

Some new nonlinear wave solutions for two (3+1)-dimensional equations[☆]



Yiren Chen, Rui Liu*

School of Mathematics, South China University of Technology, Guangzhou, 510640, P.R. China

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ABSTRACT

In this paper, two methods are employed to study the nonlinear wave solutions for two (3+1)-dimensional equations which can be reduced to the potential KdV equation.

Firstly, using the simplified Hirota's method, we present generalized multiple soliton solutions and generalized multiple singular soliton solutions in which some differentiable arbitrary functions are involved. Secondly, by means of some special orbits of the traveling wave system and integrating approach, we obtain some other nonlinear wave solutions which also include differentiable arbitrary functions. Our work extends pioneer's results.

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1. Introduction

In the nonlinear research, some higher-dimensional equations have been investigated. For instance, Jimbo and Miwa [1] introduced (3+1)-dimensional equation

$$u_{yt} + u_{xxy} - 3u_{xx}u_y - 3u_xu_{xy} - u_{xz} = 0, \quad (1.1)$$

as the second equation in the so-called Kadomtsev–Petviashvili hierarchy of equations. However, Dorizzi et al. [2] showed that Eq. (1.1) is not completely integrable in the usual sense.

Boiti et al. [3] developed an inverse scattering scheme to solve the Cauchy problem for (2+1)-dimensional equation

$$u_{yt} + u_{xxy} - 3u_{xx}u_y - 3u_xu_{xy} = 0, \quad (1.2)$$

which is reduced to the KdV equation for $y = x$.

Yajima et al. [4] presented (2+1)-dimensional equation

$$u_{tt} - u_{xx} - u_{yy} + u_xu_{xt} + u_yu_{yt} - u_{xxt} - u_{yyt} = 0, \quad (1.3)$$

as a model of ion-acoustic waves in plasmas. Kako and Yajima [5] studied soliton interactions for Eq. (1.3).

Bogoyavlenskii [6,7] discussed the inverse scattering method of solution for (2+1)-dimensional equation

$$u_{xt} + u_{xxy} - 2u_{xx}u_y - 4u_xu_{xy} = 0, \quad (1.4)$$

which, like (1.2), reduces to the KdV equation for $y = x$.

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* Corresponding author. Tel.: +86 20 22236202.

E-mail address: scliurui@scut.edu.cn (R. Liu).

Clarkson and Mansfield [8] pointed out that the Eqs. (1.1)–(1.3) can be reduced to the SWWI equation

$$u_{xxx} + \beta (u_x u_{xt} + u_t u_{xx}) - (u_{xx} + u_{xt}) = 0, \tag{1.5}$$

and Eq. (1.4) can be reduced to the SWWI equation

$$u_{xxx} + \beta (2u_x u_{xt} + u_t u_{xx}) - (u_{xx} + u_{xt}) = 0. \tag{1.6}$$

For Eq. (1.5), Clarkson and Mansfield [9] gave an interesting solution

$$u(x, t) = \frac{3}{\beta} \tanh \frac{x+f(t)}{2} + \frac{3}{\beta} \tanh \frac{x-f(t)}{2} + \frac{t}{\beta}, \tag{1.7}$$

which contains a differentiable arbitrary functions $f(t)$. For Eq. (1.6), it seems that there is not solution which is similar to (1.7).

Several other equations and their multiple soliton solutions were studied by Wazwaz [9,10], Wen and Xu [11], Zhen et al. [12] and Zuo et al. [13]. In these references, none of the solutions contains arbitrary function.

Wazwaz [14] introduced two (3+1)-dimensional equations

$$u_{yzt} + u_{xxxzy} - 6u_x u_{xyz} - 6u_{xy} u_{xz} = 0, \tag{1.8}$$

and

$$u_{xzt} + u_{xxxzy} - 2(u_{xx} u_{yz} + u_y u_{xz}) - 4(u_x u_{xyz} + u_{xy} u_{xz}) = 0, \tag{1.9}$$

as two higher-dimensional shallow water wave equations. It is easy to see that Eqs. (1.8) and (1.9) can be reduced to the potential KdV equation for $z = y = x$.

In [14], Wazwaz investigated multiple soliton solutions and multiple singular soliton solutions of Eqs. (1.8) and (1.9) respectively.

For Eq. (1.8), Wazwaz gave the solutions combined by $e^{k_i x + r_i y + s_i z - k_i^3 t}$ ($i = 1, 2, 3$), where k_i, r_i, s_i are arbitrary constants. For Eq. (1.9), Wazwaz presented the solutions combined by $e^{k_i x + r_i y + s_i z - k_i^2 r_i t}$ ($i = 1, 2, 3$), where k_i, r_i, s_i are arbitrary constants.

In this paper, we study the nonlinear wave solutions for Eqs. (1.8) and (1.9) respectively. Firstly, using Hirota's bilinear method [15,16], we obtain generalized soliton solutions and generalized singular soliton solutions. For Eq. (1.8), these solutions are combined by $\varphi_i(y, z)e^{k_i x - k_i^3 t}$, where $\varphi_i(y, z)$ ($i = 1, 2, 3$) are arbitrary differentiable functions and k_i ($i = 1, 2, 3$) are arbitrary constants. For Eq. (1.9), these solutions are combined by $\varphi_i(z)e^{k_i x + r_i y - r_i k_i^2 t}$, where $\varphi_i(z)$ ($i = 1, 2, 3$) are arbitrary differentiable functions and k_i, r_i ($i = 1, 2, 3$) are arbitrary constants. These imply that our work extends Wazwaz's results. Secondly, by means of some special orbits of the traveling wave system and integrating technology [17–22], we get some other nonlinear wave solutions which also include differentiable arbitrary functions.

This paper is organized as follows. In Section 2, our main results are presented through four propositions. In Section 3, we prove our propositions. A short conclusion is given in Section 4.

2. The new nonlinear wave solutions

In this section, we state the main results for Eqs. (1.8) and (1.9). Firstly, we state the main results for Eq. (1.8) by the following two propositions.

