Chapter 7  Symmetric Matrices and Quadratic Forms

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§ 7.1 Diagonalization of Symmetric Matrices

§ 7.2 Quadratic Forms
Symmetric Matrices

See the following matrices:

\[
\begin{bmatrix}
1 & 0 \\
0 & -3
\end{bmatrix}, \quad \begin{bmatrix}
0 & -1 & 0 \\
-1 & 5 & 8 \\
0 & 8 & -7
\end{bmatrix}, \quad \begin{bmatrix}
a & b & c \\
b & d & e \\
c & e & f
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & -3 \\
3 & 0
\end{bmatrix}, \quad \begin{bmatrix}
1 & -4 & 0 \\
-6 & 1 & -4 \\
0 & -6 & 1
\end{bmatrix}, \quad \begin{bmatrix}
5 & 4 & 3 & 2 \\
4 & 3 & 2 & 1 \\
3 & 2 & 1 & 0
\end{bmatrix}
\]

Try to figure out the main differences between the matrices above and the ones below.
Symmetric Matrices

Definition

An $n \times n$ matrix $A$ is called a symmetric matrix if

\[ a_{ij} = a_{ji}. \]

In other word, a symmetric matrix satisfies

\[ A = A^T. \]
Exercise: [P454, 1-6] Determine which of the following matrices are symmetric

1. \[
\begin{bmatrix}
3 & 5 \\
5 & -7 
\end{bmatrix}
\]

2. \[
\begin{bmatrix}
-3 & 5 \\
-5 & 3 
\end{bmatrix}
\]

3. \[
\begin{bmatrix}
2 & 2 \\
4 & 4 
\end{bmatrix}
\]

4. \[
\begin{bmatrix}
0 & 8 & 3 \\
8 & 0 & -2 \\
3 & -2 & 0 
\end{bmatrix}
\]

5. \[
\begin{bmatrix}
-6 & 2 & 0 \\
0 & -6 & 2 \\
0 & 0 & -6 
\end{bmatrix}
\]

6. \[
\begin{bmatrix}
1 & 2 & 1 & 2 \\
2 & 1 & 2 & 1 \\
1 & 2 & 1 & 2 
\end{bmatrix}
\]
Diagonalization of Symmetric Matrices

Example

If possible, diagonalize the matrix

\[ A = \begin{pmatrix} 6 & -2 & -1 \\ -2 & 6 & -1 \\ -1 & -1 & 5 \end{pmatrix} \]
**Solution:** The characteristic equation of $A$ is

$$0 = -\lambda^3 + 17\lambda^2 - 90\lambda + 144 = -(\lambda - 8)(\lambda - 6)(\lambda - 3)$$

Standard calculations produce a basis for each eigenspace:

$\lambda = 8$: $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$; 
$\lambda = 6$: $\mathbf{v}_2 = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$;  
$\lambda = 3$: $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

These three vectors form a basis for $\mathbb{R}^3$. Set a matrix $P = [\vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3]$. Then $P$ diagonalizes $A$. 
Diagonalization of Symmetric Matrices

\[ \lambda = 8: \mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}; \quad \lambda = 6: \mathbf{v}_2 = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}; \quad \lambda = 3: \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \]

It is easy to verify that \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} is an orthogonal set. We can further normalize \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \) to produce the unit eigenvectors, that is,

\[ \mathbf{u}_1 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \]
Set a new matrix $P$ and a diagonal matrix $D$ as follows

$$P = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}, \quad D = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Then $A = PDP^{-1}$, as usual. However, since $P$ has orthonormal columns, $P$ is an orthogonal matrix, and $P^{-1} = P^T$. 
Diagonalization of Symmetric Matrices

We already know the fact that eigenvectors corresponding to distinct eigenvalues are **linearly independent**.

