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SOME NEW RESULTS ON EXPLICIT TRAVELING WAVE SOLUTIONS OF $K(m,n)$ EQUATION

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Abstract. In this paper, we investigate the traveling wave solutions of $K(m,n)$ equation $u_t + a(u^m)_x + (u^n)_{xxx} = 0$ by using the bifurcation method and numerical simulation approach of dynamical systems. We obtain some new results as follows: **(i)** For $K(2,2)$ equation, we extend the expressions of the smooth periodic wave solutions and obtain a new solution, the periodic-cusp wave solution. Further, we demonstrate that the periodic-cusp wave solution may become the peakon wave solution. **(ii)** For $K(3,2)$ equation, we extend the expression of the elliptic smooth periodic wave solution and obtain a new solution, the elliptic periodic-blow-up solution. From the limit forms of the two solutions, we get other three types of new solutions, the smooth solitary wave solutions, the hyperbolic 1-blow-up solutions and the trigonometric periodic-blow-up solutions. **(iii)** For $K(4,2)$ equation, we construct two new solutions, the 1-blow-up and 2-blow-up solutions.

1. Introduction.

The role of nonlinear dispersion in the formation of patterns in liquid drops was studied by Rosenau and Hyman [1]. In [1]-[5] the studies were carried out by introducing a family of nonlinear KdV type equations of the form

$$u_t + a\,(u^m)_x + (u^n)_{xxx} = 0, \tag{1}$$

which is called $K(m,n)$ equation.

For (1), it was formally derived in [6]-[8] that the delicate interaction between nonlinear convection and genuine nonlinear dispersion generates solitary waves with compact support. Unlike soliton that narrows as the amplitude increases, the compacton's width is independent on the amplitude. Two important features of the compacton structures were found:

(1) The compacton is a soliton characterized by the absence of exponential wings.

(2) The width of the compactons is independent on the amplitude.

Three main methods, namely, the pseudo spectral method, the tri-Hamiltonian operators, and Adomian decomposition method [9, 10], have been employed as appropriate schemes to handle (1) analytically. Also, the bifurcation method [11]

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of dynamical systems was used to study the traveling wave solutions of (1). In [11] Li and Liu obtained the following results:

(1) the explicit smooth periodic wave solutions of $K(m,2)$ $(m=2,3,4)$ equations,

(2) the explicit periodic-cusp wave solutions of $K(m,3)$ $(m=2,3,4,5,7)$ equations,

(3) the explicit peakon wave solutions of $K(m,2)$ $(m=2,3,4)$ equations.

Wazwaz [12, 13, 14] used the Adomian decomposition method and sine-cosine method to study the traveling wave solutions for $K(m,1)$ and $K(n,n)$ equations. Many interesting results were obtained by Li and Wazwaz et al. Now we list some previous results which will be compared with our work. In [11]-[14] letting $\xi=x-c\,t$, the following conclusions were showed:

(1) $K(2,2)$ equation has the following explicit solutions:

(i) the smooth periodic wave solutions

$$u_1^\circ(\xi)=\frac{4\,c}{3\,a}\cos^2\left(\frac{\sqrt{a}}{4}\,\xi\right),\quad \text{for}\quad a>0, \tag{2}$$

and

$$u_2^\circ(\xi)=\frac{4\,c}{3\,a}\sin^2\left(\frac{\sqrt{a}}{4}\,\xi\right),\quad \text{for}\quad a>0, \tag{3}$$

(ii) the peakon wave solution

$$u_3^\circ(\xi)=\frac{2\,c}{3\,a}\left(1-\exp\left(\frac{\sqrt{-a}}{2}\,|\xi|\right)\right),\quad \text{for}\quad a<0, \tag{4}$$

(iii) the unbounded solutions

$$u_4^\circ(\xi)=-\frac{4\,c}{3\,a}\sinh^2\left(\frac{\sqrt{-a}}{4}\,\xi\right),\quad \text{for}\quad a<0, \tag{5}$$

and

$$u_5^\circ(\xi)=\frac{4\,c}{3\,a}\cosh^2\left(\frac{\sqrt{-a}}{4}\,\xi\right),\quad \text{for}\quad a<0. \tag{6}$$

(2) $K(3,2)$ equation has the following explicit solutions:

(i) the smooth periodic wave solution

$$u_6^\circ(\xi)=\sqrt{\frac{5\,c}{3\,a}}\,\text{cn}^2\left(\sqrt[4]{\frac{a\,c}{60}}\,\xi,\ \frac{1}{\sqrt{2}}\right),\quad \text{for}\quad a>0,\ c>0, \tag{7}$$

(ii) the peakon wave solution

$$u_7^\circ(\xi)=\sqrt{\frac{5\,c}{9\,a}}\left[\frac{3\left(1-(\sqrt{3}+\sqrt{2})^2\exp\left(-\sqrt[4]{\frac{c}{5\,a}}\,|\xi|\,\right)\right)^2}{\left(1+(\sqrt{3}+\sqrt{2})^2\exp\left(-\sqrt[4]{\frac{c}{5\,a}}\,|\xi|\,\right)\right)^2}-2\right],\quad \text{for}\quad ac>0. \tag{8}$$

(3) $K(4,2)$ equation has the following explicit solutions:

(i) the peakon wave solution

$$u_8^\circ(\xi)=\sqrt[3]{\frac{c}{2\,a}}\,\frac{\left[(3\sqrt{2}+4)\exp\left(\sqrt[6]{\frac{|a|\,c^2}{4}}\,|\xi|\right)-4\right]^2-18}{\left[(3\sqrt{2}+4)\exp\left(\sqrt[6]{\frac{|a|\,c^2}{4}}\,|\xi|\right)+2\right]^2-6},\quad \text{for}\quad a<0, \tag{9}$$

(ii) the smooth periodic wave solution

$$u_9^{\circ}(\xi)=\sqrt[3]{\frac{c}{2\,a}}\;\frac{1-\mathrm{cn}\;(\omega_0\xi+\eta_0,\;k_0)}{1+\sqrt{3}-(1-\sqrt{3})\,\mathrm{cn}\;(\omega_0\xi+\eta_0,\;k_0)},\tag{10}$$

where $a>0$, $\omega_0=\sqrt[4]{\frac{1}{3}}\;\sqrt[6]{\frac{a\,c^2}{2}}$, $\eta_0=\frac{\sqrt[4]{3}\,\pi^2\,\Gamma(\frac{1}{6})}{3\,\Gamma(\frac{2}{3})}$ and $k_0=\frac{\sqrt{3}-1}{2\sqrt{2}}$.

