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SEVERAL NEW TYPES OF SOLITARY WAVE SOLUTIONS FOR THE GENERALIZED CAMASSA-HOLM-DEGASPERIS-PROCESI EQUATION

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Abstract. In this paper, we study the nonlinear wave solutions of the generalized Camassa-Holm-Degasperis-Procesi equation $u_t - u_{xxt} + (1+b)u^2u_x = bu_xu_{xx} + uu_{xxx}$. Through phase analysis, several new types of the explicit nonlinear wave solutions are constructed. Our concrete results are: (i) For given $b > -1$, if the wave speed equals $\frac{1}{1+b}$, then the explicit expressions of the smooth solitary wave solution and the singular wave solution are given. (ii) For given $b > -1$, if the wave speed equals $1+b$, then the explicit expressions of the peakon wave solution and the singular wave solution are got. (iii) For given $b > -2$ and $b \neq -1$, if the wave speed equals $\frac{2+b}{2}$, then the explicit smooth solitary wave solution, the peakon wave solution and the singular wave solution are obtained. We also verify the correctness of these solutions by using the software Mathematica. Our work extends some previous results.

1. **Introduction.** Consider the following nonlinear equation

$$u_t - u_{xxt} + (1+b)u^2u_x = bu_xu_{xx} + uu_{xxx}, \tag{1}$$

which was proposed by Wazwaz [1], where $b \neq -1$. Eq.(1) is a generalized form of the Camassa-Holm-Degasperis-Procesi equation [2]-[4]

$$u_t - u_{xxt} + (1+b)uu_x = bu_xu_{xx} + uu_{xxx}. \tag{2}$$

When $b = 2$, Eq.(2) becomes the CH equation

$$u_t - u_{xxt} + 3uu_x = 2u_xu_{xx} + uu_{xxx}, \tag{3}$$

and Eq.(1) becomes a generalized CH equation

$$u_t - u_{xxt} + 3u^2u_x = 2u_xu_{xx} + uu_{xxx}. \tag{4}$$

While $b = 3$, Eq.(2) becomes the DP equation

$$u_t - u_{xxt} + 4uu_x = 3u_xu_{xx} + uu_{xxx}, \tag{5}$$

and Eq.(1) becomes a generalized DP equation

$$u_t - u_{xxt} + 4u^2u_x = 3u_xu_{xx} + uu_{xxx}. \tag{6}$$

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Eq.(3) was derived by Camassa and Holm [5] in 1993. It was found that Eq.(3) is of many rich structures and properties, for instance [6]-[17]. Eq.(5) was given by Degasperis and Procesi [2] in 1999, and it was also investigated in many literatures, for instance [18]-[21].

Recently, Eq.(4) and Eq.(6) have also been investigated by many authors. Liu and Qian [22] studied the peakons and their bifurcation for Eq.(4). Tian and Song [23] gave some physical explanations and obtained the peakons composed of the hyperbolic function $\tanh z$ for Eq.(4). Shen and Xu [24] discussed the bifurcations of the smooth and non-smooth traveling waves for Eq.(4). When the wave speed equals 1, Khuri [25] obtained a singular wave solution composed of the trigonometric functions for Eq.(4). When the wave speed equals 1 or 2, Wazwaz [26] obtained eleven exact traveling wave solutions composed of the trigonometric functions or the hyperbolic functions, and Liu et al [27] got a peakon solution composed of the hyperbolic functions for Eq.(4). Liu and Guo [28] investigated the periodic blow-up solutions and their limit forms for Eq.(4). When the wave speed equals 5/2, Wazwaz [26] obtained nine exact traveling wave solutions composed of the hyperbolic functions, and Liu et al [27] got a peakon solution composed of the hyperbolic functions for Eq.(6). In Ref.[29] Wang and Tang obtained four exact solutions to Eq.(4) and Eq.(6) respectively. For wave speed $\frac{2+b}{2}$, Wazwaz [1] showed that the bell-shaped solitary wave and singular wave coexist in Eq.(1) and gave their expressions as

$$\widetilde{u}_1(x,t) = -\frac{3(2+b)}{2(1+b)}\left[1-\tanh^2\left(\frac{1}{2}x-\frac{2+b}{4}t\right)\right], \tag{7}$$

and

$$\widetilde{u}_2(x,t) = -\frac{3(2+b)}{2(1+b)}\left[1-\coth^2\left(\frac{1}{2}x-\frac{2+b}{4}t\right)\right]. \tag{8}$$

In this paper, we investigate the explicit nonlinear wave solutions to Eq.(1). Some previous results [1], [22], [26]-[29] are extended. Our main results and remarks are arranged in Section 2. In Section 3 we give some preliminaries. The demonstrations of the main results are given in Sections 4, 5, 6. A short conclusion is given in the final section.

2. **Main results and remarks.**

In this section we state our main results and give some remarks. Our main results are listed in the following three propositions.

Proposition 1. *When the parameter $b > -1$ and the wave speed equals $\frac{1}{1+b}$, the smooth solitary wave solution and the singular wave solution coexist in Eq.(1). Their expressions are as follows:*

(1) *The smooth solitary wave solution is of expression*

$$u_1(x,t) = \frac{1}{(1+b)^2}\left[1+b-3(2+b)\,\mathrm{sech}^2\frac{1}{\sqrt{2(1+b)}}\left(x-\frac{t}{1+b}\right)\right]. \tag{9}$$

(2) *The singular wave solution possesses expression*

$$u_2(x,t) = \frac{1}{(1+b)^2}\left[1+b+3(2+b)\,\mathrm{csch}^2\frac{1}{\sqrt{2(1+b)}}\left(x-\frac{t}{1+b}\right)\right]. \tag{10}$$

For the figures of $u_1(x,t)$ and $u_2(x,t)$ with $b=5$, see Fig.1(a), (b).