Proposition 2.1. For arbitrarily given constants k_i and arbitrarily given differentiable functions $g(t, y), h(t, z), e(y, z), p(y, z), \varphi_i = \varphi_i(y, z)$ ($i = 1, 2, 3$), if let

$$\gamma(t, y, z) = g(t, y) + h(t, z) + e(y, z), \tag{2.1}$$

$$\theta_i = k_i x - k_i^3 t \quad (i = 1, 2, 3), \tag{2.2}$$

$$a_0 = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2 [p(y, z)]}, \tag{2.3}$$

$$a_{ij} = \left(\frac{k_i - k_j}{k_i + k_j} \right)^2 \quad (1 \leq i < j \leq 3), \tag{2.4}$$

$$a_{123} = a_{12} a_{13} a_{23}, \tag{2.5}$$

$$f = 1 + \varphi_1 e^{\theta_1} + \varphi_2 e^{\theta_2} + \varphi_3 e^{\theta_3} + a_{12} \varphi_1 \varphi_2 e^{\theta_1 + \theta_2} + a_{13} \varphi_1 \varphi_3 e^{\theta_1 + \theta_3} + a_{23} \varphi_2 \varphi_3 e^{\theta_2 + \theta_3} + a_{123} \varphi_1 \varphi_2 \varphi_3 e^{\theta_1 + \theta_2 + \theta_3}, \tag{2.6}$$

and

$$g = 1 - \varphi_1 e^{\theta_1} - \varphi_2 e^{\theta_2} - \varphi_3 e^{\theta_3} + a_{12} \varphi_1 \varphi_2 e^{\theta_1 + \theta_2} + a_{13} \varphi_1 \varphi_3 e^{\theta_1 + \theta_3} + a_{23} \varphi_2 \varphi_3 e^{\theta_2 + \theta_3} - a_{123} \varphi_1 \varphi_2 \varphi_3 e^{\theta_1 + \theta_2 + \theta_3}, \tag{2.7}$$

then Eq. (1.8) has the following generalized soliton solutions and generalized singular soliton solutions:

(1°) the generalized 1-soliton solution

$$u = \frac{-2k_1\varphi_1 e^{\theta_1}}{p(y, z) + \varphi_1 e^{\theta_1}} + \gamma(t, y, z); \tag{2.8}$$

(2°) the generalized 2-soliton solution

$$u = \frac{-2(k_1\varphi_1 e^{\theta_1} + k_2\varphi_2 e^{\theta_2} + a_0(k_1 + k_2)\varphi_1\varphi_2 e^{\theta_1+\theta_2})}{p(y, z) + \varphi_1 e^{\theta_1} + \varphi_2 e^{\theta_2} + a_0\varphi_1\varphi_2 e^{\theta_1+\theta_2}} + \gamma(t, y, z); \tag{2.9}$$

(3°) the generalized 3-soliton solution

$$u = -2 \frac{\partial f}{\partial x} / f + \gamma(t, y, z); \tag{2.10}$$

(4°) the generalized singular 1-soliton solution

$$u = \frac{2k_1\varphi_1 e^{\theta_1}}{p(y, z) - \varphi_1 e^{\theta_1}} + \gamma(t, y, z); \tag{2.11}$$

(5°) the generalized singular 2-soliton solution

$$u = \frac{2(k_1\varphi_1 e^{\theta_1} + k_2\varphi_2 e^{\theta_2} - a_0(k_1 + k_2)\varphi_1\varphi_2 e^{\theta_1+\theta_2})}{p(y, z) - \varphi_1 e^{\theta_1} - \varphi_2 e^{\theta_2} + a_0\varphi_1\varphi_2 e^{\theta_1+\theta_2}} + \gamma(t, y, z); \tag{2.12}$$

(6°) the generalized singular 3-soliton solution

$$u = -2 \frac{\partial g}{\partial x} / g + \gamma(t, y, z). \tag{2.13}$$

Proposition 2.2. For arbitrarily given constants c, k and arbitrarily given differentiable functions $\psi(y, z), p(t, y), q(t, z), r(y, z)$, if let

$$\lambda(t, y, z) = p(t, y) + q(t, z) + r(y, z), \tag{2.14}$$

$$\alpha = \sqrt{\left| \frac{c}{k} \right|}, \tag{2.15}$$

and

$$\beta = \frac{1}{2} \sqrt{\left| \frac{c}{k^3} \right|}, \tag{2.16}$$

then Eq. (1.8) has the following nonlinear wave solutions:

(1°) the generalized fractional function solutions

$$u = -\frac{c(kx + \psi(y, z) - ct)}{6k^2} - \frac{2k}{kx + \psi(y, z) - ct} + \lambda(t, y, z); \tag{2.17}$$

and

$$u = \frac{c(kx + \psi(y, z) + ct)}{6k^2} - \frac{2k}{kx + \psi(y, z) + ct} + \lambda(t, y, z); \tag{2.18}$$

(2°) the generalized hyperbolic tanh function solutions

(i) if $k > 0$ and $c > 0$, then

$$u = -\alpha \tanh \beta [kx + \psi(y, z) - ct] + \lambda(t, y, z); \tag{2.19}$$

(ii) if $k < 0$ and $c < 0$, then

$$u = \alpha \tanh \beta [kx + \psi(y, z) - ct] + \lambda(t, y, z); \tag{2.20}$$

(iii) if $k > 0$ and $c < 0$, then

$$u = -\alpha \tanh \beta [kx + \psi(y, z) + ct] + \lambda(t, y, z); \tag{2.21}$$

(iv) if $k < 0$ and $c > 0$, then

$$u = \alpha \tanh \beta [kx + \psi(y, z) + ct] + \lambda(t, y, z); \tag{2.22}$$

(3°) the generalized hyperbolic coth function solutions

(i) if $k > 0$ and $c > 0$, then

$$u = -\alpha \coth \beta [kx + \psi(y, z) - ct] + \lambda(t, y, z); \tag{2.23}$$

(ii) if $k < 0$ and $c < 0$, then

$$u = \alpha \coth \beta [kx + \psi(y, z) - ct] + \lambda(t, y, z); \tag{2.24}$$

(iii) if $k > 0$ and $c < 0$, then

$$u = -\alpha \coth \beta [kx + \psi(y, z) + ct] + \lambda(t, y, z); \tag{2.25}$$

(iv) if $k < 0$ and $c > 0$, then

$$u = \alpha \coth \beta [kx + \psi(y, z) + ct] + \lambda(t, y, z); \tag{2.26}$$

(4°) the generalized tangent function solutions

(i) if $k > 0$ and $c > 0$, then

$$u = \alpha \tan \beta [kx + \psi(y, z) + ct] + \lambda(t, y, z); \tag{2.27}$$

(ii) if $k < 0$ and $c < 0$, then

$$u = -\alpha \tan \beta [kx + \psi(y, z) + ct] + \lambda(t, y, z); \tag{2.28}$$

(iii) if $k > 0$ and $c < 0$, then

$$u = \alpha \tan \beta [kx + \psi(y, z) - ct] + \lambda(t, y, z); \tag{2.29}$$

(iv) if $k < 0$ and $c > 0$, then

$$u = -\alpha \tan \beta [kx + \psi(y, z) - ct] + \lambda(t, y, z); \tag{2.30}$$

(5°) the generalized cotangent function solutions

(i) if $k > 0$ and $c > 0$, then

$$u = -\alpha \cot \beta [kx + \psi(y, z) + ct] + \lambda(t, y, z); \tag{2.31}$$

(ii) if $k < 0$ and $c < 0$, then

$$u = \alpha \cot \beta [kx + \psi(y, z) + ct] + \lambda(t, y, z); \tag{2.32}$$

(iii) if $k > 0$ and $c < 0$, then

$$u = -\alpha \cot \beta [kx + \psi(y, z) - ct] + \lambda(t, y, z); \tag{2.33}$$

(iv) if $k < 0$ and $c > 0$, then

$$u = \alpha \cot \beta [kx + \psi(y, z) - ct] + \lambda(t, y, z). \tag{2.34}$$

Secondly, we state the main results for Eq. (1.9) by the following two propositions.