In this paper, we use the bifurcation method and numerical simulation approach to study the traveling wave solutions of $K(m,2)$ $(m=2,3,4)$ equations. We obtain some new results which are listed roughly as follows.

(1) For $K(2,2)$ equation, our results are as follows:

(i) We obtain the general expressions $u_1(\xi)$ and $u_2(\xi)$ (see (15), (16)) of the explicit smooth periodic wave solutions. The previous solutions $u_1^{\circ}(\xi)$ and $u_2^{\circ}(\xi)$ become the special cases of $u_1(\xi)$ and $u_2(\xi)$ respectively.

(ii) We construct a new solution, the periodic-cusp wave solution $u_3(\xi)$ (see (21)-(24)). It is showed the previous solution $u_2^{\circ}(\xi)$ is a limit form of $u_3(\xi)$.

(2) For $K(3,2)$ equation, we complete the following works:

(i) We give the general expressions $u_4(\xi)$ and $u_{10}(\xi)$ (see (28) and (39)) of explicit smooth periodic wave solutions. The previous solution $u_6^{\circ}(\xi)$ becomes a special case of $u_4(\xi)$.

(ii) We get two new solutions $u_5(\xi)$ and $u_{11}(\xi)$ (see (29) and (40)) which are called periodic-blow-up solutions.

(iii) From the limits of $u_4(\xi)$, $u_5(\xi)$, $u_{10}(\xi)$ and $u_{11}(\xi)$, we obtain six new solutions $u_6(\xi)$, $u_7(\xi)$, $u_9(\xi)$, $u_{12}(\xi)$, $u_{13}(\xi)$ and $u_{15}(\xi)$ (see (36), (37), (38), (46), (47) and (49)).

(3) For $K(4,2)$ equation, we obtain two new solutions, the blow-up solutions $u_{16}(\xi)$ and $u_{17}(\xi)$ (see (51) and (52)).

(4) With the help of the software Mathematica, we verify the correctness of these solutions by substituting their expressions into the equations $K(m,2)$ $(m=2,3,4)$ respectively (see Remark 3).

We organize this paper as follows. In Section 2, we list our main results which are included in three propositions. In Sections 3, 4, 5, we give the proofs of the three propositions respectively. In Section 6, we give a short conclusion.

2. Main results and remarks.

In this section we list our main results and give some remarks. To state conveniently, for given constant wave speed c, let

$$\xi=x-c\,t.\tag{11}$$

Via the following three propositions we state our main results.

Proposition 2.1. *Consider $K(2,2)$ equation*

$$u_t+a(u^2)_x+(u^2)_{xxx}=0,\tag{12}$$

and its traveling wave equation

$$2\varphi\varphi''(\xi)+2(\varphi'(\xi))^2+a\varphi^2-c\varphi=g.\tag{13}$$

For arbitrarily given parameter a and constants c, g, basing on two different cases, we have the following results.

(1) *If a, c and g satisfy that $a>0$ and*

$$g>-\frac{2c^2}{9a},\tag{14}$$

then (12) has two trigonometric smooth periodic wave solutions

$$u_1(\xi) = \frac{1}{3a}\left[2\,c \pm \sqrt{18ag+4\,c^2}\cos\left(\frac{\sqrt{a}}{2}\,\xi\right)\right] \tag{15}$$

and

$$u_2(\xi) = \frac{1}{3a}\left[2\,c \pm \sqrt{18ag+4\,c^2}\sin\left(\frac{\sqrt{a}}{2}\,\xi\right)\right]. \tag{16}$$

(2) *Assume a, c and g satisfy that $a < 0$ and*

$$0 < g < \frac{2c^2}{9|a|}. \tag{17}$$

Denote

$$\alpha = \frac{4c^2}{9} + 2ag, \tag{18}$$

$$\beta = \frac{2c}{3} + \sqrt{2|a|g}, \tag{19}$$

$$T = \frac{2}{\sqrt{|a|}}\ln\left|\frac{3\sqrt{2|a|g}+2c}{\sqrt{4c^2+18ag}}\right|, \tag{20}$$

$$\varphi_1(\xi) = -\frac{1}{2\alpha\,\beta}\left[\beta^2\exp\left(-\frac{\sqrt{|a|}}{2}\,\xi\right) + \alpha\exp\left(\frac{\sqrt{|a|}}{2}\,\xi\right) - \frac{4c\,\beta}{3}\right]$$
$$\text{for} \quad c>0 \quad \text{and} \quad -T \le \xi \le T, \tag{21}$$

and

$$\varphi_2(\xi) = -\frac{1}{2\alpha\,\beta}\left[\beta^2\exp\left(\frac{\sqrt{|a|}}{2}\,\xi\right) + \alpha\exp\left(-\frac{\sqrt{|a|}}{2}\,\xi\right) - \frac{4c\,\beta}{3}\right]$$
$$\text{for} \quad c<0 \quad \text{and} \quad -T \le \xi \le T. \tag{22}$$

Then (12) has a periodic-cusp wave solution

$$u_3(\xi) = \varphi_1(\xi + 2nT) \ \text{ for } \ c>0,\ n = 0, \pm1, \pm2, \cdots, \tag{23}$$

$$u_3(\xi) = \varphi_2(\xi + 2nT) \ \text{ for } \ c<0,\ n = 0, \pm1, \pm2, \cdots. \tag{24}$$

For the figures of $u_3(\xi)$ and its numerical simulation with $a = -2$, $c = 9$ and $g = 1$, see Fig.1(a_i) and (b_i) ($i = 1, 2$). When g tends to $\frac{2c^2}{9|a|}$, $u_3(\xi)$ tends to the peakon wave solution $u_3^\circ(\xi)$. For the process of $u_3(\xi)$ changing to $u_3^\circ(\xi)$ with $a = -2$, $c = 9$, $g = 8.5, 8.99, 8.99999, 9$, see Fig.2(a), (b), (c), (d).

Remark 1. When $g = 0$, $u_1(\xi)$ and $u_2(\xi)$ become $u_1^\circ(\xi)$ and $u_2^\circ(\xi)$ (see (2) and (3)) respectively. This implies that $u_1^\circ(\xi)$ and $u_2^\circ(\xi)$ are the special cases of $u_1(\xi)$ and $u_2(\xi)$.

Proposition 2.2. *Consider $K(3,2)$ equation*

$$u_t + a(u^3)_x + (u^2)_{xxx} = 0, \tag{25}$$

and its traveling wave equation

$$2\varphi\varphi''(\xi) + 2(\varphi'(\xi))^2 + a\varphi^3 - c\varphi = g. \tag{26}$$

For arbitrarily given parameter a and constants c, g, basing on two different cases, there are the following results.