Proposition 2. *When the parameter $b > -1$ and the wave speed equals $1 + b$, the peakon wave solution and the singular wave solution coexist in Eq.(1). Their expressions are as follows:*

(1°) *The peakon wave solution has expression*

$$u_1^\circ(x,t) = \frac{6(2+b)}{\left(\sqrt{6}+\sqrt{1+b}\,|x-(1+b)t\,|\right)^2} - 1. \tag{11}$$

(2°) *The singular wave solution is of expression*

$$u_2^\circ(x,t) = \frac{6(2+b)}{(1+b)\,[x-(1+b)t]^2} - 1. \tag{12}$$

For the figures of $u_1^\circ(x,t)$ and $u_2^\circ(x,t)$ with $b = 5$, see Fig.2(a), (b).

Proposition 3. *When the parameter $b > -2$, $b \neq -1$ and the wave speed equals $\frac{2+b}{2}$, the smooth solitary wave solution, the peakon wave solution and the singular wave solution coexist in Eq.(1). Their expressions are as follows:*

(1*) *The smooth solitary wave solution has expression*

$$u_1^*(x,t) = -\frac{3(2+b)}{2(1+b)}\,\mathrm{sech}^2\,\frac{1}{2}\left(x - \frac{2+b}{2}t\right). \tag{13}$$

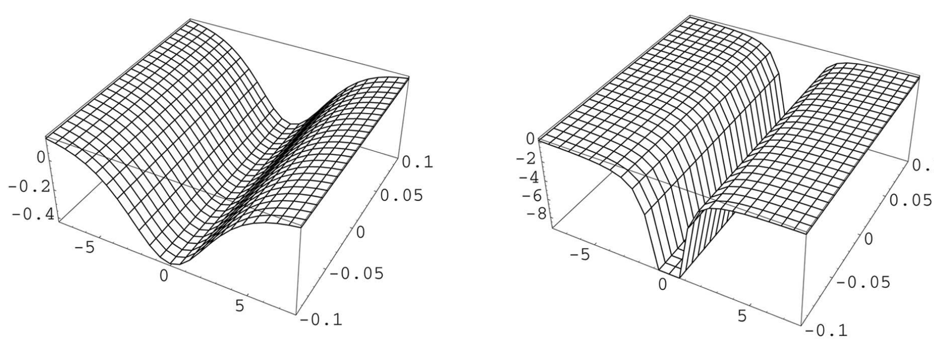
(2*) *The peakon wave solution is of expression*

$$u_2^*(x,t) = \frac{3(2+b)(2+3b+b^2)}{2(1+b)\left[\sqrt{3(2+b)}\cosh\frac{1}{2}\left(x-\frac{2+b}{2}t\right)+\sqrt{8+6b+b^2}\sinh\frac{1}{2}\left|x-\frac{2+b}{2}t\right|\right]^2}. \tag{14}$$

(3*) *The singular wave solution possesses expression*

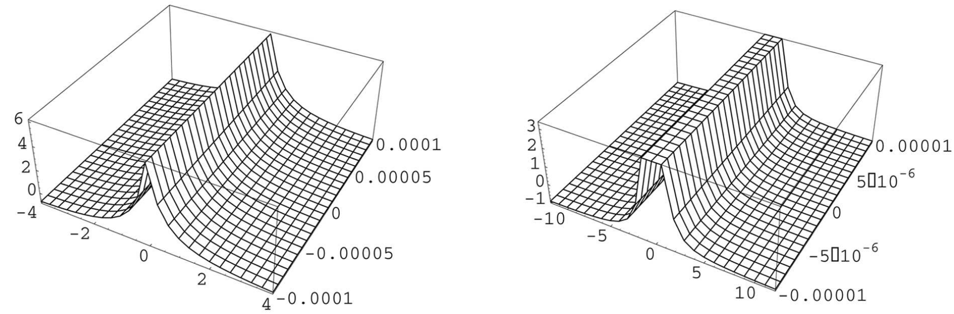
$$u_3^*(x,t) = \frac{3(2+b)}{2(1+b)}\,\mathrm{csch}^2\,\frac{1}{2}\left(x - \frac{2+b}{2}t\right). \tag{15}$$

For the figures of $u_i^*(x,t)$ $(i = 1, 2, 3)$ with $b = 5$, see Fig.3(a), (b), (c).



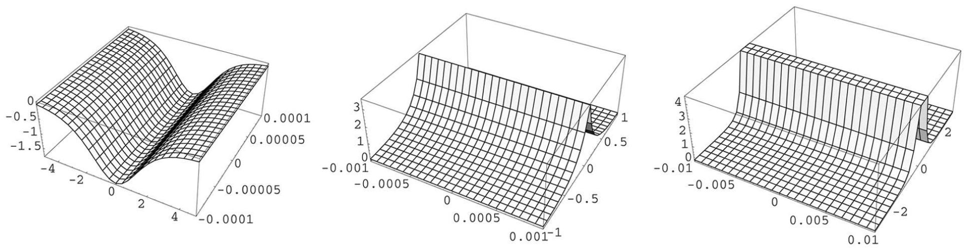
(a) $u = u_1(x,t)$ (b) $u = u_2(x,t)$

Fig.1 The figures of $u = u_1(x,t)$ and $u = u_2(x,t)$



(a) $u = u_1^\circ(x,t)$ (b) $u = u_2^\circ(x,t)$

Fig.2 The figures of $u = u_1^\circ(x,t)$ and $u = u_2^\circ(x,t)$



(a) $u = u_1^*(x,t)$ (b) $u = u_2^*(x,t)$ (c) $u = u_3^*(x,t)$

Fig.3 The figures of $u = u_1^*(x,t)$, $u = u_2^*(x,t)$, and $u = u_3^*(x,t)$

Remark 1. The expressions $u_1^*(x,t)$ and $u_3^*(x,t)$ are the same with Wazwaz's results $\widetilde{u}_1(x,t)$ and $\widetilde{u}_2(x,t)$ obtained by using tanh and sine-cosine method. Meanwhile, other five expressions, $u_1(x,t)$, $u_2(x,t)$, $u_1^\circ(x,t)$, $u_2^\circ(x,t)$ and $u_2^*(x,t)$ are new exact solutions of Eq.(1).