From this example, it seems that the eigenvectors corresponding to distinct eigenvalues are also **orthogonal**.
Theorem (Orthogonality of eigenvectors of symmetric matrix)

If matrix $A$ is symmetric, then any two eigenvectors from different eigenspaces are orthogonal.
**Proof:** Let \( \vec{v}_1 \) and \( \vec{v}_2 \) be eigenvectors that correspond to distinct eigenvalues, say, \( \lambda_1 \) and \( \lambda_2 \). To show that \( \vec{v}_1 \cdot \vec{v}_2 = 0 \), compute

\[
\lambda_1 \mathbf{v}_1 \cdot \mathbf{v}_2 = (\lambda_1 \mathbf{v}_1)^T \mathbf{v}_2 = (A\mathbf{v}_1)^T \mathbf{v}_2 \\
= (\mathbf{v}_1^T A^T) \mathbf{v}_2 = \mathbf{v}_1^T (A\mathbf{v}_2) \\
= \mathbf{v}_1^T (\lambda_2 \mathbf{v}_2) \\
= \lambda_2 \mathbf{v}_1^T \mathbf{v}_2 = \lambda_2 \mathbf{v}_1 \cdot \mathbf{v}_2
\]

From \( (\lambda_1 - \lambda_2) \vec{v}_1 \cdot \vec{v}_2 = 0 \), we know \( \vec{v}_1 \cdot \vec{v}_2 = 0 \) since \( \lambda_1 \neq \lambda_2 \).
Diagonalization of Symmetric Matrices

**Definition**

A matrix $A$ is said to be **orthogonally diagonalizable** if there is an orthogonal matrix $P$ such that $P^{-1}AP$ is a diagonal matrix $D$.

That is

$$A = PDP^{-1} = PDP^T$$

since $P^{-1} = P^T$. 
Theorem (Orthogonally diagonalizable matrix)

An $n \times n$ matrix $A$ is orthogonally diagonalizable if and only if $A$ is a symmetric matrix.

Proof: $\implies$

$$A^T = (PDP^T)^T = P^T D^T P^T = PDP^T = A$$
Example (Orthogonally diagonalizable matrix)

Orthogonally diagonalize the matrix $A = \begin{pmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 5 \end{pmatrix}$.

That is, find an orthogonal matrix $P$ and a diagonal matrix $D$, such that $A = PDP^{-1} = PDP^T$. 

Diagonalization of Symmetric Matrices
Diagonalization of Symmetric Matrices

Solution:

Step 1  The characteristic equation of $A$ is

$$0 = -\lambda^3 + 12\lambda^2 - 21\lambda - 98 = -(\lambda - 7)^2(\lambda + 2)$$

Step 2  Standard calculations produce a basis for each eigenspace:

$\lambda = 7$: $v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix}$; $\lambda = -2$: $v_3 = \begin{bmatrix} -1 \\ -1/2 \\ 1 \end{bmatrix}$

Although $\vec{v}_1$ and $\vec{v}_2$ are linearly independent, they are not orthogonal.
Step 3 Recall the orthogonal projection in Section 6.2 that the projection of $\vec{v}_2$ onto $\vec{v}_1$ is

$$\frac{\vec{v}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1.$$ 

and the component of $\vec{v}_2$ orthogonal to $\vec{v}_1$ is

$$z_2 = v_2 - \frac{v_2 \cdot v_1}{v_1 \cdot v_1} v_1 = \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix} - \frac{-1/2}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/4 \\ 1 \\ 1/4 \end{bmatrix}$$
Then \( \{ \vec{v}_1, \vec{z}_2 \} \) is an orthogonal set in the eigenspace for \( \lambda = 7 \). (\( \vec{z}_2 \) is a linear combination of the eigenvectors \( \vec{v}_1 \) and \( \vec{v}_2 \), so \( \vec{z}_2 \) is in the eigenspace for \( \lambda = 7 \).)

Since the eigenspace is two dimensional, the orthogonal set \( \{ \vec{v}_1, \vec{z}_2 \} \) is an orthogonal basis for the eigenspace.