(1°) *If a, c and g satisfy that $a > 0$, $c > 0$ and $|g| < g_1$, where*

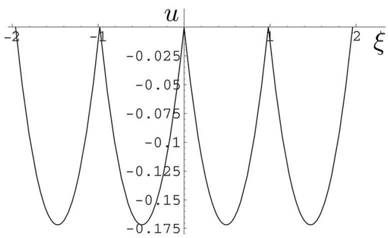
$$g_1 = \sqrt{\frac{80\,c^3}{729\,a}}, \tag{27}$$

then (25) has an elliptic smooth periodic wave solution

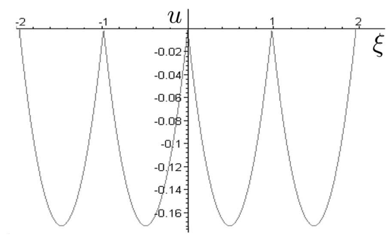
$$u_4(\xi) = a_1 - (a_1 - b_1)\,\mathrm{sn}^2(\eta_1 \xi, k_1), \tag{28}$$

and an elliptic periodic-blow-up solution

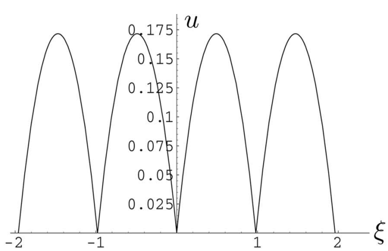
$$u_5(\xi) = a_1 - (a_1 - c_1)\,\mathrm{sn}^{-2}(\eta_1 \xi, k_1), \tag{29}$$



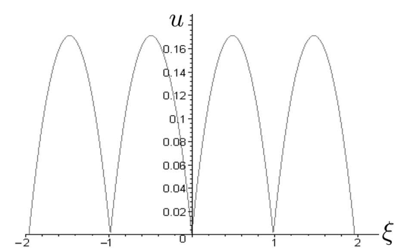
(a_1) graph of $u_3(\xi)$ for $c = 9$



(b_1) simulation of $u_3(\xi)$ for $c = 9$

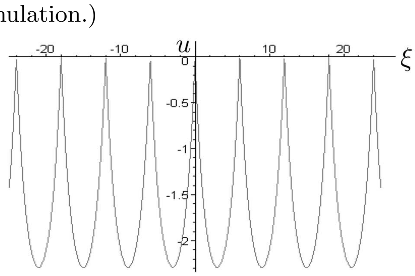


(a_2) graph of $u_3(\xi)$ for $c = -9$

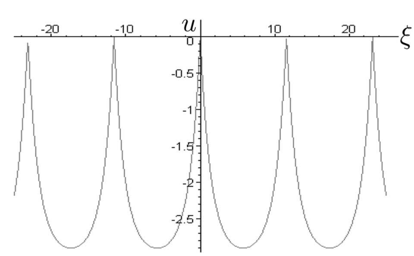


(b_2) simulation of $u_3(\xi)$ for $c = -9$

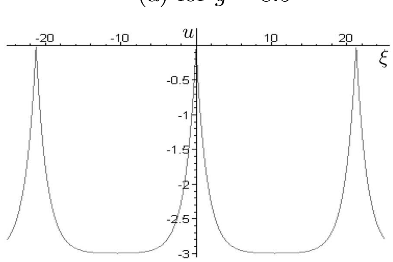
Fig.1 The figures and the numerical simulation of $u = u_3(\xi)$ when $a = -2$, $c = \pm 9$ and $g = 1$. (It is displayed that the figure of $u = u_3(\xi)$ is identical with its numerical simulation.)



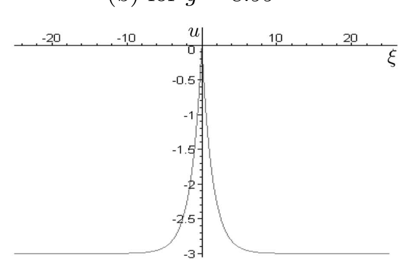
(a) for $g = 8.5$



(b) for $g = 8.99$



(c) for $g = 8.99999$



(d) for $g = 9$

Fig.2 The changing process of $u_3(\xi)$ to $u_3^{\circ}(\xi)$. (It is displayed that the periodic-cusp wave becomes the peakon wave.)

where

$$a_1 = \frac{2\sqrt[3]{100}\,ac + \sqrt[3]{10}\,\sqrt[3]{(27a^2g+\Delta)}^{\,2}}{6a\sqrt[3]{27a^2g+\Delta}}, \tag{30}$$

$$b_1 = \frac{\sqrt[3]{5}\,\left(2\sqrt[3]{10}\,(-1+\sqrt{3}\,\mathrm{i})\,ac - (1+\sqrt{3}\,\mathrm{i})\sqrt[3]{(27a^2g+\Delta)}^{\,2}\right)}{6\sqrt[3]{4}\,a\,\sqrt[3]{27a^2g+\Delta}}, \tag{31}$$

$$c_1 = \frac{\sqrt[3]{5}\,\left(-2\sqrt[3]{10}\,(1+\sqrt{3}\,\mathrm{i})\,ac + (-1+\sqrt{3}\,\mathrm{i})\sqrt[3]{(27a^2g+\Delta)}^{\,2}\right)}{6\sqrt[3]{4}\,a\,\sqrt[3]{27a^2g+\Delta}}, \tag{32}$$

$$\Delta = \sqrt{a^3\,(-80\,c^3 + 729\,ag^2)}, \tag{33}$$

$$k_1 = \sqrt{\frac{a_1-b_1}{a_1-c_1}}, \tag{34}$$

and

$$\eta_1 = \sqrt{\frac{a(a_1-c_1)}{20}}. \tag{35}$$

These two solutions $u_4(\xi)$ and $u_5(\xi)$ possess the following limit properties:

(i) *When $g \to g_1$, the elliptic smooth periodic wave solution $u_4(\xi)$ tends to a hyperbolic smooth solitary wave solution*

$$u_6(\xi) = \sqrt{\frac{5\,c}{9\,a}}\left(-1 + 3\,\mathrm{sech}^2\left(\sqrt[4]{\frac{ac}{80}}\xi\right)\right), \tag{36}$$

and the elliptic periodic-blow-up solution $u_5(\xi)$ tends to a hyperbolic blow-up solution

$$u_7(\xi) = -\sqrt{\frac{5\,c}{9\,a}}\left(1 + 3\,\mathrm{csch}^2\left(\sqrt[4]{\frac{ac}{80}}\xi\right)\right). \tag{37}$$