Remark 2. By using the software Mathematica, we have tested the correctness of these solutions. The commands are as follows:

$$\mathrm{u_1(x,t)} = \frac{1}{(1+b)^2}\left(1+b-3\ (2+b)\ \mathrm{Sech}\left[\sqrt{\frac{1}{2\ (1+b)}}\left(x-\frac{t}{1+b}\right)\right]^2\right)$$

$$\mathrm{u_2(x,t)} = \frac{1}{(1+b)^2}\left(1+b+3\ (2+b)\ \mathrm{Csch}\left[\sqrt{\frac{1}{2\ (1+b)}}\left(x-\frac{t}{1+b}\right)\right]^2\right)$$

$$\mathrm{u_3(x,t)} = \frac{6\ (2+b)}{\left(\sqrt{6}+\sqrt{1+b}\ (x-(1+b)\ t)\right)^2}-1$$

$$\mathrm{u_4(x,t)} = \frac{6\ (2+b)}{\left(\sqrt{6}-\sqrt{1+b}\ (x-(1+b)\ t)\right)^2}-1$$

$$\mathrm{u_5(x,t)} = \frac{6\ (2+b)}{(1+b)\ (x-(1+b)\ t)^2}-1$$

$$\mathrm{u_6(x,t)} = -\frac{3\ (2+b)}{2\ (1+b)}\ \mathrm{Sech}\left[\frac{x}{2}-\frac{(2+b)\ t}{4}\right]^2$$

$$\mathrm{u_7(x,t)} = \frac{3\ (2+b)\ (2+3\ b+b^2)}{2\ (1+b)\ \left(\sqrt{3\ (2+b)}\ \mathrm{Cosh}\left[\frac{x}{2}-\frac{(2+b)\ t}{4}\right]+\sqrt{8+6\ b+b^2}\,\mathrm{Sinh}\left[\frac{x}{2}-\frac{(2+b)\ t}{4}\right]\right)^2}$$

