "Integrable and non-integrable equations with peakons," *Nonlin. Phys.*: *Th. Exp. II*, eds. Ablowitz, M. J., Boiti, M. & Pempinelli, F. (World Scientific, Singapore), pp. 37–43.

Guha, P. [2007] "Euler–Poincare formalism of (two component) Degasperis–Procesi and Holm–Staley type systems," *J. Nonlin. Math. Phys.* **14**, 398–429.

Guo, B. L. & Liu, Z. R. [2005] "Periodic cusp wave solutions and single-solitons for the *b*-equation," *Chaos Solit. Fract.* **23**, 1451–1463.

He, B., Rui, W. G. & Chen, C. *et al.* [2008a] "Exact traveling wave solutions for a generalized Camassa–Holm equation using the integral bifurcation method," *Appl. Math. Comput.* **206**, 141–149.

He, B., Rui, W. G. & Li, S. L. *et al.* [2008b] "Bounded traveling wave solutions for a modified form of generalized Degasperis–Procesi equation," *Appl. Math. Comput.* **206**, 113–123.

Holm, D. D. & Staley, M. F. [2003] "Nonlinear balance and exchange of stability in dynamics of solitons, peakons ramps/cliffs and leftons in a 1 + 1 nolinear evolutionary PDEs," *Phys. Lett. A* **308**, 437–444.

Johnson, R. S. [2002] "Camassa–Holm, Korteweg–de Vries and related models for water waves," *J. Fluid Mech.* **455**, 63–82.

Khuri, S. A. [2005] "New ansatz for obtaining wave solutions of the generalized Camassa–Holm equation," *Chaos Solit. Fract.* **25**, 705–710.

Lenells, J. [2002] "The scattering approach for the Camassa–Holm equation," *J. Nonlin. Math. Phys.* **9**, 389–393.

Liu, Z. R. & Qian, T. F. [2001] "Peakons and their bifurcation in a generalized Camassa–Holm equation," *Int. J. Bifurcation and Chaos* **11**, 781–792.

Liu, Z. R., Wand, R. Q. & Jing, Z. J. [2004] "Peaked wave solutions of Camassa–Holm equation," *Chaos Solit. Fract.* **19**, 77–92.

Liu, Z. R., Kayed, A. M. & Chen, C. [2006] "Periodic waves and their limits for the Camassa–Holm equation," *Int. J. Bifurcation and Chaos* **16**, 2261–2274.

Liu, Z. R. & Ouyang, Z. Y. [2007] "A note on solitary waves for modified forms of Camassa–Holm and Degasperis–Procesi equations," *Phys. Lett. A* **366**, 377–381.

Liu, Z. R. & Guo, B. L. [2008] "Periodic blow-up solutions and their limit forms for the generalized Camassa–Holm equation," *Progr. Nat. Sci.* **18**, 259–266.

Liu, Z. R. & Pan, J. [2009] "Coexistence of multifarious explicit nonlinear wave solutions for modified forms of Camassa–Holm and Degaperis–Procesi equations," *Int. J. Bifurcation and Chaos* **19**, 2267–2282.

Liu, R. [2010] "Several new types of solitary wave solutions for the generalized Camassa–Holm–Degasperis–Procesi equation," *Commun. Pur. Appl. Anal.* **9**, 77–90.

Lundmark, H. & Szmigielski, J. [2003] "Multi-peakon solutions of the Degasperis–Procesi equation," *Inverse Prob.* **19**, 1241–1245.

Lundmark, H. & Szmigielski, J. [2005] "Degasperis–Procesi peakons and the discrete cubic string," *Int. Math. Res. Papers* **2**, 53–116.

Reyes, E. G. [2002] "Geometric integrability of the Camassa–Holm equation," *Lett. Math. Phys.* **59**, 117–131.

Shen, J. W. & Xu, W. [2005] "Bifurcations of smooth and non-smooth traveling wave solutions in the generalized Camassa–Holm equation," *Chaos Solit. Fract.* **26**, 1149–1162.

Tian, L. X. & Song, X. Y. [2004] "New peaked solitary wave solutions of the generalized Camassa–Holm equation," *Chaos Solit. Fract.* **21**, 621–637.

Wang, Q. D. & Tang, M. Y. [2008] "New exact solutions for two nonlinear equations," *Phys. Lett. A* **372**, 2995–3000.

Wazwaz, A. M. [2006] "Solitary wave solutions for modified forms of Degasperis–Procesi and Camassa–Holm equations," *Phys. Lett. A* **352**, 500–504.

Wazwaz, A. M. [2007] "New solitary wave solutions to the modified forms of Degasperis–Procesi and Camassa–Holm equations," *Appl. Math. Comput.* **186**, 130–141.

Yomba, E. [2008a] "The sub-ODE method for finding exact traveling wave solutions of generalized nonlinear Camassa–Holm, and generalized nonlinear Schrodinger equations," *Phys. Lett. A* **372**, 215–222.

Yomba, E. [2008b] "A generalized auxiliary equation method and its application to nonlinear Klein–Gordon and generalized nonlinear Camassa–Holm equations," *Phys. Lett. A* **372**, 1048–1060.

Zhang, L. J., Li, Q. C. & Huo, X. W. [2007] "Bifurcations of smooth and nonsmooth traveling wave solutions in a generalized Degasperis–Procesi equation," *J. Comput. Appl. Math.* **205**, 174–185.