**Step 4** Normalizing the orthogonal basis \( \{ \vec{v}_1, \vec{z}_2 \} \), we obtain the following orthonormal basis for the eigenspace for \( \lambda = 7 \).

\[
\mathbf{u}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -1/\sqrt{18} \\ 4/\sqrt{18} \\ 1/\sqrt{18} \end{bmatrix}
\]
An orthonormal basis for the eigenspace for $\lambda = -2$ is

$$
\mathbf{u}_3 = \frac{1}{\|2\mathbf{v}_3\|} 2\mathbf{v}_3 = \frac{1}{3} \begin{bmatrix}
-2 \\
-1 \\
2
\end{bmatrix} = \begin{bmatrix}
-2/3 \\
-1/3 \\
2/3
\end{bmatrix}
$$

$\mathbf{u}_3$ is definitely orthogonal to $\mathbf{u}_1$ and $\mathbf{u}_2$. (Why?)
Step 5 Hence $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an orthonormal set. Let

$$P = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{18} & -2/3 \\ 0 & 4/\sqrt{18} & -1/3 \\ 1/\sqrt{2} & 1/\sqrt{18} & 2/3 \end{bmatrix}, \quad D = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Then $P$ orthogonally diagonalizes $A$, and $A = PDP^{-1} = PDPT$. 
The set of eigenvalues of a matrix $A$ is also called the spectrum (谱) of $A$.

For all the eigenvalues of a symmetric matrix $A$, we have the following theorem.
Theorem (The Spectral Theorem for Symmetric Matrices)

An $n \times n$ symmetric matrix $A$ has the following properties

1. $A$ has $n$ real eigenvalues (counting multiplicities).

2. The dimension of the eigenspace for each eigenvalue $\lambda$ equals the multiplicity of $\lambda$.

3. The eigenspaces are orthogonal to each other (eigenvectors corresponding to different eigenspaces are orthogonal).

4. $A$ is orthogonally diagonalizable. That is, we can find an orthogonal matrix $P$ and a diagonal matrix $D$, such that $A = PDP^{-1} = PDP^T$. 
The Spectral Decomposition

Proposition (The diagonalization of symmetric matrix)

For an $n \times n$ symmetric matrix $A$, suppose $A = PDP^{-1}$.

1. The columns of $P$ are orthonormal eigenvectors $\vec{u}_1, ..., \vec{u}_n$ of $A$,
2. The eigenvalues $\lambda_1, ..., \lambda_n$ of $A$ are in the diagonal matrix $D$.
3. $P^{-1} = P^T$. 
The Spectral Decomposition

Since $P^{-1} = P^T$, we have

$$A = PDP^T = \begin{bmatrix} \mathbf{u}_1 & \cdots & \mathbf{u}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \mathbf{u}_1^T \\ \vdots \\ \mathbf{u}_n^T \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 \mathbf{u}_1 & \cdots & \lambda_n \mathbf{u}_n \end{bmatrix} \begin{bmatrix} \mathbf{u}_1^T \\ \vdots \\ \mathbf{u}_n^T \end{bmatrix}$$

Therefore, $A$ can be written as

$$A = \lambda_1 \mathbf{u}_1 \mathbf{u}_1^T + \lambda_2 \mathbf{u}_2 \mathbf{u}_2^T + \cdots + \lambda_n \mathbf{u}_n \mathbf{u}_n^T$$
The following representation

\[ A = \lambda_1 \vec{u}_1 \vec{u}_1^T + \lambda_2 \vec{u}_2 \vec{u}_2^T + \cdots + \lambda_n \vec{u}_n \vec{u}_n^T \]

is called a spectral decomposition of \( A \).

It breaks up \( A \) into \( n \) pieces determined by the spectrum (eigenvalues) of \( A \).