$$\mathrm{u_8(x,t)} = \frac{3\ (2+b)\ (2+3\ b+b^2)}{2\ (1+b)\ \left(\sqrt{3\ (2+b)}\ \mathrm{Cosh}\left[\frac{x}{2}-\frac{(2+b)\ t}{4}\right]-\sqrt{8+6\ b+b^2}\,\mathrm{Sinh}\left[\frac{x}{2}-\frac{(2+b)\ t}{4}\right]\right)^2}$$

```
u9(x, t) = 3 (2+b)/(2 (1+b)) Csch[x/2 − (2+b) t/4]^2
u=u1(x, t)
ut=D[u, t]
ux=D[u, x]
uxx=D[ux, x]
uxxt=D[uxx, t]
uxxx=D[uxx, x]
Simplify[ut−uxxt+(b+1) u^2 ux−b ux uxx−u uxxx]
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3. **Preliminaries.**

In order to derive our main results, let

$$\xi = x - ct, \tag{16}$$

and $u = \varphi(\xi)$, where c is a constant and is called the wave speed. Thus Eq.(1) becomes a third order ordinary differential equation

$$-c\,\varphi' + c\,\varphi''' + (1+b)\varphi^2\varphi' = b\varphi'\varphi'' + \varphi\varphi'''. \tag{17}$$

Integrating (17) once, it follows that

$$\varphi''(\varphi - c) = \frac{1+b}{3}\varphi^3 - c\varphi + g - \frac{b-1}{2}(\varphi')^2, \tag{18}$$

where g is the constant of integration .

Letting $y = \varphi'$, it yields the following planar system

$$\begin{cases} \dfrac{\mathrm{d}\varphi}{\mathrm{d}\xi} = y, \\ (\varphi - c)\dfrac{\mathrm{d}y}{\mathrm{d}\xi} = \dfrac{1+b}{3}\varphi^3 - c\,\varphi + g - \dfrac{b-1}{2}y^2. \end{cases} \tag{19}$$

Multiplying $(19)_1$ by $\varphi - c$, we have

$$\begin{cases} (\varphi - c)\dfrac{\mathrm{d}\varphi}{\mathrm{d}\xi} = (\varphi - c)y, \\ (\varphi - c)\dfrac{\mathrm{d}y}{\mathrm{d}\xi} = \dfrac{1+b}{3}\varphi^3 - c\,\varphi + g - \dfrac{b-1}{2}y^2. \end{cases} \tag{20}$$

By using the transformation $\mathrm{d}\tau = \frac{\mathrm{d}\xi}{\varphi - c}$, (20) can be rewritten as the Hamiltonian system

$$\begin{cases} \dfrac{\mathrm{d}\varphi}{\mathrm{d}\tau} = (\varphi - c)y, \\ \dfrac{\mathrm{d}y}{\mathrm{d}\tau} = \dfrac{1+b}{3}\varphi^3 - c\,\varphi + g - \dfrac{b-1}{2}y^2. \end{cases} \tag{21}$$

Let

$$a_0 = \frac{6g - 6c^2 + 2(1+b)c^3}{3(b-1)}, \tag{22}$$

$$a_1 = \frac{2(1+b)c^2 - 2c}{b}, \tag{23}$$

$$a_2 = 2c, \tag{24}$$

$$a_3 = \frac{2(1+b)}{3(2+b)}, \tag{25}$$

and

$$H(\varphi, y) = (\varphi - c)^{b-1}\left[a_0 + a_1(\varphi - c) + a_2(\varphi - c)^2 + a_3(\varphi - c)^3 - y^2\right]. \tag{26}$$

It is easy to check that

$$H(\varphi, y) = h \tag{27}$$

is the first integral for both systems (19) and (21). Therefore both systems (19) and (21) have the same topological phase portraits except line $\varphi = c$. This implies that one can understand the phase portraits of system (19) from that of system (21).

Now we begin to study some phase portraits of system (21). Let

$$z = f(\varphi), \tag{28}$$

where

$$f(\varphi) = \frac{1+b}{3}\varphi^3 - c\varphi + g. \tag{29}$$

Thus we have

$$f'(\varphi) = (1+b)\varphi^2 - c. \tag{30}$$

Solving $f'(\varphi) = 0$, it follows that

$$\varphi = \pm\sqrt{\frac{c}{1+b}}.$$

When $g = 0$, we have

$$f\left(\pm\sqrt{\frac{c}{1+b}}\right) = \mp\frac{2c}{3}\sqrt{\frac{c}{1+b}}. \tag{31}$$

Let

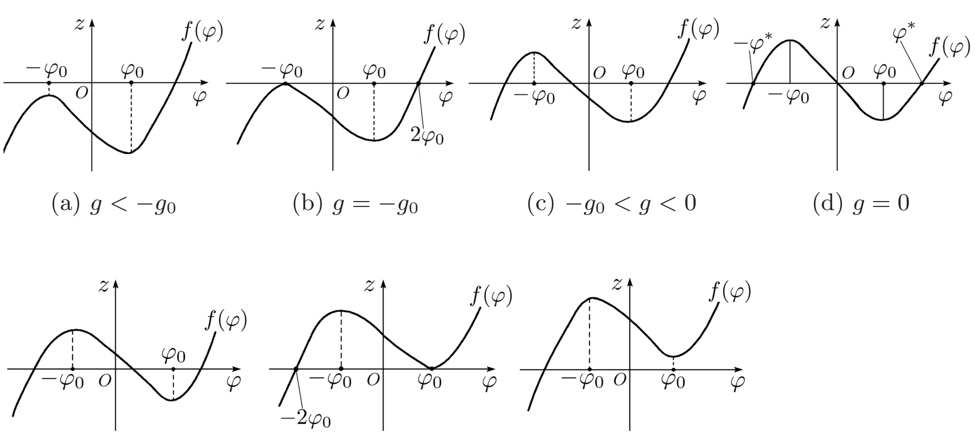
$$\varphi^* = \sqrt{\frac{3c}{1+b}}, \tag{32}$$

$$\varphi_0 = \sqrt{\frac{c}{1+b}}, \tag{33}$$

and

$$g_0 = \frac{2|c|}{3}\sqrt{\frac{c}{1+b}}. \tag{34}$$

Thus for given $b > -1$ and $c > 0$, we draw the graph of $z = f(\varphi)$ as Fig.4. For given $b < -1$ and $c < 0$, the graph is similar.



(a) $g < -g_0$ (b) $g = -g_0$ (c) $-g_0 < g < 0$ (d) $g = 0$

(e) $0 < g < g_0$ (f) $g = g_0$ (g) $g > g_0$

Fig.4 The graph of $z = f(\varphi)$ for given $b > -1$ and $c > 0$

On the other hand, it is seen that $(\widetilde{\varphi}, 0)$ is a singular point of system (21) if and only if $f(\widetilde{\varphi}) = 0$. At the singular point $(\widetilde{\varphi}, 0)$, it is easy to see that the linearized system of system (21) has the eigenvalues