For the changing process of $u_4(\xi)$ to $u_5(\xi)$ with $a = 2$, $c = 3$ and $g \to g_1 =$ 1.2171612389, *see Fig.3, 4.*

(ii) *When $g \to -g_1$, the elliptic smooth periodic wave solution $u_4(\xi)$ tends to a trivial solution $u_8(\xi) = \sqrt{\frac{5\,c}{9\,a}}$, and the elliptic periodic-blow-up solution $u_5(\xi)$ tends to a trigonometric periodic-blow-up solution*

$$u_9(\xi) = \sqrt{\frac{5\,c}{9\,a}}\left(1 - 3\,\csc^2\left(\sqrt[4]{\frac{ac}{80}}\,\xi\right)\right). \tag{38}$$

(2°) *If a, c and g satisfy that $a < 0$, $c < 0$ and $|g| < g_1$, the (25) has an elliptic smooth periodic wave solution*

$$u_{10}(\xi) = c_2 + (b_2 - c_2)\,\mathrm{sn}^2(\eta_2\xi, k_2), \tag{39}$$

and an elliptic periodic-blow-up solution

$$u_{11}(\xi) = c_2 + (a_2 - c_2)\,\mathrm{sn}^{-2}(\eta_2\xi, k_2), \tag{40}$$

where

$$a_2 = \frac{\sqrt[3]{5}\left(-2\sqrt[3]{10}\,(1+\sqrt{3}\,\mathrm{i})ac + (-1+\sqrt{3}\,\mathrm{i})\sqrt[3]{(27a^2g+\Delta)}^{\,2}\right)}{6\sqrt[3]{4}\,a\,\sqrt[3]{27a^2g+\Delta}}, \tag{41}$$

$$b_2 = \frac{\sqrt[3]{5}\left(2\sqrt[3]{10}\,(-1+\sqrt{3}\,\mathrm{i})ac - (1+\sqrt{3}\,\mathrm{i})\sqrt[3]{(27a^2g+\Delta)}^{\,2}\right)}{6\sqrt[3]{4}\,a\,\sqrt[3]{27a^2g+\Delta}}, \tag{42}$$

$$c_2 = \frac{2\sqrt[3]{100}\,ac + \sqrt[3]{10}\,\sqrt[3]{(27a^2g+\Delta)}^2}{6\,a\sqrt[3]{27a^2g+\Delta}}, \tag{43}$$

$$k_2 = \sqrt{\frac{b_2 - c_2}{a_2 - c_2}}, \tag{44}$$

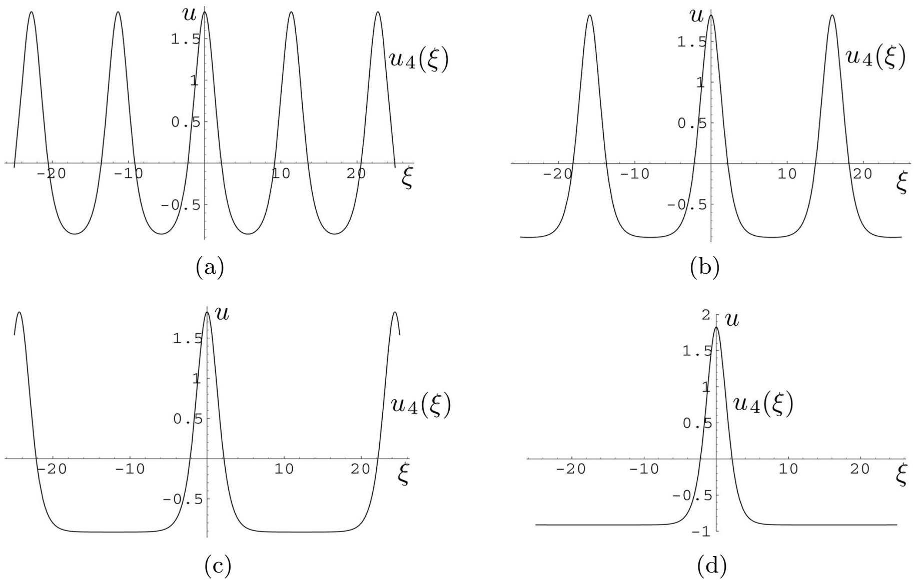


Fig.3 The changing process of $u_4(\xi)$ with $a = 2$, $c = 3$ and (a) $g = 1.21$, (b) $g = 1.2171$, (c) $g = 1.21716123$, (d) $g = 1.2171612389$. (It is displayed that the smooth periodic wave becomes the smooth solitary wave.)

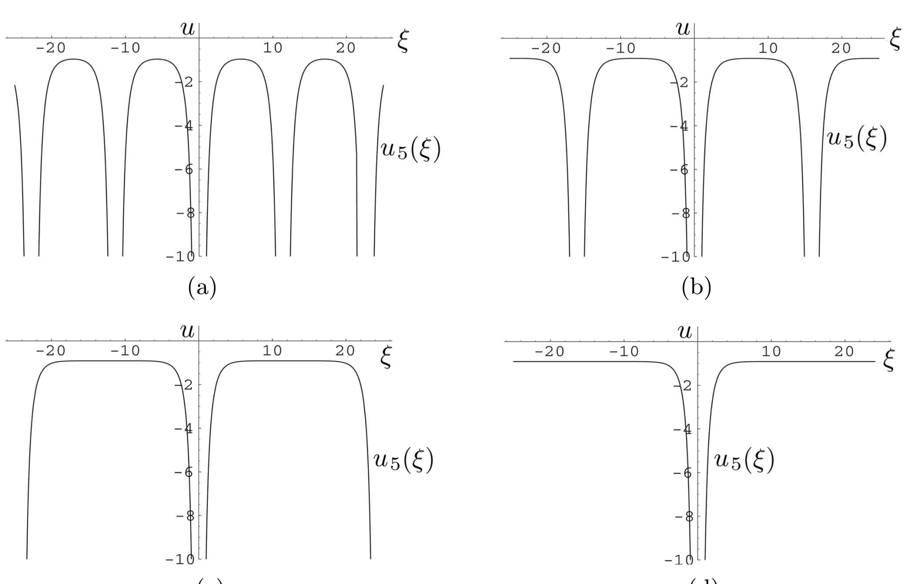


Fig.4 The changing process of $u_5(\xi)$ with $a = 2$, $c = 3$ and (a) $g = 1.21$, (b) $g = 1.2171$, (c) $g = 1.21716123$, (d) $g = 1.2171612389$. (It is displayed that the periodic-blow-up wave becomes a blow-up wave.) and

$$\eta_2 = \sqrt{\frac{|a|\,(a_2 - c_2)}{20}}. \tag{45}$$

These two solutions $u_{10}(\xi)$ and $u_{11}(\xi)$ have the following limit properties:

(i) *When $g \to g_1$, the elliptic smooth periodic wave solution $u_{10}(\xi)$ tends to a hyperbolic smooth solitary wave solution*

$$u_{12}(\xi) = \sqrt{\frac{5\,c}{9\,a}}\left(1 - 3\,\mathrm{sech}^2\left(\sqrt[4]{\frac{a\,c}{80}}\,\xi\right)\right), \tag{46}$$

and the elliptic periodic-blow-up solution $u_{11}(\xi)$ tends to a hyperbolic blow-up solution

$$u_{13}(\xi) = \sqrt{\frac{5\,c}{9\,a}}\left(1 + 3\,\mathrm{csch}^2\left(\sqrt[4]{\frac{a\,c}{80}}\,\xi\right)\right). \tag{47}$$

(ii) *When $g \to -g_1$, the elliptic smooth periodic wave solution $u_{10}(\xi)$ tends to a trivial solution*

$$u_{14}(\xi) = -\sqrt{\frac{5\,c}{9\,a}}, \tag{48}$$

and the elliptic periodic-blow-up solution $u_{11}(\xi)$ tends to a trigonometric periodic-blow-up solution

$$u_{15}(\xi) = \sqrt{\frac{5\,c}{9\,a}}\left(-1 + 3\csc^2\left(\sqrt[4]{\frac{a\,c}{80}}\,\xi\right)\right). \tag{49}$$

Remark 2. When $a > 0$, $c > 0$, $g = 0$, it follows that $a_1 = \sqrt{\frac{5\,c}{3\,a}}$, $b_1 = 0$, $c_1 = -\sqrt{\frac{5\,c}{3\,a}}$, $k_1 = \frac{1}{\sqrt{2}}$, and $\eta_1 = \sqrt[4]{\frac{a\,c}{60}}$. Thus $u_4(\xi)$ becomes $u_6^\circ(\xi)$. This implies that $u_6^\circ(\xi)$ is a special case of $u_4(\xi)$.

Proposition 2.3. *For arbitrarily given constants $a < 0$ and $c \neq 0$, $K(4,2)$ equation*

$$u_t + a(u^4)_x + b(u^2)_{xxx} = 0, \tag{50}$$

has a 1-blow-up solution

$$u_{16}(\xi) = \sqrt[3]{\frac{c}{2\,|a|}}\left(\frac{4(\sqrt{6}-2) + \mathrm{e}^{-\eta_3|\xi|} + (2\sqrt{6}-5)\,\mathrm{e}^{\eta_3|\xi|}}{2(\sqrt{6}-2) - \mathrm{e}^{-\eta_3|\xi|} - (2\sqrt{6}-5)\,\mathrm{e}^{\eta_3|\xi|}}\right), \tag{51}$$

and a 2-blow-up solution

$$u_{17}(\xi) = \sqrt[3]{\frac{c}{2\,|a|}}\left(\frac{-4(\sqrt{6}-2) + \mathrm{e}^{-\eta_3|\xi|} + (2\sqrt{6}-5)\,\mathrm{e}^{\eta_3|\xi|}}{-2(\sqrt{6}-2) - \mathrm{e}^{-\eta_3|\xi|} - (2\sqrt{6}-5)\,\mathrm{e}^{\eta_3|\xi|}}\right), \tag{52}$$

where

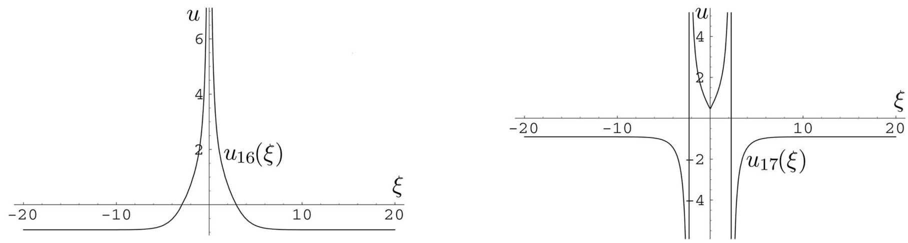
$$\eta_3 = \sqrt[6]{\frac{|a|\,c^2}{4}}. \tag{53}$$

For the figures of $u_{16}(\xi)$ and $u_{17}(\xi)$ with $a = -3$, and $c = 2$, see Fig.5.

Remark 3. We have verified the correctness of these solutions by using the software Mathematica. For example, when $a = \frac{1}{3}$, $c = 3$, $g = -1$, $m = 2$ and $n = 2$, the commands for verifying $u_1(\xi)$ are as follows:

$\mathrm{u} = 6 + \sqrt{30}\ \mathrm{Cos}\left[\frac{\sqrt{3}}{6}\,(\mathrm{x} - \mathrm{c\,t})\right]$

Simplify[D[u, t]+a D[u^2, x]+D[u^2, {x, 3}]]



(a) (b)

Fig.5 The figures of $u_{16}(\xi)$ and $u_{17}(\xi)$ with $a=-3$ and $c=2$.
(a) The figure of $u_{16}(\xi)$. (b) The figure of $u_{17}(\xi)$.