Proposition 2.3. For arbitrarily given constants k_i, r_i and arbitrarily given differentiable functions $\gamma(z, t), p(z), \varphi_i = \varphi_i(z) (i = 1, 2, 3)$, if let

$$\omega_i = k_i x + r_i y - k_i^2 r_i t \quad (i = 1, 2, 3), \tag{2.35}$$

$$b_0 = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2 [p(z)]}, \tag{2.36}$$

$$b_{ij} = \left(\frac{k_i - k_j}{k_i + k_j} \right)^2 \quad (1 \leq i < j \leq 3), \tag{2.37}$$

$$b_{123} = b_{12} b_{13} b_{23}, \tag{2.38}$$

$$f = 1 + \varphi_1 e^{\omega_1} + \varphi_2 e^{\omega_2} + \varphi_3 e^{\omega_3} + b_{12} \varphi_1 \varphi_2 e^{\omega_1 + \omega_2} + b_{13} \varphi_1 \varphi_3 e^{\omega_1 + \omega_3} + b_{23} \varphi_2 \varphi_3 e^{\omega_2 + \omega_3} + b_{123} \varphi_1 \varphi_2 \varphi_3 e^{\omega_1 + \omega_2 + \omega_3}, \tag{2.39}$$

and

$$g = 1 - \varphi_1 e^{\omega_1} - \varphi_2 e^{\omega_2} - \varphi_3 e^{\omega_3} + b_{12} \varphi_1 \varphi_2 e^{\omega_1 + \omega_2} + b_{13} \varphi_1 \varphi_3 e^{\omega_1 + \omega_3} + b_{23} \varphi_2 \varphi_3 e^{\omega_2 + \omega_3} - b_{123} \varphi_1 \varphi_2 \varphi_3 e^{\omega_1 + \omega_2 + \omega_3}, \tag{2.40}$$

then Eq. (1.9) has the following generalized soliton solutions and generalized singular soliton solutions:

(1°) the generalized 1-soliton solution

$$u = \frac{-2k_1 \varphi_1 e^{\omega_1}}{p(z) + \varphi_1 e^{\omega_1}} + \gamma(z, t); \tag{2.41}$$

(2°) the generalized 2-soliton solution

$$u = \frac{-2(k_1\varphi_1e^{\omega_1} + k_2\varphi_2e^{\omega_2} + b_0(k_1 + k_2)\varphi_1\varphi_2e^{\omega_1+\omega_2})}{p(z) + \varphi_1e^{\omega_1} + \varphi_2e^{\omega_2} + b_0\varphi_1\varphi_2e^{\omega_1+\omega_2}} + \gamma(z, t); \tag{2.42}$$

(3°) the generalized 3-soliton solution

$$u = -2 \frac{\partial f}{\partial x} / f + \gamma(z, t); \tag{2.43}$$

(4°) the generalized singular 1-soliton solution

$$u = \frac{2k_1\varphi_1e^{\omega_1}}{p(z) - \varphi_1e^{\omega_1}} + \gamma(z, t); \tag{2.44}$$

(5°) the generalized singular 2-soliton solution

$$u = \frac{2(k_1\varphi_1e^{\omega_1} + k_2\varphi_2e^{\omega_2} - b_0(k_1 + k_2)\varphi_1\varphi_2e^{\omega_1+\omega_2})}{p(z) - \varphi_1e^{\omega_1} - \varphi_2e^{\omega_2} + b_0\varphi_1\varphi_2e^{\omega_1+\omega_2}} + \gamma(z, t); \tag{2.45}$$

(6°) the generalized singular 3-soliton solution

$$u = -2 \frac{\partial g}{\partial x} / g + \gamma(z, t). \tag{2.46}$$

Proposition 2.4. For arbitrarily given constants c, k, r and arbitrarily given differentiable functions $\psi(z), \lambda(z, t)$, if let

$$\alpha = \sqrt{\left| \frac{c}{r} \right|}, \tag{2.47}$$

and

$$\beta = \frac{1}{2} \sqrt{\left| \frac{c}{k^2r} \right|}, \tag{2.48}$$

then Eq. (1.9) has the following nonlinear wave solutions:

(1°) the generalized fractional function solutions

$$u = -\frac{c(kx + ry + \psi(z) - ct)}{6kr} - \frac{2k}{kx + ry + \psi(z) - ct} + \lambda(z, t); \tag{2.49}$$

and

$$u = \frac{c(kx + ry + \psi(z) + ct)}{6kr} - \frac{2k}{kx + ry + \psi(z) + ct} + \lambda(z, t); \tag{2.50}$$

(2°) the generalized hyperbolic tanh function solutions

(i) if $rc > 0$ and $k > 0$, then

$$u = -\alpha \tanh \beta [kx + ry + \psi(z) - ct] + \lambda(z, t); \tag{2.51}$$

(ii) if $rc > 0$ and $k < 0$, then

$$u = \alpha \tanh \beta [kx + ry + \psi(z) - ct] + \lambda(z, t); \tag{2.52}$$

(iii) if $rc < 0$ and $k > 0$, then

$$u = -\alpha \tanh \beta [kx + ry + \psi(z) + ct] + \lambda(z, t); \tag{2.53}$$

(iv) if $rc < 0$ and $k < 0$, then

$$u = \alpha \tanh \beta [kx + ry + \psi(z) + ct] + \lambda(z, t); \tag{2.54}$$

(3°) the generalized hyperbolic coth function solutions

(i) if $rc > 0$ and $k > 0$, then

$$u = -\alpha \coth \beta [kx + ry + \psi(z) - ct] + \lambda(z, t); \tag{2.55}$$

(ii) if $rc > 0$ and $k < 0$, then

$$u = \alpha \coth \beta [kx + ry + \psi(z) - ct] + \lambda(z, t); \tag{2.56}$$

(iii) if $rc < 0$ and $k > 0$, then

$$u = -\alpha \coth \beta [kx + ry + \psi(z) + ct] + \lambda(z, t); \tag{2.57}$$

(iv) if $rc < 0$ and $k < 0$, then

$$u = \alpha \coth \beta [kx + ry + \psi(z) + ct] + \lambda(z, t); \quad (2.58)$$