$$\lambda_{\pm}(\widetilde{\varphi}, 0) = \pm\sqrt{(\widetilde{\varphi} - c)f'(\widetilde{\varphi})}. \tag{35}$$

From (27) and (35) we see that the singular point $(\widetilde{\varphi}, 0)$ is of the following properties:

(i) If $(\widetilde{\varphi} - c)f'(\widetilde{\varphi}) > 0$, then $(\widetilde{\varphi}, 0)$ is a saddle point of system (21).

(ii) If $(\widetilde{\varphi} - c)f'(\widetilde{\varphi}) = 0$, then $(\widetilde{\varphi}, 0)$ is a degenerate saddle point of system (21).

(iii) If $(\widetilde{\varphi} - c)f'(\widetilde{\varphi}) < 0$, then $(\widetilde{\varphi}, 0)$ is a center point of system (21).

Let

$$y_0 = \frac{2(1+b)c^3 - 6c^2 + 6g}{3(b-1)}. \tag{36}$$

Similarly, if $y_0 > 0$, then $(c, -\sqrt{y_0})$ and $(c, \sqrt{y_0})$ are two saddle points of system (21) and

$$H(c, \pm\sqrt{y_0}) = 0. \tag{37}$$

Next we study the bifurcation phase portraits of system (21) by using these information, and derive our results through some phase portraits.

4. **The demonstrations on Proposition 1.** In this section we suppose $b > -1$, $c > 0$ and take $g = g_0$. For other cases, the demonstrations are similar. From Fig.4(f) we see that system (21) has two singular points $(-2\varphi_0, 0)$ and $(\varphi_0, 0)$ on φ-axis. And $f'(\varphi)$ satisfies $f'(-2\varphi_0) > 0$, $f'(\varphi_0) = 0$. For $c \neq \frac{1}{1+b}$, we have $y_0 > 0$. Therefore system (21) has two saddle points $(c, \pm\sqrt{y_0})$ on line $\varphi = c$. For given $b > -1$ and different c, we have the following inequalities and properties:

(i) When $c > \frac{1}{1+b}$, it follows that

$$-2\varphi_0 < \varphi_0 < c \ \text{ and } \ y_0 > 0, \tag{38}$$

$$H(-2\varphi_0, 0) < H(\varphi_0, 0) < 0, \quad \text{for even number } b, \tag{39}$$

and

$$H(-2\varphi_0, 0) > H(\varphi_0, 0) > 0, \quad \text{for odd number } b. \tag{40}$$

Hence, $(-2\varphi_0, 0)$ is a center point and $(\varphi_0, 0)$ is a degenerate saddle point.

(ii) When $c = \frac{1}{1+b}$, it follows that

$$-2\varphi_0 < \varphi_0 = c \ \text{ and } \ y_0 = 0, \tag{41}$$

$$H(-2\varphi_0, 0) < H(\varphi_0, 0) = 0, \quad \text{for even number } b, \tag{42}$$

and

$$H(-2\varphi_0, 0) > H(\varphi_0, 0) = 0, \quad \text{for odd number } b. \tag{43}$$

Thus $(-2\varphi_0, 0)$ is a center point. $(\varphi_0, 0)$ is a multiple singular point, that is, the degenerate saddle point. And the saddle points $(c, \sqrt{y_0})$, $(c, -\sqrt{y_0})$ coincide together.

(iii) When $0 < c < \frac{1}{1+b}$, it follows that

$$-2\varphi_0 < c < \varphi_0 \ \text{ and } \ y_0 > 0, \tag{44}$$

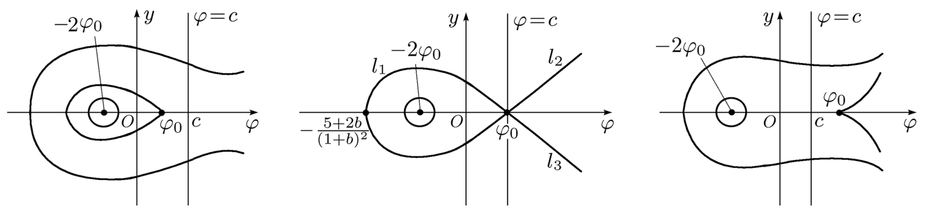
$$H(-2\varphi_0, 0) < 0 < H(\varphi_0, 0), \quad \text{for even number } b, \tag{45}$$

and

$$H(-2\varphi_0, 0) > H(\varphi_0, 0) > 0, \quad \text{for odd number } b. \tag{46}$$

Therefore, $(-2\varphi_0, 0)$ is a center point and $(\varphi_0, 0)$ is a degenerate saddle point.

Thus we obtain the bifurcation phase portraits of system (21) as Fig.5.



(a) $c > 1/(1+b)$ (b) $c = 1/(1+b)$ (c) $0 < c < 1/(1+b)$

Fig.5 The bifurcation phase portraits of system (21) when $g = g_0$

From Fig.5(b) it is seen that when $c = \frac{1}{1+b}$, there is a homoclinic orbit l_1 connecting with singular point $\left(\frac{1}{1+b}, 0\right)$ and passing point $\left(-\frac{5+2b}{(1+b)^2}, 0\right)$. The expression of l_1 is

$$y = \pm\sqrt{\frac{2(1+b)}{3(2+b)}}\left(\frac{1}{1+b} - \varphi\right)\left(\varphi + \frac{5+2b}{(1+b)^2}\right)^{1/2}, \text{ where } -\frac{5+2b}{(1+b)^2} \le \varphi < \frac{1}{1+b}. \tag{47}$$

It is easy to show that a homoclinic orbit of system (21) corresponds to a solitary wave of Eq.(1). In order to obtain the solitary wave solution, we substitute (47) into $\frac{\mathrm{d}\varphi}{y} = \mathrm{d}\xi$. This yields equation