3. **The demonstrations on Proposition 2.1.** In this section we derive the explicit expressions of the smooth periodic wave solutions and the periodic-cusp wave solution for $K(2,2)$ equation. Substituting $u=\varphi(\xi)$ with $\xi=x-ct$ into (12), it follows that

$$-c\varphi'+2a\varphi\varphi'+6\varphi'\varphi''+2\varphi\varphi'''=0. \tag{54}$$

Integrating (54) once, we have

$$2\varphi\varphi''+2(\varphi')^2+a\varphi^2-c\varphi=g. \tag{55}$$

Letting $y=\varphi'$, it yields the following planar system

$$\begin{cases} \frac{\mathrm{d}\varphi}{\mathrm{d}\xi}=y, \\ \frac{\mathrm{d}y}{\mathrm{d}\xi}=\frac{1}{2\varphi}(g+c\varphi-a\varphi^2-2y^2), \end{cases} \tag{56}$$

which has the first integral

$$2y^2\varphi^2=g\varphi^2+\frac{2c}{3}\varphi^3-\frac{a}{2}\varphi^4+h. \tag{57}$$

Solving equation $g+c\varphi-a\varphi^2=0$, we get two roots

$$\varphi_{\pm}^{*}=\frac{1}{2a}\left(c\pm\sqrt{c^2+4ag}\right),\quad \text{where } a>0 \text{ and } g>-\frac{c^2}{4a}. \tag{58}$$

Similarly, solving equation $g+\frac{2c}{3}\varphi^3-\frac{a}{2}\varphi^2=0$, we obtain

$$\varphi_{\pm}^{\circ}=\frac{1}{3a}\left(2c\pm\sqrt{4c^2+18ag}\right),\quad \text{where } a>0 \text{ and } g>-\frac{2c^2}{9}. \tag{59}$$

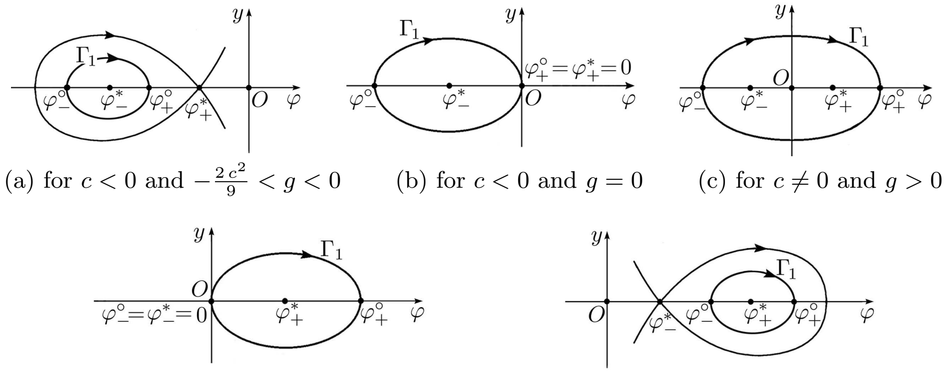
When $a>0$, according to the qualitative theory of differential equations, we draw the special closed orbit Γ_1 which passes $(\varphi_{+}^{\circ},0)$ and $(\varphi_{-}^{\circ},0)$ as Fig.6.

On φ-y plane the orbit Γ_1 has expression

$$y=\pm\sqrt{\frac{a}{2}}\sqrt{g+\frac{2c}{3}\varphi-\varphi^2},\quad \text{where } \varphi_{-}^{\circ}\le\varphi\le\varphi_{+}^{\circ}. \tag{60}$$

Substituting (60) into the first equation of (56) and integrating it along the orbit Γ_1, we obtain the smooth periodic wave solutions $u_1(\xi)$ and $u_2(\xi)$ as (15) and (16).

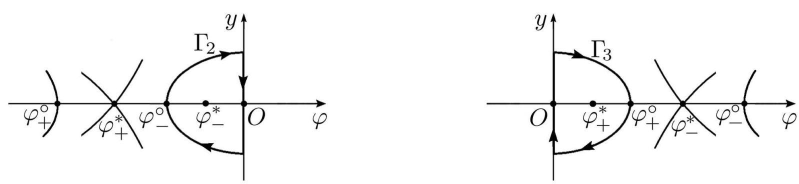
When $a<0$, similarly we get the boundaries (denoted as Γ_2 and Γ_3) of periodic orbits (see Fig.7 (a), (b)).



(a) for $c < 0$ and $-\frac{2\,c^2}{9} < g < 0$ (b) for $c < 0$ and $g = 0$ (c) for $c \neq 0$ and $g > 0$

(d) for $c > 0$ and $g = 0$ (e) for $c > 0$ and $-\frac{2\,c^2}{9} < g < 0$

Fig.6 The graph of the special closed orbit Γ_1 when $a > 0$



(a) for $c > 0$ and $0 < g < \frac{2\,c^2}{9\,|a|}$ (b) for $c < 0$ and $0 < g < \frac{2\,c^2}{9\,|a|}$

Fig.7 The graphs of the special closed orbits Γ_2 and Γ_3 when $a < 0$

Since Γ_2 and Γ_3 also have expression (60), substituting it into the first equation of (56) and integrating, we obtain the periodic-cusp wave solution $u_3(\xi)$ as (23) and (24).

Clearly, when $g \to \frac{2\,c^2}{9\,|a|}$, it follows that $\alpha \to 0$, $\beta \to \frac{4\,c}{3}$ and $T \to +\infty$. This implies that the periodic-cusp wave solution $u_3(\xi)$ tends to the peakon wave solution $u_3^{\circ}(\xi)$ when $g \to \frac{2\,c^2}{9\,|a|}$. These complete the demonstrations on Proposition 2.1.

4. **The demonstrations on Proposition 2.2.** In this section, firstly we derive the explicit expressions of the smooth periodic wave solutions and the periodic-blow-up solutions for $K(3,2)$ equation. Secondly we show their limit forms. Similar to the derivations in Sections 3, substituting $u = \varphi(\xi)$ with $\xi = x - c\,t$ into $K(3,2)$ equation and integrating it, we have the following planar system

$$\begin{cases} \frac{\mathrm{d}\varphi}{\mathrm{d}\xi} = y, \\ \frac{\mathrm{d}y}{\mathrm{d}\xi} = \frac{g + c\,\varphi - a\,\varphi^3 - 2y^2}{2\varphi}, \end{cases} \tag{61}$$

which has the first integral

$$2\varphi^2 y^2 = h + g\varphi^2 + \frac{2\,c}{3}\varphi^3 - \frac{2a}{5}\varphi^5. \tag{62}$$

When the integral constant h is zero, (62) becomes

$$2\varphi^2 y^2 = \varphi^2\left(g + \frac{2\,c}{3}\varphi - \frac{2a}{5}\varphi^3\right). \tag{63}$$

Solving equation $g + \frac{2\,c}{3}\varphi - \frac{2a}{5}\varphi^3 = 0$, we get three roots. When $a > 0$ and $c > 0$, these three roots are written as a_1, b_1 and c_1 in (30)-(32). When $a < 0$ and $c < 0$, the three roots are written as a_2, b_2 and c_2 in (41)-(43).