(4°) the generalized tangent function solutions

(i) if $rc > 0$ and $k > 0$, then

$$u = \alpha \tan \beta [kx + ry + \psi(z) + ct] + \lambda(z, t); \quad (2.59)$$

(ii) if $rc > 0$ and $k < 0$, then

$$u = -\alpha \tan \beta [kx + ry + \psi(z) + ct] + \lambda(z, t); \quad (2.60)$$

(iii) if $rc < 0$ and $k > 0$, then

$$u = \alpha \tan \beta [kx + ry + \psi(z) - ct] + \lambda(z, t); \quad (2.61)$$

(iv) if $rc < 0$ and $k < 0$, then

$$u = -\alpha \tan \beta [kx + ry + \psi(z) - ct] + \lambda(z, t); \quad (2.62)$$

(5°) the generalized cotangent function solutions

(i) if $rc > 0$ and $k > 0$, then

$$u = -\alpha \cot \beta [kx + ry + \psi(z) + ct] + \lambda(z, t); \quad (2.63)$$

(ii) if $rc > 0$ and $k < 0$, then

$$u = \alpha \cot \beta [kx + ry + \psi(z) + ct] + \lambda(z, t); \quad (2.64)$$

(iii) if $rc < 0$ and $k > 0$, then

$$u = -\alpha \cot \beta [kx + ry + \psi(z) - ct] + \lambda(z, t); \quad (2.65)$$

(iv) if $rc < 0$ and $k < 0$, then

$$u = \alpha \cot \beta [kx + ry + \psi(z) - ct] + \lambda(z, t). \quad (2.66)$$

Remark 2.1. Since these solutions contain arbitrary functions, we add the word “generalized” to each type of solutions.

3. The derivations of our main results

Firstly, employing the simplified Hirota’s method, we derive the results of [Proposition 2.1](#). For arbitrarily given constants k_i ($i = 1, 2, 3$) and arbitrarily given differentiable functions $p(y, z)$, $\varphi_i = \varphi_i(y, z)$ ($i = 1, 2, 3$) and $\gamma(t, y, z)$ mentioned in [Proposition 2.1](#), the derivations of [Proposition 2.1](#) are composed of seven steps as follows:

Step 1. Determining the solutions of the linear equation

Consider the linear equation

$$u_{yzt} + u_{xxyz} = 0, \quad (3.1)$$

which is constructed by the linear part of Eq. (1.8). Assume that Eq. (3.1) has solution of form

$$u = \varphi_i e^{k_i x - c_i t} \quad (i = 1, 2, 3), \quad (3.2)$$

where c_i are to be determined.

Substituting (3.2) into Eq. (3.1), we get

$$c_i = k_i^3 \quad (i = 1, 2, 3). \quad (3.3)$$

This implies that

$$u = \varphi_i e^{k_i x - k_i^3 t} \quad (i = 1, 2, 3), \quad (3.4)$$

are the solutions of Eq. (3.1).

Step 2. Constructing generalized 1-soliton solution

Let

$$\theta_1 = k_1 x - k_1^3 t, \quad (3.5)$$

$$F_1 = p(y, z) + \varphi_1 e^{\theta_1}, \quad (3.6)$$

and suppose Eq. (1.8) has solution of form

$$u = R \frac{\partial F_1}{\partial x} / F_1 + \gamma(t, y, z), \quad (3.7)$$

where R is to be determined. Substituting (3.7) into Eq. (1.8), we obtain $R = -2$. This means that the function in (2.8) is a solution of Eq. (1.8).

Step 3. Constructing generalized 2-soliton solution

Let

$$\theta_2 = k_2 x - k_2^3 t, \quad (3.8)$$

$$F_2 = F_1 + \varphi_2 e^{\theta_2} + d_{12} \varphi_1 \varphi_2 e^{\theta_1 + \theta_2}, \quad (3.9)$$

and assume that Eq. (1.8) has solution of form

$$u = -2 \frac{\partial F_2}{\partial x} / F_2 + \gamma(t, y, z), \quad (3.10)$$

where d_{12} is to be determined. Substituting (3.10) into Eq. (1.8), it follows that

$$d_{12} = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2 [p(y, z)]}. \quad (3.11)$$

This shows that the function in (2.9) is a solution of Eq. (1.8).

Step 4. Constructing generalized 3-soliton solution

Let

$$\theta_3 = k_3 x - k_3^3 t, \quad (3.12)$$

$$a_{ij} = \left(\frac{k_i - k_j}{k_i + k_j} \right)^2 \quad (1 \leq i < j \leq 3), \quad (3.13)$$

$$F_3 = 1 + \varphi_1 e^{\theta_1} + \varphi_2 e^{\theta_2} + \varphi_3 e^{\theta_3} + a_{12} \varphi_1 \varphi_2 e^{\theta_1 + \theta_2} + a_{13} \varphi_1 \varphi_3 e^{\theta_1 + \theta_3} + a_{23} \varphi_2 \varphi_3 e^{\theta_2 + \theta_3} + d_{123} \varphi_1 \varphi_2 \varphi_3 e^{\theta_1 + \theta_2 + \theta_3}, \quad (3.14)$$

and suppose Eq. (1.8) has solution of form

$$u = -2 \frac{\partial F_3}{\partial x} / F_3 + \gamma(t, y, z), \quad (3.15)$$

where d_{123} is to be determined. Substituting (3.15) into Eq. (1.8), we have

$$d_{123} = a_{12} a_{13} a_{23}. \quad (3.16)$$

This implies that the function in (2.10) is a solution of Eq. (1.8).

Step 5. Constructing generalized singular 1-soliton solution

Let

$$G_1 = p(y, z) - \varphi_1 e^{\theta_1}, \quad (3.17)$$

and suppose Eq. (1.8) has solution of form

$$u = N \frac{\partial G_1}{\partial x} / G_1 + \gamma(t, y, z), \quad (3.18)$$

where N is to be determined. Substituting (3.18) into Eq. (1.8), we get $N = -2$. This shows that the function in (2.11) is a solution of Eq. (1.8).

Step 6. Constructing generalized singular 2-soliton solution

Let

$$G_2 = p(y, z) - \varphi_1 e^{\theta_1} - \varphi_2 e^{\theta_2} + \rho_{12} \varphi_1 \varphi_2 e^{\theta_1 + \theta_2}, \quad (3.19)$$

and suppose Eq. (1.8) has solution of form

$$u = -2 \frac{\partial G_2}{\partial x} / G_2 + \gamma(t, y, z), \quad (3.20)$$

where ρ_{12} is to be determined. Substituting (3.20) into Eq. (1.8), we obtain $\rho_{12} = a_0$. This means that the function in (2.12) is a solution of Eq. (1.8).