$$\frac{\pm\mathrm{d}\varphi}{\left(\frac{1}{1+b} - \varphi\right)\sqrt{\varphi + \frac{5+2b}{(1+b)^2}}} = \sqrt{\frac{2(1+b)}{3(2+b)}}\mathrm{d}\xi. \tag{48}$$

Integrating (48) along l_1, it follows that

$$\int_{-\frac{5+2b}{(1+b)^2}}^{\varphi} \frac{\mathrm{d}s}{\left(\frac{1}{1+b} - s\right)\sqrt{s + \frac{5+2b}{(1+b)^2}}} = \sqrt{\frac{2(1+b)}{3(2+b)}}|\xi|. \tag{49}$$

Solving Eq.(49) for φ, we have

$$\varphi = -\frac{5+2b}{(1+b)^2} + \frac{3(2+b)}{(1+b)^2}\tanh^2\frac{\xi}{\sqrt{2(1+b)}}. \tag{50}$$

Noting that $u = \varphi(\xi)$ and $\xi = x - \frac{t}{1+b}$, we get the smooth solitary wave solution $u_1(x,t)$ as (9).

On the other hand, from Fig.5(b) it is also seen there are two heteroclinic orbits l_2 and l_3 connecting with singular point $\left(\frac{1}{1+b}, 0\right)$. Their expressions are

$$y = \pm\sqrt{\frac{2(1+b)}{3(2+b)}}\left(\varphi - \frac{1}{1+b}\right)\left(\varphi + \frac{5+2b}{(1+b)^2}\right)^{1/2}, \text{ where } \frac{1}{1+b} < \varphi < +\infty. \tag{51}$$

Substituting (51) into $\frac{\mathrm{d}\varphi}{y} = \mathrm{d}\xi$ and integrating it along these two heteroclinic orbits, it yields equation

$$\int_{\varphi}^{+\infty} \frac{\mathrm{d}s}{\left(s - \frac{1}{1+b}\right)\sqrt{s + \frac{5+2b}{(1+b)^2}}} = \sqrt{\frac{2(1+b)}{3(2+b)}}|\xi|. \tag{52}$$

Completing the integral and solving the equation for φ, it follows that

$$\varphi = \frac{1}{(1+b)^2}\left[1 + b + 3(2+b)\,\mathrm{csch}^2\frac{\xi}{\sqrt{2(1+b)}}\right]. \tag{53}$$

Since $u = \varphi(\xi)$ and $\xi = x - \frac{t}{1+b}$, we get the singular wave solution $u_2(x, t)$ as (10). These complete the demonstrations on Proposition 1.

5. **The demonstrations on Proposition 2.** In this section we suppose $b > 1$, $c > 0$ and take the integral constant $g = -g_0$. For other cases, the demonstrations are similar. From Fig.4(b) it is seen that system (21) has two singular points $(-\varphi_0, 0)$ and $(2\varphi_0, 0)$ on φ-axis, and $f'(-\varphi_0) = 0$, $f'(2\varphi_0) > 0$. When $y_0 > 0$, system (21) has two saddle points $(c, \pm\sqrt{y_0})$ on line $\varphi = c$. Also we have the following inequalities:

(i) When $c > 1 + b$, it follows that

$$-\varphi_0 < 2\varphi_0 < c \ \text{ and } \ y_0 > 0, \tag{54}$$

$$H(2\varphi_0, 0) < H(-\varphi_0, 0) < 0 \quad \text{for even number } b, \tag{55}$$

and

$$H(2\varphi_0, 0) > H(-\varphi_0, 0) > 0 \quad \text{for odd number } b. \tag{56}$$

(ii) When $c = 1 + b$, it follows that

$$-\varphi_0 < 2\varphi_0 < c \ \text{ and } \ y_0 > 0, \tag{57}$$

$$H(2\varphi_0, 0) < H(-\varphi_0, 0) = 0 \quad \text{for even number } b, \tag{58}$$

and

$$H(2\varphi_0, 0) > H(-\varphi_0, 0) = 0, \quad \text{for odd number } b. \tag{59}$$

(iii) When $\frac{4}{1+b} < c < 1 + b$, it follows that

$$-\varphi_0 < 2\varphi_0 < 0 \ \text{ and } \ y_0 > 0, \tag{60}$$

$$H(2\varphi_0, 0) < 0 < H(-\varphi_0, 0) \quad \text{for even number } b, \tag{61}$$

and

$$H(2\varphi_0, 0) > 0 > H(-\varphi_0, 0) \quad \text{for odd number } b. \tag{62}$$

(iv) When $c = \frac{4}{1+b}$, it follows that

$$-\varphi_0 < 2\varphi_0 = c \ \text{ and } \ y_0 = 0. \tag{63}$$

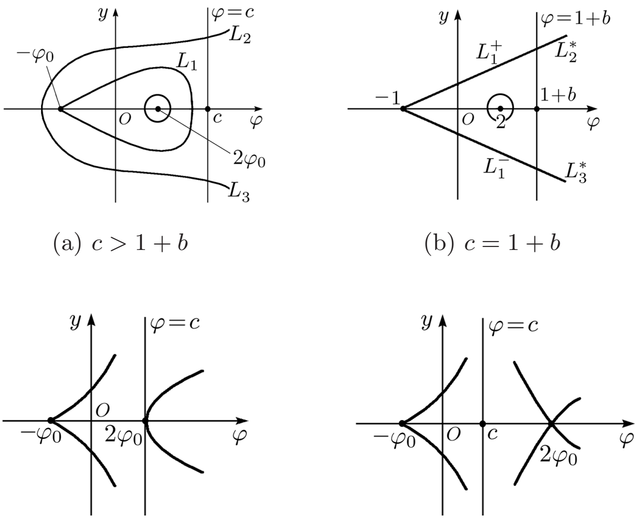
(v) When $0 < c < \frac{4}{1+b}$, it follows that

$$-\varphi_0 < c < 2\varphi_0 \ \text{ and } \ y_0 < 0. \tag{64}$$

Similar to the analysis in Section 4, we got the bifurcation phase portraits of system (21) as Fig.6.