Each term is an \( n \times n \) matrix of rank 1. Every column of \( \lambda_i \vec{u}_i \vec{u}_i^T \) is a multiple of \( \vec{u}_i \).
Homework:

Section 7.1 p. 431: 19, 20;
Quadratic Forms

\[3x_1^2 - 4x_1 x_2 + 6x_2^2\]
\[2x_1^2 + 10x_1 x_2 + 2x_2^2\]
\[x_1^2 - 6x_1 x_2 + 9x_2^2\]
\[x_1^2 + x_2^2 + x_3^2 + x_4^2 + 9x_1 x_2 - 12x_1 x_4 + 12x_2 x_3 + 9x_3 x_4\]
\[11x_1^2 - x_2^2 - 12x_1 x_2 - 12x_1 x_3 - 12x_1 x_4 - 2x_3 x_4\]
A quadratic form is defined as

\[ Q(\vec{x}) = \vec{x}^T A \vec{x}, \]

where \( A \) is an \( n \times n \) symmetric matrix, and \( \vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}. \]
Example

Let \( \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \). Compute \( Q(\vec{x}) = \vec{x}^T A \vec{x} \) for the following matrices:

(1) \( A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \), (2) \( A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \)

(3) \( A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 3 \\ 0 & 3 & 1 \end{pmatrix} \), (4) \( A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 3 \\ 1 & 3 & 1 \end{pmatrix} \).
Quadratic Forms

\[
Q(\vec{x}) = \vec{x}^T A \vec{x}
\]

(1) \( A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad 2x_1^2 + 2x_2^2 + x_3^2 \)

(2) \( A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad 2x_1^2 + 2x_2^2 + x_3^2 - 2x_1x_2 \)

(3) \( A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad 2x_1^2 + 2x_2^2 + x_3^2 - 2x_1x_2 + 2x_1x_3 \)
Quadratic Forms

(4) \[ A = \begin{pmatrix} 2 & -1 & 1 \\
-1 & 2 & 3 \\
1 & 3 & 1 \end{pmatrix} \]

\[ Q(\vec{x}) = \vec{x}^T A \vec{x} = 2x_1^2 + 2x_2^2 + x_3^2 - 2x_1x_2 + 2x_1x_3 + 6x_2x_3 \]
Example:

Find the symmetric matrices $A$, such that the following quadratic forms $Q(x)$ transform to $\vec{x}^T A \vec{x}$

$$3x_1^2 - 4x_1 x_2 + 6x_2^2$$
$$2x_1^2 + 10x_1 x_2 + 2x_2^2$$
$$x_1^2 - 6x_1 x_2 + 9x_2^2$$
Example:

For a quadratic form $Q(\vec{x}) = 2x_1^2 - 16x_1x_2 - 10x_2^2$. Compute the value of $Q(\vec{x})$ for

$\vec{x} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 2 \\ -2 \end{pmatrix}$, $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$
Example:

For a quadratic form $Q(\vec{x}) = 2x_1^2 - 16x_1x_2 - 10x_2^2$. Compute the value of $Q(\vec{x})$ for

$$\vec{x} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

$Q(-3, 1) = 56$
$Q(2, -2) = 32$
$Q(1, -3) = -40$
Change of Variables in a Quadratic Form

Definition (Change of Variables)

If $\vec{x}$ represents a variable vector in $\mathbb{R}^n$, then a change of variable is an equation of the form

$$\vec{x} = P\vec{y}$$

or equivalently

$$\vec{y} = P^{-1}\vec{x}$$

where $P$ is an invertible matrix and $\vec{y}$ is a new variable in $\mathbb{R}^n$. 
Proposition (Change of Variables in a Quadratic Form)

Since

\[ \vec{x}^T A \vec{x} = (P\vec{y})^T A (P\vec{y}) = \vec{y}^T P^T A P \vec{y} = \vec{y}^T (P^T A P) \vec{y} \]

1. The original quadratic form \( \vec{x}^T A \vec{x} \) is transformed into a new quadratic form \( \vec{y}^T (P^T A P) \vec{y} \).

2. The original symmetric matrix \( A \) is transformed into a new symmetric matrix \( P^T A P \).
Example (Change of Variables in a Quadratic Form)

Find a change of variable that transforms the quadratic form

\[ Q(\vec{x}) = x_1^2 - 8x_1x_2 - 5x_2^2 \]

into a new quadratic form with no cross-product term.
Solution:

The symmetric matrix of the quadratic form \( Q(\vec{x}) = x_1^2 - 8x_1x_2 - 5x_2^2 \) is

\[
A = \begin{pmatrix} 1 & -4 \\ -4 & -5 \end{pmatrix}
\]

First step: orthogonally diagonalize \( A \).