Step 7. Constructing generalized singular 3-soliton solution

Let

$$G_3 = 1 - \varphi_1 e^{\theta_1} - \varphi_2 e^{\theta_2} - \varphi_3 e^{\theta_3} + a_{12}\varphi_1\varphi_2 e^{\theta_1+\theta_2} + a_{13}\varphi_1\varphi_3 e^{\theta_1+\theta_3} + a_{23}\varphi_2\varphi_3 e^{\theta_2+\theta_3} + \rho_{123}\varphi_1\varphi_2\varphi_3 e^{\theta_1+\theta_2+\theta_3}, \quad (3.21)$$

and assume that Eq. (1.8) has solution of form

$$u = -2 \frac{\partial G_3}{\partial x} / G_3 + \gamma(t, y, z), \quad (3.22)$$

where ρ_{123} is to be determined. Substituting (3.15) into Eq. (1.8), we get $\rho_{123} = -a_{123}$. This implies that the function in (2.13) is a solution of Eq. (1.8).

Hereto, we complete the derivation of Proposition 2.1.

Remark 3.1. The 1- and 2-soliton solutions contain arbitrary differentiable function $p(y, z)$ and this phenomenon has not been found in previous works. But we have not found this phenomenon in 3-soliton solution.

Secondly, by using the method of dynamical systems, we derive the results of Proposition 2.2. For arbitrarily given constants c, k and arbitrarily given differentiable functions $\psi(y, z)$ and $\lambda(t, y, z)$ mentioned in Proposition 2.2, let

$$\delta = \sqrt{\left| \frac{2}{k} \right|}, \quad (3.23)$$

$$\alpha = \sqrt{\left| \frac{c}{k} \right|}, \quad (3.24)$$

$$\beta = \frac{1}{2} \sqrt{\left| \frac{c}{k^3} \right|}, \quad (3.25)$$

$$v_0 = \frac{c}{6k^2}, \quad (3.26)$$

$$v_1 = \frac{c}{2k^2}, \quad (3.27)$$

$$\xi = kx + \psi(y, z) - ct, \quad (3.28)$$

and

$$\eta = kx + \psi(y, z) + ct. \quad (3.29)$$

Based on ξ and η , our derivations contain the following two parts.

Part 1. The derivations based on ξ

Substituting

$$u = f(\xi) + \lambda(t, y, z), \quad (3.30)$$

into Eq. (1.8), it follows that

$$[-6k^2 f''(\xi) f'''(\xi) - 6k^2 f'(\xi) f''''(\xi) - cf''''(\xi) + k^3 f^{(5)}(\xi)] \psi_y \psi_z + [-cf''(\xi) - 6k^2 f'(\xi) f''(\xi) + k^3 f^{(4)}(\xi)] \psi_{yz} = 0. \quad (3.31)$$

Since $\psi(y, z)$ is arbitrary, Eq. (3.31) holds if and only if

$$-6k^2 (f'')^2(\xi) - 6k^2 f'(\xi) f''''(\xi) - cf''''(\xi) + k^3 f^{(5)}(\xi) = 0, \quad (3.32)$$

and

$$-cf''(\xi) - 6k^2 f'(\xi) f''(\xi) + k^3 f^{(4)}(\xi) = 0. \quad (3.33)$$

Note that Eq. (3.32) comes from taking the derivative on both sides of Eq. (3.33). Therefore we only consider Eq. (3.33). Integrating Eq. (3.33) once, it follows that

$$f'''(\xi) = h_0 + \frac{cf'(\xi)}{k^3} + \frac{3}{k} f'(\xi)^2, \quad (3.34)$$

where h_0 is an integral constant. If let

$$v = f'(\xi), \quad (3.35)$$

then Eq. (3.34) becomes

$$v''(\xi) = h_0 + \frac{c}{k^3} v(\xi) + \frac{3}{k} v(\xi)^2. \quad (3.36)$$

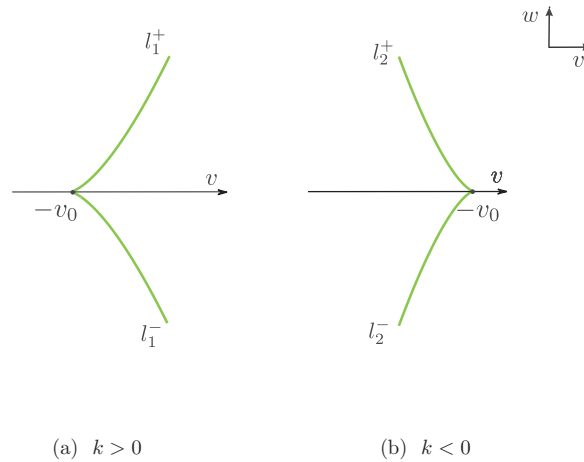


Fig. 1. The graphs of (3.38) when $h_0 = \frac{c^2}{12k^5}$ and $h_1 = \frac{c^3}{216k^7}$.

Putting $w = v'(\xi)$, we get a planar system

$$\begin{cases} \frac{dv}{d\xi} = w, \\ \frac{dw}{d\xi} = h_0 + \frac{c}{k^3}v + \frac{3}{k}v^2, \end{cases} \tag{3.37}$$

with the first integral

$$w^2 = 2 \left(h_1 + h_0 v + \frac{c}{2k^3} v^2 + \frac{1}{k} v^3 \right), \tag{3.38}$$

where h_1 is another integral constant. The subsequent derivations are processed by the following three steps.

Step 1. Constructing generalized fractional function solution with ξ

Taking $h_0 = \frac{c^2}{12k^5}$ and $h_1 = \frac{c^3}{216k^7}$, then (3.38) possesses graphs as Fig. 1. In Fig. 1, the curves possess expressions as follows:

$$l_1^\pm : w = \pm \delta (v + v_0)^{\frac{3}{2}}, \quad -v_0 < v < +\infty; \tag{3.39}$$

and

$$l_2^\pm : w = \pm \delta (-v - v_0)^{\frac{3}{2}}, \quad -\infty < v < -v_0. \tag{3.40}$$

Substituting (3.39) and (3.40) into the first equation of (3.37) respectively, and integrating them along the corresponding curves, we have

$$\int_v^{+\infty} \frac{ds}{(s + v_0)^{\frac{3}{2}}} = \delta |\xi| \quad (\text{along } l_1^\pm), \tag{3.41}$$

and

$$\int_{-\infty}^v \frac{ds}{(-v_0 - s)^{\frac{3}{2}}} = \delta |\xi| \quad (\text{along } l_2^\pm). \tag{3.42}$$

Completing the integrals and solving the equations for v , we get

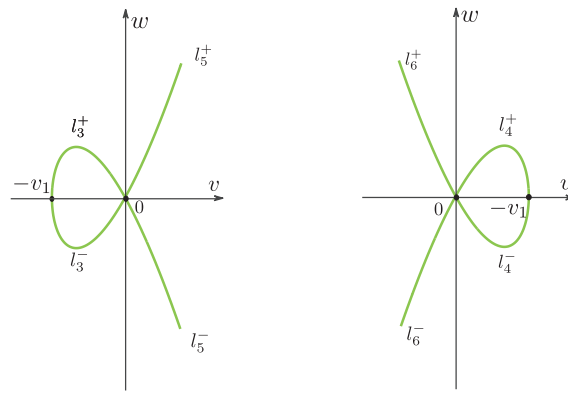
$$v = -v_0 + \frac{2k}{\xi^2}. \tag{3.43}$$

Via (3.35) and (3.43), it follows that

$$f'(\xi) = -v_0 + \frac{2k}{\xi^2}. \tag{3.44}$$

Integrating Eq. (3.44) and taking the integral constant as zero, we obtain

$$f(\xi) = -v_0 \xi - \frac{2k}{\xi}. \tag{3.45}$$



(a) $k > 0$ and $c > 0$ (b) $k < 0$ and $c < 0$

Fig. 2. The graphs of (3.38) when $h_0 = h_1 = 0$ and $kc > 0$.