From Fig.6(a) one sees that there is a homoclinic orbit L_1 connecting with singular point $(-\varphi_0, 0)$, and there are two heteroclinic orbits L_2, L_3 which connect with the singular points $(c, \sqrt{y_0})$ and $(c, -\sqrt{y_0})$ respectively. When c tends to $1 + b$, the homoclinic orbit L_1 becomes a novel homoclinic orbit composed of three curve segments L_1^+, L_1^- and $\varphi = 1+b$ (see Fig.6(b)), where L_1^+ and L_1^- possess expressions respectively

$$L_1^+: \quad y = \sqrt{\frac{2(1+b)}{3(2+b)}}(1+\varphi)^{3/2} \quad \text{where} \quad -1 < \varphi \le 1 + b, \tag{65}$$



(a) $c > 1+b$ (b) $c = 1+b$ (c) $\frac{4}{1+b} < c < 1+b$

(d) $c = \frac{4}{1+b}$ (e) $0 < c < \frac{4}{1+b}$

Fig.6 The bifurcation phase portraits of system (21) when $g = -g_0$

and

$$L_1^- : \quad y = -\sqrt{\frac{2(1+b)}{3(2+b)}}(1+\varphi)^{3/2} \quad \text{where} \quad -1 < \varphi \le 1+b. \tag{66}$$

Meanwhile, the heteroclinic orbits L_2 and L_3 become L_2^* and L_3^*.

Substituting (65) and (66) into the $\frac{\mathrm{d}\varphi}{y} = \mathrm{d}\xi$ we have

$$\frac{\mathrm{d}\varphi}{(1+\varphi)^{3/2}} = \sqrt{\frac{2(1+b)}{3(2+b)}}\mathrm{d}\xi, \tag{67}$$

and

$$-\frac{\mathrm{d}\varphi}{(1+\varphi)^{3/2}} = \sqrt{\frac{2(1+b)}{3(2+b)}}\mathrm{d}\xi. \tag{68}$$

Integrating (67) and (68) along L_1^+ and L_1^- respectively, it follows that

$$\int_{\varphi}^{1+b} \frac{\mathrm{d}s}{(1+s)^{3/2}} = \sqrt{\frac{2(1+b)}{3(2+b)}}|\xi|. \tag{69}$$

Solving Eq.(69) for φ, we get

$$\varphi = \frac{6(2+b)}{\left(\sqrt{6} + \sqrt{(1+b)}|\xi|\right)^2} - 1. \tag{70}$$

Noting that $u = \varphi(\xi)$ and $\xi = x-(1+b)t$, we obtain the peakon solution $u_1^\circ(x,t)$ as (11).

On the other hand, from Fig.6(b) one also sees that the heteroclinic orbits L_2^* and L_3^* have the same expressions as L_1^+ and L_1^- except the definition intervals.

Similarly, we have

$$\int_{\varphi}^{+\infty} \frac{\mathrm{d}s}{(1+s)^{3/2}} = \sqrt{\frac{2(1+b)}{3(2+b)}}|\xi|. \tag{71}$$

Solving Eq.(71) for φ, we get

$$\varphi = \frac{6(2+b)}{(1+b)\xi^2} - 1. \tag{72}$$

Noting that $u = \varphi(\xi)$ and $\xi = x - (b+1)t$, we obtain the singular wave solution $u_2^{\circ}(x,t)$ as (12). These complete the demonstrations on Proposition 2.

6. **The demonstrations on Proposition 3.** In this section we suppose $b > -1$, $c > 0$ and take the integral constant $g = 0$. For other cases, the demonstrations are similar. From Fig.4(d) it is seen that system (21) has three singular points $(-\varphi^*, 0)$, $(0,0)$ and $(\varphi^*, 0)$ on φ-axis, and $f'(-\varphi^*) > 0$, $f'(0) < 0$, $f'(\varphi^*) > 0$. When $y_0 > 0$, system (21) has two saddle points $(c, \pm\sqrt{y_0})$ on line $\varphi = c$. Also we have the following inequalities:

(i) When $c > \frac{2+b}{2}$, it follows that

$$-\varphi^* < 0 < \varphi^* < c \ \text{ and } \ y_0 > 0, \tag{73}$$

$$H(-\varphi^*, 0) \leq H(\varphi^*, 0) < H(0,0) < 0 \quad \text{for even number } b, \tag{74}$$

and

$$H(-\varphi^*, 0) \geq H(\varphi^*, 0) > H(0,0) > 0 \quad \text{for odd number } b. \tag{75}$$

(ii) When $c = \frac{2+b}{2}$, it follows that

$$-\varphi^* < 0 < \varphi^* < c \ \text{ and } \ y_0 > 0, \tag{76}$$

$$H(-\varphi^*, 0) \leq H(\varphi^*, 0) < H(0,0) = 0 \quad \text{for even number } b, \tag{77}$$

and

$$H(-\varphi^*, 0) \geq H(\varphi^*, 0) > H(0,0) = 0 \quad \text{for odd number } b. \tag{78}$$

(iii) When $\frac{3}{1+b} < c < \frac{2+b}{2}$, it follows that

$$-\varphi^* < 0 < \varphi^* < c \ \text{ and } \ y_0 > 0, \tag{79}$$

$$H(-\varphi^*, 0) \leq H(\varphi^*, 0) < 0 < H(0,0) \quad \text{for even number } b, \tag{80}$$

and

$$H(-\varphi^*, 0) > H(\varphi^*, 0) > 0 > H(0,0) \quad \text{for odd number } b. \tag{81}$$

(iv) When $c = \frac{3}{1+b}$, it follows that

$$-\varphi^* < 0 < \varphi^* = c \ \text{ and } \ y_0 = 0. \tag{82}$$

(v) When $0 < c < \frac{3}{1+b}$, it follows that