On the other hand, solving equation

$$\begin{cases} y = 0, \\ g + c\,\varphi - a\,\varphi^3 - 2y^2 = 0, \end{cases} \tag{64}$$

we get three singular points $(\varphi_i, 0)$ $(i = 1, 2, 3)$ of system (61), where

$$\Omega = \sqrt{3a^3(-4\,c^3 + 27ag^2)}, \tag{65}$$

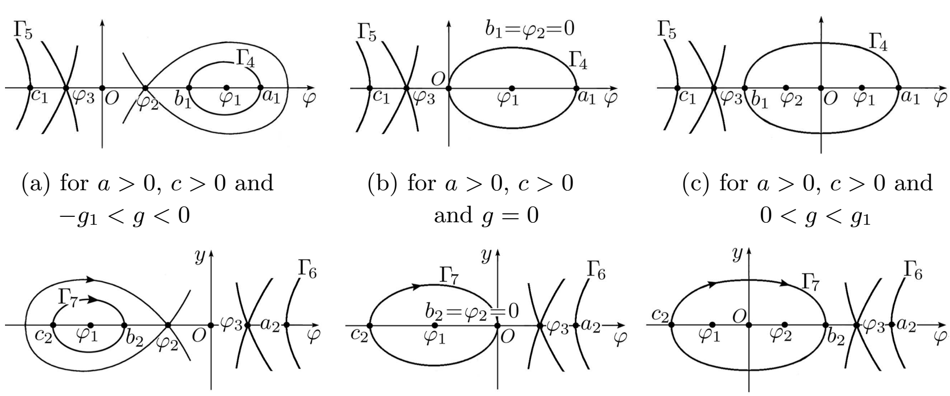
$$\varphi_1 = \frac{2(3\,\mathrm{i} + \sqrt{3})a\,c + \sqrt[3]{2}\,\sqrt[6]{3}\,(1 - \sqrt{3}\,\mathrm{i})\,\sqrt[3]{(-9\,a^2g + \Omega)^2}}{2\,\sqrt[3]{4}\,\sqrt[6]{3^5}\,a\,\sqrt[3]{-9\,a^2g + \Omega}}, \tag{66}$$

$$\varphi_2 = \frac{2(-3\,\mathrm{i} + \sqrt{3})a\,c + \sqrt[3]{2}\,\sqrt[6]{3}\,(1 + \sqrt{3}\,\mathrm{i})\,\sqrt[3]{(-9\,a^2g + \Omega)^2}}{2\,\sqrt[3]{4}\,\sqrt[6]{3^5}\,a\,\sqrt[3]{-9\,a^2g + \Omega}}, \tag{67}$$

and

$$\varphi_3 = \frac{2\,\sqrt[3]{3}\,a\,c + \sqrt[3]{2}\,\sqrt[3]{(-9\,a^2g + \Omega)^2}}{\sqrt[3]{36}\,a\,\sqrt[3]{-9\,a^2g + \Omega}}. \tag{68}$$

According to the qualitative theory of differential equations, we draw the special orbits Γ_i $(i = 4, 5, 6, 7)$ as Fig.8, where Γ_4 passes $(a_1, 0)$ and $(b_1, 0)$, Γ_5 passes $(c_1, 0)$, Γ_6 passes $(a_2, 0)$, Γ_7 passes $(b_2, 0)$ and $(c_2, 0)$.



(a) for $a > 0$, $c > 0$ and $-g_1 < g < 0$

(b) for $a > 0$, $c > 0$ and $g = 0$

(c) for $a > 0$, $c > 0$ and $0 < g < g_1$

(d) for $a < 0$, $c < 0$ and $-g_1 < g < 0$

(e) for $a < 0$, $c < 0$ and $g = 0$

(f) for $a < 0$, $c < 0$ and $0 < g < g_1$

Fig.8 The graphs of the special orbits Γ_i $(i = 4, 5, 6, 7)$ when $ac > 0$

On φ-y plane, Γ_i $(i = 4, 5, 6, 7)$ possess expressions as follows:

$$\Gamma_4: \quad y = \pm\sqrt{\frac{a}{5}}\,\sqrt{(a_1 - \varphi)(\varphi - b_1)(\varphi - c_1)}, \quad \text{where } \ b_1 \le \varphi \le a_1, \tag{69}$$

$$\Gamma_5: \quad y = \pm\sqrt{\frac{a}{5}}\,\sqrt{(a_1 - \varphi)(b_1 - \varphi)(c_1 - \varphi)}, \quad \text{where } -\infty < \varphi \le c_1, \tag{70}$$

$$\Gamma_6: \quad y = \pm\sqrt{\frac{|a|}{5}}\,\sqrt{(\varphi - a_2)(\varphi - b_2)(\varphi - c_2)}, \quad \text{where } \ a_2 \le \varphi < \infty, \tag{71}$$

and

$$\Gamma_7: \quad y = \pm\sqrt{\frac{|a|}{5}}\,\sqrt{(a_2 - \varphi)(b_2 - \varphi)(\varphi - c_2)}, \quad \text{where } \ c_2 \le \varphi \le b_2. \tag{72}$$

Substituting the expressions of Γ_i $(i = 4,5,6,7)$ into the first equation and integrating it along the orbits Γ_i $(i = 4,5,6,7)$, we have

$$\int_{b_1}^{\varphi} \frac{\mathrm{d}s}{\sqrt{(a_1 - s)(s - b_1)(s - c_1)}} = \sqrt{\frac{a}{5}}\,|\xi|, \quad c_1 < b_1 \le \varphi < a_1 \text{ (along } \Gamma_4), \tag{73}$$

$$\int_{-\infty}^{\varphi} \frac{\mathrm{d}s}{\sqrt{(a_1 - s)(b_1 - s)(c_1 - s)}} = \sqrt{\frac{a}{5}}\,|\xi|, \quad \varphi \le c_1 < b_1 < a_1 \text{ (along } \Gamma_5), \tag{74}$$

$$\int_{\varphi}^{\infty} \frac{\mathrm{d}s}{\sqrt{(s - a_2)(s - b_2)(s - c_2)}} = \sqrt{\frac{|a|}{5}}\,|\xi|, \quad c_2 < b_2 < a_2 \le \varphi \text{ (along } \Gamma_6), \tag{75}$$

and

$$\int_{c_2}^{\varphi} \frac{\mathrm{d}s}{\sqrt{(a_2 - s)(b_2 - s)(s - c_2)}} = \sqrt{\frac{|a|}{5}}\,|\xi|, \quad c_2 < \varphi \le b_2 < a_2 \text{ (along } \Gamma_7), \tag{76}$$

Completing the four integrals above and noting that $u = \varphi(\xi)$, we obtain $u_4(\xi)$, $u_5(\xi)$, $u_{10}(\xi)$ and $u_{11}(\xi)$ as (28), (29), (39) and (40).