These imply that

$$u = -v_0 \xi - \frac{2k}{\xi} + \lambda(t, y, z) \tag{3.46}$$

is a solution of Eq. (1.8), that is, the function in (2.17) is a solution of Eq. (1.8).

Step 2. Constructing generalized hyperbolic tanh and coth function solutions with ξ

Taking $h_0 = h_1 = 0$ and $k \cdot c > 0$, then the graphs of (3.38) are as Fig. 2.

In Fig. 2, the curves possess the following expressions:

$$l_3^\pm : w = \mp \delta v (v + v_1)^{\frac{1}{2}}, \quad -v_1 \leq v < 0; \tag{3.47}$$

$$l_4^\pm : w = \pm \delta v (-v - v_1)^{\frac{1}{2}}, \quad 0 < v \leq -v_1; \tag{3.48}$$

$$l_5^\pm : w = \pm \delta v (v + v_1)^{\frac{1}{2}}, \quad 0 < v < +\infty; \tag{3.49}$$

and

$$l_6^\pm : w = \mp \delta v (-v - v_1)^{\frac{1}{2}}, \quad -\infty < v < 0. \tag{3.50}$$

Substituting (3.47) into $\frac{dv}{w} = d\xi$ and integrating it along l_3^\pm , we have

$$-\int_{-v_1}^v \frac{ds}{s(s+v_1)^{\frac{1}{2}}} = \delta |\xi|. \tag{3.51}$$

Completing the integral, it follows that

$$\operatorname{arctanh} \left(\frac{\sqrt{c+2k^2v}}{\sqrt{c}} \right) = -\beta |\xi|. \tag{3.52}$$

From (3.52), we get

$$v = -\frac{c}{2k^2} \operatorname{sech}^2 \beta \xi, \tag{3.53}$$

that is,

$$f'(\xi) = -\frac{c}{2k^2} \operatorname{sech}^2 \beta \xi. \tag{3.54}$$

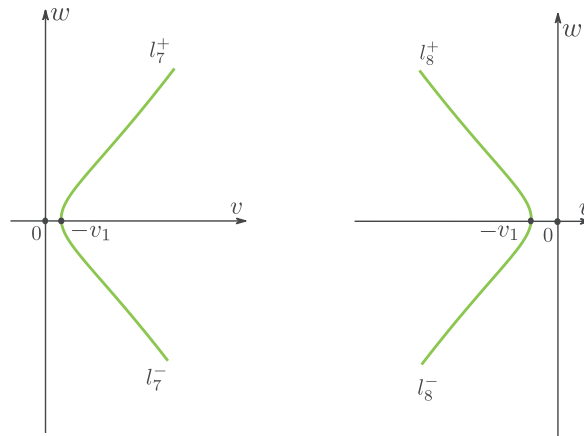
Further we obtain

$$f(\xi) = -\alpha \tanh \beta \xi. \tag{3.55}$$

This implies that the function in (2.19) is a solution of Eq. (1.8).

Substituting (3.48)–(3.50) into $\frac{dv}{w} = d\xi$ respectively, and integrating them along the corresponding curves, we have

$$\int_v^{-v_1} \frac{ds}{s(-s-v_1)^{\frac{1}{2}}} = \delta |\xi| \quad (\text{along } l_4^\pm), \tag{3.56}$$



(a) $k > 0$ and $c < 0$

(b) $k < 0$ and $c > 0$

Fig. 3. The graphs of (3.38) when $h_0 = h_1 = 0$ and $kc < 0$.

$$\int_v^{+\infty} \frac{ds}{s(s+v_1)^{\frac{1}{2}}} = \delta |\xi| \quad (\text{along } l_7^\pm), \tag{3.57}$$

and

$$-\int_{-\infty}^v \frac{ds}{s(-s-v_1)^{\frac{1}{2}}} = \delta |\xi| \quad (\text{along } l_8^\pm). \tag{3.58}$$

Solving v from (3.56), we have

$$v = -\alpha \operatorname{sech}^2 \beta \xi. \tag{3.59}$$

From (3.59) and $v = f(\xi)$, we get

$$f(\xi) = \alpha \tanh \beta \xi. \tag{3.60}$$

Via (3.60) and $u = f(\xi) + \lambda(t, y, z)$, it is seen that the function in (2.20) is a solution of Eq. (1.8).

Similarly, solving v from (3.57), (3.58), and via $f(\xi) = v$, $u = f(\xi) + \lambda(t, y, z)$, we see that the functions in (2.23) and (2.24) are solutions of Eq. (1.8).

Step 3. Constructing generalized tangent and cotangent function solutions with ξ

When $h_0 = h_1 = 0$ and $kc < 0$, the graphs of (3.38) are as Fig. 3.

In Fig. 3, the expressions of the curves are as follows:

$$l_7^\pm : w = \pm \delta v (v + v_1)^{\frac{1}{2}}, \quad -v_1 \leq v < +\infty; \tag{3.61}$$

and

$$l_8^\pm : w = \mp \delta v (-v - v_1)^{\frac{1}{2}}, \quad -\infty < v \leq -v_1. \tag{3.62}$$

Substituting (3.61) and (3.62) into $\frac{dv}{w} = d\xi$ respectively, and integrating them along the corresponding curves, we have

$$\int_v^{+\infty} \frac{ds}{s(s+v_1)^{\frac{1}{2}}} = \delta |\xi| \quad (\text{along } l_7^\pm), \tag{3.63}$$

and

$$\int_{-\infty}^v \frac{ds}{s(-s-v_1)^{\frac{1}{2}}} = -\delta |\xi| \quad (\text{along } l_8^\pm). \tag{3.64}$$

Completing the integrals, (3.63) and (3.64) become

$$\arctan \left(\frac{\sqrt{v+v_1}}{\sqrt{-v_1}} \right) = \beta |\xi|, \tag{3.65}$$

and

$$\arctan \left(\frac{\sqrt{-v-v_1}}{\sqrt{v_1}} \right) = \beta |\xi|. \tag{3.66}$$

Solving v from (3.65) and (3.66) respectively, and via $f(\xi) = v$, $u = f(\xi) + \lambda(t, y, z)$, we obtain the functions in (2.29) and (2.30).