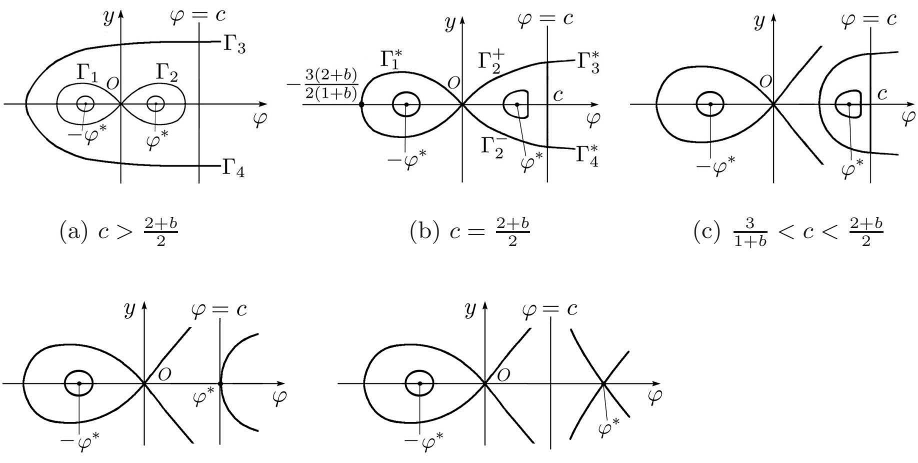
$$-\varphi^* < 0 < c < \varphi^* \ \text{ and } \ y_0 < 0. \tag{83}$$

Similar to the analysis in Section 4, we obtain the bifurcation phase portraits of system (21) as Fig.7.

From Fig.7(a) it is seen that when $c > \frac{2+b}{2}$, there are two homoclinic orbits Γ_1 and Γ_2 connecting with the singular point $(0,0)$ and two heteroclinic orbits Γ_3 and Γ_4 which connect with the singular points $(c, \sqrt{y_0})$ and $(c, -\sqrt{y_0})$ respectively. When c tends to $\frac{2+b}{2}$, Γ_1 becomes the homoclinic orbit Γ_1^*, Γ_2 becomes a novel homoclinic orbit composed of three curve segments Γ_2^+, Γ_2^- and $\varphi = \frac{2+b}{2}$. Meanwhile Γ_3 and

Γ_4 become heteroclinic orbits Γ_3^* and Γ_4^* respectively. The homoclinic orbit Γ_1^* is of expression

$$y = \pm\sqrt{\frac{2(1+b)}{3(2+b)}}\sqrt{\varphi + \frac{3(2+b)}{2(1+b)}}\,\varphi \quad \text{where} \quad -\frac{3(2+b)}{2(1+b)} \le \varphi < 0. \tag{84}$$



(a) $c > \frac{2+b}{2}$ (b) $c = \frac{2+b}{2}$ (c) $\frac{3}{1+b} < c < \frac{2+b}{2}$

(d) $c = \frac{3}{1+b}$ (e) $0 < c < \frac{3}{1+b}$

Fig.7 The bifurcation phase portraits of system (21) when $g = 0$

Substituting (84) into $\frac{\mathrm{d}\varphi}{y} = \mathrm{d}\xi$ and integrating it along Γ_1^*, it follows that

$$\int_{-\frac{3(2+b)}{2(1+b)}}^{\varphi} \frac{\mathrm{d}s}{s\sqrt{s + \frac{3(2+b)}{2(1+b)}}} = -\sqrt{\frac{2(1+b)}{3(2+b)}}|\xi|. \tag{85}$$

Solving Eq.(85) for φ, we have

$$\varphi = -\frac{3(2+b)}{2(1+b)}\operatorname{sech}^2\frac{1}{2}\xi. \tag{86}$$

Since $\xi = x - \frac{2+b}{2}t$ and $u = \varphi(\xi)$, we obtain the bell-shaped solitary wave solution $u_1^*(x,t)$ as (13).

Note that the novel homoclinic orbit also has expression (84), where $0 < \varphi \le \frac{2+b}{2}$. Similarly, we have

$$\int_{\varphi}^{c} \frac{\mathrm{d}s}{s\sqrt{s + \frac{3(2+b)}{2(1+b)}}} = \sqrt{\frac{2(1+b)}{3(2+b)}}|\xi|. \tag{87}$$

Solving Eq.(87) for φ and noting that $u = \varphi(\xi)$ with $\xi = x - \frac{2+b}{2}t$, we obtain the peakon solution $u_2^*(x,t)$ as (14).

On the other hand, the heteroclinic orbits Γ_3^* and Γ_4^* also have expression (84), where $0 < \varphi < +\infty$. Similarly, we have

$$\int_{\varphi}^{+\infty} \frac{\mathrm{d}s}{s\sqrt{s + \frac{3(2+b)}{2(1+b)}}} = \sqrt{\frac{2(1+b)}{3(2+b)}}|\xi|. \tag{88}$$

Solving Eq.(88) for φ and noting that $\xi = x - \frac{2+b}{2}t$ and $u = \varphi(\xi)$, we obtain the singular wave solution $u_3^*(x,t)$ as (15). These complete the demonstrations on Proposition 3.

7. **Conclusion.** In this paper, through the phase analysis, not only has the coexistence of several types of nonlinear wave solutions been shown, but their explicit expressions have been given to Eq.(1). The correctness of these solutions have been tested by the software Mathematica. Among our results, the solutions $u_1^*(x,t)$ and $u_3^*(x,t)$ had been obtained by Wazwaz [1] via other method, tanh and sine-cosine method. Other five solutions $u_1(x,t)$, $u_2(x,t)$, $u_1^\circ(x,t)$, $u_2^\circ(x,t)$ and $u_2^*(x,t)$ are new. Some previous results become our special cases. For instance, when $b = 2, 3$, $u_2^*(x,t)$ becomes the peakon solution which had been obtained in [26], and $u_1^\circ(x,t)$, $u_2^\circ(x,t)$ respectively become the peakon wave solution and the singular wave solution which had been given in [27]. These imply that our work extends some previous results. In Eq.(3) and Eq.(5), the stability of the peakons and the solitary waves had been proved by Constantin et al [30], [31]. But the stability of the solutions given in this paper waits for further study.

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