COMMUNICATIONS ON
PURE AND APPLIED ANALYSIS
Volume **12**, Number **2**, March **2013**

doi:10.3934/cpaa.2013.12.1029

pp. **1029–1047**

THE EXPLICIT NONLINEAR WAVE SOLUTIONS OF THE GENERALIZED b-EQUATION

Liu Rui

Department of Mathematics, South China University of Technology
Guangzhou 510640, People's Republic of China

(Communicated by Tong Yang)

Abstract. In this paper, we study the nonlinear wave solutions of the generalized b-equation involving two parameters b and k. Let c be constant wave speed, $c_5 = \frac{1}{2}\left(1+b-\sqrt{(1+b)(1+b-8k)}\right)$, $c_6 = \frac{1}{2}\left(1+b+\sqrt{(1+b)(1+b-8k)}\right)$. We obtain the following results:

1. If $-\infty < k < \frac{1+b}{8}$ and $c \in (c_5, c_6)$, then there are three types of explicit nonlinear wave solutions, hyperbolic smooth solitary wave solution, hyperbolic peakon wave solution and hyperbolic blow-up solution.

2. If $-\infty < k < \frac{1+b}{8}$ and $c = c_5$ or c_6, then there are two types of explicit nonlinear wave solutions, fractional peakon wave solution and fractional blow-up solution.

3. If $k = \frac{1+b}{8}$ and $c = \frac{b+1}{2}$, then there are two types of explicit nonlinear wave solutions, fractional peakon wave solution and fractional blow-up solution.

Not only is the existence of these solutions shown, but their concrete expressions are presented. We also reveal the relationships among these solutions. Besides, the correctness of these solutions is tested by using the software Mathematica.

1. **Introduction.** Consider the nonlinear equation

$$u_t + 2ku_x - u_{xxt} + (1+b)u^2u_x = bu_xu_{xx} + uu_{xxx}, \tag{1}$$

where b, k are two real parameters, and $b > 1$ is a positive integer. Eq.(1) is called generalized b-equation because it is the generalized form of the following b-equation

$$u_t + 2ku_x - u_{xxt} + (1+b)uu_x = bu_xu_{xx} + uu_{xxx}, \tag{2}$$

which was presented by Degasperis, Holm and Hone [1, 2]. Clearly, when $b = 2,\ 3$, Eq.(2) respectively changes into the CH equation

$$u_t + 2ku_x - u_{xxt} + 3uu_x = 2u_xu_{xx} + uu_{xxx}, \tag{3}$$

and the DP equation

$$u_t + 2ku_x - u_{xxt} + 4uu_x = 3u_xu_{xx} + uu_{xxx}. \tag{4}$$

Camassa and Holm [3] showed that the CH equation is integrable and has peaked solitons. Cooper and Shepard [4] derived an approximate solitary wave solution of

2000 *Mathematics Subject Classification.* Primary: 34A20, 34C35, 35B65; Secondary: 58F05, 76B25.

Key words and phrases. Explicit nonlinear wave solution, generalized b-equation, solitary, dynamical system, bifurcation method.

Supported by the Fundamental Research Funds for the Central Universities (No.2012ZM0057) and the National Natural Science Foundation of China (No.11171115).

the CH equation by using some variational functions. Constantin [5, 6] gave the mathematical description of the existence of interacting solitary waves and showed that the peakons are stable for the CH equation. Boyd [7] derived a perturbation series which converges even at the peakon limit, and gave three analytical representations for the spatially periodic generalization of the peakon, called "the coshoidal wave" in the CH equation. Recently, the CH equation has been studied successively by many authors (see for instance in Refs. [8]-[12]).

The DP equation was given by Degasperis and Procesi [13]. Lundmark and Szmigielski [14, 15] presented an inverse scattering approach for computing the n-peakon solutions of the DP equation and gave the concrete expressions of the 3-peakon solutions. Chen and Tang [16] showed that in DP equation there are the kink-like waves. Guha [17] proposed an Euler-Poincare formalism of the DP equation.

The solutions of the b-equation were studied numerically for various values of b by Holm and Staley [18]. For arbitrary $b > 1$, Guo and Liu [19] showed that Eq.(2) has periodic cusp waves and constructed their expressions.

To study the bifurcation of peakon waves, Liu and Qian [20] suggested a generalized CH equation

$$u_t + 2ku_x - u_{xxt} + 3u^2u_x = 2u_xu_{xx} + uu_{xxx}. \tag{5}$$

Similarly, to investigate the change of peakon waves, Wazwaz [21, 22] proposed a generalized DP equation