Now we show the limit forms of $u_4(\xi)$, $u_5(\xi)$, $u_{10}(\xi)$ and $u_{11}(\xi)$ when $|g| \to g_1$. From (30)-(35) and (41)-(45), we see that when $g \to g_1$, it follows that

$$a_1 \to 2\sqrt{\frac{5c}{9a}}, \tag{77}$$

$$c_1 \to -\sqrt{\frac{5c}{9a}} \quad \text{and} \quad b_1 \to -\sqrt{\frac{5c}{9a}}, \tag{78}$$

$$a_2 \to \sqrt{\frac{5c}{9a}} \quad \text{and} \quad b_2 \to \sqrt{\frac{5c}{9a}}, \tag{79}$$

$$c_2 \to -2\sqrt{\frac{5c}{9a}}, \tag{80}$$

$$k_2 \to 1 \quad \text{and} \quad k_1 \to 1 \tag{81}$$

and

$$\eta_2 \to \sqrt[4]{\frac{ac}{80}} \quad \text{and} \quad \eta_1 \to \sqrt[4]{\frac{ac}{80}}. \tag{82}$$

Note that

$$\operatorname{sn}(\xi, 1) = \tanh \xi, \tag{83}$$

$$\operatorname{sech}^2 \xi = 1 - \tanh^2 \xi, \tag{84}$$

$$\coth \xi = \frac{1}{\tanh \xi}, \tag{85}$$

and

$$\operatorname{csch}^2 \xi = \coth^2 \xi - 1. \tag{86}$$

Thus from (28), (29), (39), (40) and (77)-(86), we obtain the limit forms $u_6(\xi)$, $u_7(\xi)$, $u_{12}(\xi)$ and $u_{13}(\xi)$ (see (36), (37), (46) and (47)).

On the other hand, when $g \to -g_1$, it follows that

$$a_1 \to \sqrt{\frac{5c}{9a}} \quad \text{and} \quad b_1 \to \sqrt{\frac{5c}{9a}}, \tag{87}$$

$$c_1 \to -2\sqrt{\frac{5c}{9a}}, \tag{88}$$

$$a_2 \to 2\sqrt{\frac{5c}{9a}}, \tag{89}$$

$$c_2 \to -\sqrt{\frac{5c}{9a}} \text{ and } b_2 \to -\sqrt{\frac{5c}{9a}}, \tag{90}$$

$$k_1 \to 0 \text{ and } k_2 \to 0 \tag{91}$$

and

$$\eta_1 \to \sqrt[4]{\frac{ac}{80}} \text{ and } \eta_2 \to \sqrt[4]{\frac{ac}{80}}. \tag{92}$$

Note that

$$\operatorname{sn}(\xi, 0) = \sin\xi = \frac{1}{\csc\xi}. \tag{93}$$

From (28), (29), (39), (40) and (87)-(93), we obtain the limit forms $u_8(\xi)$, $u_9(\xi)$, $u_{14}(\xi)$ and $u_{15}(\xi)$ (see (38), (48) and (49)).

These complete the demonstration on Proposition 2.2.

5. **The demonstrations on Proposition 2.3.** In this section we derive the explicit expressions of the blow-up solutions for $K(4,2)$ equation. Similar to the derivations in Sections 3, substituting $u = \varphi(\xi)$ with $\xi = x - ct$ into $K(4,2)$ equation and integrating it, we have the following planar system

$$\begin{cases} \frac{d\varphi}{d\xi} = y, \\ \frac{dy}{d\xi} = \frac{g + c\varphi - a\varphi^4 - 2y^2}{2\varphi}, \end{cases} \tag{94}$$

which has the first integral

$$2\varphi^2 y^2 = h + g\varphi^2 + \frac{2c}{3}\varphi^3 - \frac{a}{3}\varphi^6, \tag{95}$$

where h is the integral constant.

When $a < 0$, $h = 0$ and $g = \frac{\sqrt[3]{c^4}}{\sqrt[3]{16|a|}}$, (95) becomes

$$y^2 = \sqrt{\frac{|a|}{6}}(\varphi + p)^2(\varphi^2 + g\varphi + r), \tag{96}$$

where

$$p = \sqrt[3]{\frac{c}{2a}}, \tag{97}$$

$$q = -\sqrt[3]{\frac{4c}{a}}, \tag{98}$$

and

$$r = 3\sqrt[3]{\left(\frac{c}{2a}\right)^2}. \tag{99}$$

Substituting (96) into the first equation of (94) and integrating it from $-\infty$ to φ, or φ to ∞, we have

$$\int_{-\infty}^{\varphi} \frac{ds}{(s+p)\sqrt{s^2 + qs + r}} = -\sqrt{\frac{|a|}{6}}\,|\xi| \tag{100}$$

and

$$\int_{\varphi}^{\infty} \frac{ds}{(s+p)\sqrt{s^2 + qs + r}} = \sqrt{\frac{|a|}{6}}\,|\xi|. \tag{101}$$

Completing the two integrals above and noting that $u = \varphi(\xi)$, we obtain two blow-up solutions $u_{16}(\xi)$ and $u_{17}(\xi)$ as (51)-(52). This completes the demonstration on Proposition 2.3.

6. **Conclusion.** In this paper we have studied $K(m, 2)$ $(m = 2, 3, 4)$ equations. We have got some new results which were listed in Proposition 2.1, 2.2, 2.3. Not only have some previous works become our special cases, but many new explicit solutions have been obtained. So far, for the three equations above, we have the following conclusions on the explicit traveling wave solutions:

(i) When $a > 0$, $K(2, 2)$ equation has two types of explicit traveling wave solutions, the smooth periodic wave solution and the periodic-cusp wave solution.

(ii) When $a < 0$, $K(2, 2)$ equation has three types of explicit traveling wave solutions, the periodic-cusp wave solution, the peakon wave solution and the blow-up solution.

(iii) When $a \neq 0$, $K(3, 2)$ equation has six types of explicit traveling wave solutions, the smooth periodic wave solution, the periodic-cusp wave solution, the periodic-blow-up solution, the 1-blow-up solution, the smooth solitary wave solution and the peakon wave solution.

(iv) When $a > 0$, $K(4, 2)$ equation has one type of explicit traveling wave solution, that is, the smooth periodic wave solution.

(v) When $a < 0$, $K(4, 2)$ equation three types of explicit traveling wave solutions, that is, the peakon wave solution, the 1-blow-up and 2-blow-up solutions.

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