Note that if $u = f(\xi) + \lambda(t, y, z)$ is a solution of Eq. (1.8), then so is $u = f(\xi + \frac{\pi}{2}) + \lambda(t, y, z)$. Therefore, from (2.29) and (2.30) we get (2.33) and (2.34). These complete the derivations based on $\xi = kx + \psi(y, z) - ct$.

Part 2. The derivations based on η

Substituting

$$u = g(\eta) + \lambda(t, y, z), \tag{3.67}$$

into Eq. (1.8), it follows that

$$[-6k^2g''(\eta)g'(\eta) - 6k^2g'(\eta)g'''(\eta) + cg'''(\eta) + k^3g^{(5)}(\eta)]\psi_y\psi_z + [cg'(\eta) - 6k^2g'(\eta)g''(\eta) + k^3g^{(4)}(\eta)]\psi_{yz} = 0. \tag{3.68}$$

Clearly, Eq. (3.68) holds if and only if

$$-6k^2(g'')^2(\eta) - 6k^2g'(\eta)g'''(\eta) + cg'''(\eta) + k^3g^{(5)}(\eta) = 0, \tag{3.69}$$

and

$$cg''(\eta) - 6k^2g'(\eta)g''(\eta) + k^3g^{(4)}(\eta) = 0. \tag{3.70}$$

Since Eq. (3.69) comes from taking the derivative on both sides of Eq. (3.70), we only study Eq. (3.70). Integrating Eq. (3.70) once, we have

$$\frac{c}{k^3}g' - \frac{3}{k}g'^2 + g''' = r_0, \tag{3.71}$$

where r_0 is an integral constant. Letting

$$\mu = g', \tag{3.72}$$

Eq. (3.71) becomes

$$\mu'' = r_0 - \frac{c}{k^3}\mu + \frac{3}{k}\mu^2. \tag{3.73}$$

Letting $\Omega = \mu'$, from (3.73) it yields a planar system

$$\begin{cases} \frac{d\mu}{d\eta} = \Omega, \\ \frac{d\Omega}{d\eta} = r_0 - \frac{c}{k^3}\mu + \frac{3}{k}\mu^2, \end{cases} \tag{3.74}$$

with the first integral

$$\Omega^2 = 2 \left(r_1 + r_0\mu - \frac{c}{2k^3}\mu^2 + \frac{1}{k}\mu^3 \right), \tag{3.75}$$

where r_1 is another integral constant. Let

$$\mu_0 = \frac{c}{6k^2}, \tag{3.76}$$

and

$$\mu_1 = \frac{c}{2k^2}. \tag{3.77}$$

The following derivations are divided into three steps, too.

Step 1. Constructing generalized fractional function solution with η

Taking $r_0 = \frac{c^2}{12k^5}$ and $r_1 = -\frac{c^3}{216k^7}$, then (3.75) possesses graphs as Fig. 4.

In Fig. 4, the curves possess expressions as follows:

$$\Gamma_1^\pm : \Omega = \pm\delta(\mu - \mu_0)^{\frac{3}{2}}, \quad \mu_0 < \mu < +\infty; \tag{3.78}$$

and

$$\Gamma_2^\pm : \Omega = \pm\delta(-\mu + \mu_0)^{\frac{3}{2}}, \quad -\infty < \mu < \mu_0. \tag{3.79}$$

Substituting (3.78) and (3.79) into $\frac{d\mu}{\Omega} = d\eta$ respectively, and integrating them along the corresponding curves, we have

$$\int_{\mu}^{+\infty} \frac{ds}{(s - \mu_0)^{\frac{3}{2}}} = \delta |\eta| \quad (\text{along } \Gamma_1^\pm), \tag{3.80}$$

and

$$\int_{-\infty}^{\mu} \frac{ds}{(\mu_0 - s)^{\frac{3}{2}}} = \delta |\eta| \quad (\text{along } \Gamma_2^\pm). \tag{3.81}$$

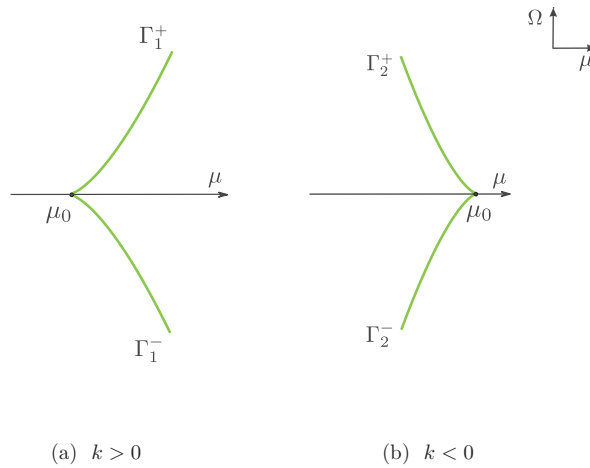


Fig. 4. The graphs of (3.75) when $r_0 = \frac{c^2}{12k^3}$ and $r_1 = -\frac{c^3}{216k^2}$.

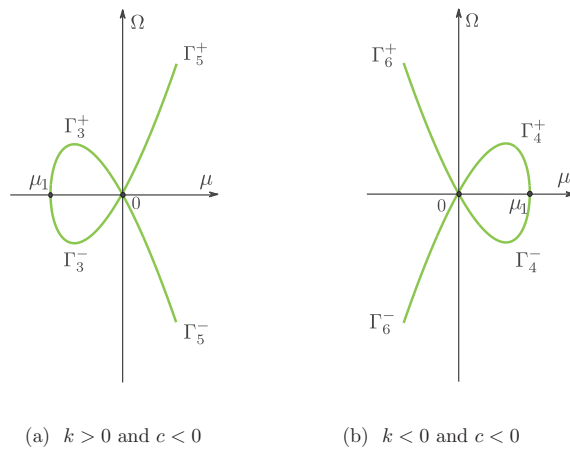


Fig. 5. The graphs of (3.75) when $r_0 = r_1 = 0$ and $kc < 0$.

Solving μ from one of (3.80) or (3.81), and via (3.67) and (3.72), we get the function in (2.18).

Step 2. Constructing generalized hyperbolic tanh and coth function solutions with η

Taking $r_0 = r_1 = 0$ and $kc < 0$, then the graphs of (3.75) are as Fig. 5.