Its eigenvalues can be computed as \( \lambda = 3 \) and \( \lambda = -7 \).

The corresponding unit eigenvectors are:

\[
\lambda = 3: \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix}; \quad \lambda = -7: \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}
\]
These vectors are automatically orthogonal and provide an orthonormal basis for $\mathbb{R}^2$. Let

$$P = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 0 \\ 0 & -7 \end{bmatrix}$$

Then $A = PDP^{-1}$ and $D = P^{-1}AP = P^TAP$. 
Second step: find a change of variable $\vec{x} = P\vec{y}$, where

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{and} \quad \vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Then

$$x_1^2 - 8x_1x_2 - 5x_2^2 = \mathbf{x}^T A \mathbf{x} = (P \mathbf{y})^T A (P \mathbf{y})$$

$$= \mathbf{y}^T P^T A P \mathbf{y} = \mathbf{y}^T D \mathbf{y}$$

$$= 3y_1^2 - 7y_2^2$$
Theorem (Change of Variables in a Quadratic Form)

Let \( A \) be an \( n \times n \) symmetric matrix. Then there is an orthogonal change of variable, \( \mathbf{x} = P\mathbf{y} \), which transforms the quadratic form \( \mathbf{x}^T A \mathbf{x} \) into a new quadratic form \( \mathbf{y}^T D \mathbf{y} \) with no cross-product term.

\[
\mathbf{x}^T A \mathbf{x} = (P\mathbf{y})^T A (P\mathbf{y}) = \mathbf{y}^T (P^T A P) \mathbf{y} = \mathbf{y}^T D \mathbf{y}
\]
Classifying Quadratic Forms

(a) $z = 3x_1^2 + 7x_2^2$

(b) $z = 3x_1^2$

(c) $z = 3x_1^2 - 7x_2^2$

(d) $z = -3x_1^2 - 7x_2^2$
Definition (Classifying Quadratic Forms)

A quadratic form $Q$ is

1. **Positive definite** (正定的) if $Q(\vec{x}) > 0$ for all $x \neq \vec{0}$.

2. **Negative definite** (负定的) if $Q(\vec{x}) < 0$ for all $x \neq \vec{0}$.

3. **Indefinite definite** (不定的) if $Q(\vec{x})$ assumes both positive and negative values.
Theorem (Classifying Quadratic Forms)

Let $A$ be an $n \times n$ symmetric matrix. Then a quadratic form $\mathbf{x}^T A \mathbf{x}$ is:

1. Positive definite if and only if the eigenvalues of $A$ are all positive, i.e. all $\lambda > 0$.
2. Negative definite if and only if the eigenvalues of $A$ are all negative, i.e. all $\lambda < 0$.
3. Indefinite definite if and only if $A$ has both positive and negative eigenvalues, i.e. some $\lambda > 0$ and some $\lambda < 0$. 
Proof

\[ Q(\vec{x}) = \vec{x}^T A \vec{x} = \vec{y}^T D \vec{y} = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2 \]

where \( \lambda_i \) are the eigenvalues of \( A \). Therefore, the signs of the quadratic form is controlled by the eigenvalues \( \lambda_i \).
Example (Quadratic form)

Determine whether

\[ Q(\vec{x}) = 3x_1^2 + 2x_2^2 + x_3^2 + 4x_1x_2 + 4x_2x_3 \]

is positive definite.

The eigenvalues of \( A \) is 5, 2, and \(-1\). Therefore, \( Q \) is an indefinite quadratic form, not positive definite.
Determine whether

\[ Q(\vec{x}) = 3x_1^2 + 2x_2^2 + x_3^2 + 4x_1x_2 + 4x_2x_3 \]

is positive definite.

The eigenvalues of \( A \) is 5, 2, and \(-1\). Therefore, \( Q \) is an indefinite quadratic form, not positive definite.
Homework:

Section 7.2 p. 439: 8, 9, 10, 11;

(Justify which type the above quadratic forms are)