In Fig. 5, the expressions of the curves are as follows:

$$\Gamma_3^\pm : \Omega = \mp \delta \mu (\mu - \mu_1)^{\frac{1}{2}}, \quad \mu_1 \leq \mu < 0; \tag{3.82}$$

$$\Gamma_4^\pm : \Omega = \pm \delta \mu (-\mu + \mu_1)^{\frac{1}{2}}, \quad 0 < \mu \leq \mu_1; \tag{3.83}$$

$$\Gamma_5^\pm : \Omega = \pm \delta \mu (\mu - \mu_1)^{\frac{1}{2}}, \quad 0 < \mu < +\infty; \tag{3.84}$$

and

$$\Gamma_6^\pm : \Omega = \mp \delta \mu (-\mu + \mu_1)^{\frac{1}{2}}, \quad -\infty < \mu < 0. \tag{3.85}$$

Substituting (3.82)–(3.85) into $\frac{d\mu}{\Omega} = d\eta$ respectively, and integrating them along the corresponding curves, we have

$$-\int_{\mu_1}^{\mu} \frac{ds}{s(s - \mu_1)^{\frac{1}{2}}} = \delta |\eta| \quad (\text{along } \Gamma_3^\pm), \tag{3.86}$$

$$\int_{\mu}^{\mu_1} \frac{ds}{s(-s + \mu_1)^{\frac{1}{2}}} = \delta |\eta| \quad (\text{along } \Gamma_4^\pm), \tag{3.87}$$

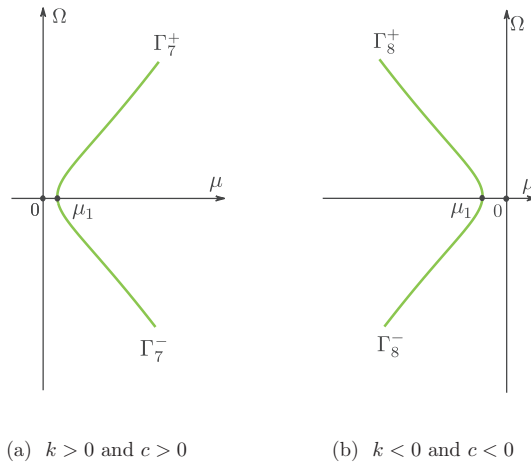


Fig. 6. The graphs of (3.75) when $r_0 = r_1 = 0$ and $kc > 0$.

$$\int_{\mu}^{+\infty} \frac{ds}{s(s - \mu_1)^{\frac{1}{2}}} = \delta |\eta| \quad (\text{along } \Gamma_7^{\pm}), \tag{3.88}$$

and

$$-\int_{-\infty}^{\mu} \frac{ds}{s(-s + \mu_1)^{\frac{1}{2}}} = \delta |\eta| \quad (\text{along } \Gamma_8^{\pm}). \tag{3.89}$$

Solving μ respectively from (3.86)–(3.89), and via (3.67) and (3.72), we get the functions in (2.21), (2.22), (2.25) and (2.26).

Step 3. Constructing generalized tangent and cotangent function solutions with η

Taking $r_0 = r_1 = 0$ and $kc > 0$, the graphs of (3.75) are as Fig. 6.

In Fig. 6, the expressions of the curves are as follows:

$$\Gamma_7^{\pm} : \Omega = \pm \delta \mu (\mu - \mu_1)^{\frac{1}{2}}, \quad \mu_1 \leq \mu < +\infty; \tag{3.90}$$

and

$$\Gamma_8^{\pm} : \Omega = \mp \delta \mu (-\mu + \mu_1)^{\frac{1}{2}}, \quad -\infty < \mu \leq \mu_1. \tag{3.91}$$

Substituting (3.90) and (3.91) into $\frac{d\mu}{w} = d\eta$ respectively, and integrating them along the corresponding curves, we have

$$\int_{\mu}^{+\infty} \frac{ds}{s(s - \mu_1)^{\frac{1}{2}}} = \delta |\eta| \quad (\text{along } \Gamma_7^{\pm}), \tag{3.92}$$

and

$$-\int_{-\infty}^{\mu} \frac{ds}{s(-s + \mu_1)^{\frac{1}{2}}} = \delta |\eta| \quad (\text{along } \Gamma_8^{\pm}). \tag{3.93}$$

Solving μ respectively from (3.92) and (3.93), and via (3.67), (3.72), we get the functions in (2.27) and (2.28).

Note that if $u = g(\eta) + \lambda(t, y, z)$ is a solution of Eq. (1.8), then so is $u = g(\eta + \frac{\pi}{2}) + \lambda(t, y, z)$. Therefore, from (2.27) and (2.28) we obtain the functions in (2.31) and (2.32).

Similarly to the derivations of Propositions 2.1 and 2.2, we can derive the Propositions 2.3 and 2.4. Hereto, we have finished the derivations for our main results.

4. Conclusions

In this paper, employing two methods we have studied the nonlinear wave solutions for Eqs. (1.8) and (1.9). We have obtained many new expressions of the solutions which were listed in Propositions 2.1–2.4. It is interesting that these expressions contain some arbitrary functions. This property seems very special. Proposition 2.1 extends the results of reference [14] for Eq. (1.8), that is, when the arbitrary functions $\varphi_i(y, z)$ are replaced by the functions $e^{r_i y + s_i z}$ ($i = 1, 2, 3$), $p(y, z) \equiv 0$ and $\gamma(t, y, z) \equiv 0$, the generalized multiple soliton solutions and generalized multiple singular soliton solutions can be reduced to the results of reference [14]. Proposition 2.3 extends the results of reference [14] for Eq. (1.9), that is, when the arbitrary functions $\varphi_i(z)$ are replaced by the functions $e^{s_i z}$ ($i = 1, 2, 3$), $p(z) \equiv 0$ and $\gamma(z, t) \equiv 0$, the generalized multiple soliton solutions and generalized multiple singular soliton solutions can be reduced to the results of reference [14].

In Proposition 2.1, the generalized soliton solutions and the generalized singular soliton solutions for Eq. (1.8) contain the factor $k_i x - c_i t$ with $c_i = k_i^3$, that is, there is relation between the coefficients of x and t . In Proposition 2.2, the nonlinear wave solutions also contain the factor $kx - ct$ or $kx + ct$, but k and c are independent.

Similarly, in Proposition 2.3, the generalized soliton solutions and the generalized singular soliton solutions for Eq. (1.9) contain the factor $k_i x + r_i y - c_i t$ with $c_i = k_i^2 r_i$, that is, there is relation between the coefficients of x , y and t . In Proposition 2.4, the nonlinear wave solutions also contain the factor $kx + ry - ct$ or $kx + ry + ct$, but k , r and c are independent.

These imply that these two methods are effective in constructing the nonlinear wave solutions of Eqs. (1.8) and (1.9). But there are respective merits and demerits. It is worth noting that there might be more efficient method to construct the nonlinear wave solutions of Eqs. (1.8) and (1.9), which could be our next goal.

Finally, the correctness of all the solutions listed in the four propositions are validated by the mathematical software.

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