$$u_t - u_{xxt} + 4u^2u_x = 3u_xu_{xx} + uu_{xxx} \tag{6}$$

and a generalized b-equation

$$u_t - u_{xxt} + (b+1)u^2u_x = bu_xu_{xx} + uu_{xxx}. \tag{7}$$

Since the CH and the DP equations possess rich structure and property, many authors were interested in their modified forms, Eqs.(5), (6) and (7). Tian and Song [23] gave some physical explanation for Eq.(5). Shen and Xu [24] discussed the existence of both smooth and non-smooth travelling waves for Eq.(5). Letting c denote the constant wave speed of travelling waves, for some special values of c, the exact travelling wave solutions were studied for Eq.(5) and Eq.(6). When $c = 1$, Khuri [25] obtained a singular wave solution composed of triangle functions for Eq.(5). When $c = 1$ and $c = 2$ respectively, Wazwaz [22] obtained eleven exact travelling wave solutions composed of triangle functions or hyperbolic functions for Eq.(5), while Liu et al [26] got a peakon solution which is composed of hyperbolic functions for Eq.(5). He et al [27] used integral bifurcation method to obtain some exact solutions for Eq.(5). Liu and Guo [28] investigated the periodic blow-up solutions and their limit forms for Eq.(5). When $c = 5/2$, Wazwaz [22] obtained nine exact travelling wave solutions composed of hyperbolic functions for Eq.(6), and Liu et al [26] got a peakon solution which is composed of hyperbolic functions for Eq.(6). Zhang et al [29] used the bifurcation theory of dynamical systems to show the existence of some travelling waves for Eq.(5). Wang and Tang [30] obtained two exact solutions for Eq.(5) when $c = \frac{1}{3}$ and $c = 3$ respectively, and gave two exact solutions for Eq.(6) when $c = \frac{1}{4}$ and $c = 4$ respectively. Yomba [31, 32] gave two methods, the sub-ODE method and the generalized auxiliary equation method, to look for the exact travelling wave solutions for Eq.(5) and Eq.(6). He et al [33] used the bifurcation method of dynamical systems to obtain some exact solutions for

Eq.(6). Liu and Pan [34] studied the coexistence of multifarious explicit nonlinear wave solutions for Eq.(5) and Eq.(6).

When the wave speed $c = \frac{2+b}{2}$, Wazwaz [22] obtained two solitary wave solutions for Eq.(7). When $c = \frac{1}{1+b}$, $\frac{2+b}{2}$, $1 + b$ respectively, Liu [35] investigated the solitary wave solution for Eq.(7).

In this paper, let c be constant wave speed and $c_5 = \frac{1}{2}\left(1+b-\sqrt{(1+b)(1+b-8k)}\right)$ and $c_6 = \frac{1}{2}\left(1+b+\sqrt{(1+b)(1+|!b-8k)}\right)$. we extend the previous results in the following three aspects. (i) When $-\infty < k < \frac{1+b}{8}$ and $c \in (c_5, c_6)$, there are three types of explicit nonlinear wave solutions, hyperbolic smooth solitary wave solution, hyperbolic peakon wave solution and hyperbolic blow-up solution. (ii) When $-\infty < k < \frac{1+b}{8}$ and $c = c_5$ or c_6, there are two types of explicit nonlinear wave solutions, fractional peakon wave solution and fractional blow-up solution. (iii) When $k = \frac{1+b}{8}$ and $c = \frac{b+1}{2}$, there are two types of explicit nonlinear wave solutions, fractional peakon wave solution and fractional blow-up solution. Not only is the existence of these solutions shown, but their concrete expressions are presented. Also the relationships among these solutions are revealed, and the correctness of these solutions is tested as well by using the software Mathematica.

We organize this paper as follows. In Section 2, we derive the traveling wave system of Eq.(1) and draw its bifurcation phase portraits which are the basis for constructing nonlinear wave solutions. In Section 3, the smooth solitary wave solution is derived. In Section 4, the hyperbolic peakon wave solution is built. In Section 5, the fractional peakon wave solutions are given. In Section 6, the relationships among the solitary wave solutions are revealed. In Section 7, the blow-up solutions are constructed. A brief conclusion is given in Section 8.

2. **Preliminaries.** For a given constant $c > 0$, substituting $u = \varphi(\xi)$ with $\xi = x - c\,t$ into Eq.(1), it follows that

$$-c\,\varphi' + 2k\varphi' + c\,\varphi''' + (1+b)\varphi^2\varphi' = b\varphi'\varphi'' + \varphi\varphi'''. \tag{8}$$

Integrating (8) once, we have

$$(\varphi - c)\varphi'' = g + (2k - c)\varphi + \frac{1+b}{3}\varphi^3 - \frac{b-1}{2}(\varphi')^2, \tag{9}$$

where g is the constant of integration .

Letting $y = \varphi'$, it yields the following planar system

$$\begin{cases} \dfrac{\mathrm{d}\varphi}{\mathrm{d}\xi} = y, \\ \dfrac{\mathrm{d}y}{\mathrm{d}\xi} = \dfrac{g + (2k-c)\varphi + \frac{1+b}{3}\varphi^3 - \frac{b-1}{2}y^2}{\varphi - c}. \end{cases} \tag{10}$$

By using the transformation $\mathrm{d}\tau = \frac{\mathrm{d}\xi}{\varphi - c}$, (10) can be written as the planar system

$$\begin{cases} \dfrac{\mathrm{d}\varphi}{\mathrm{d}\tau} = (\varphi - c)y, \\ \dfrac{\mathrm{d}y}{\mathrm{d}\tau} = g + (2k-c)\varphi + \dfrac{1+b}{3}\varphi^3 - \dfrac{b-1}{2}y^2. \end{cases} \tag{11}$$

Let

$$a_0 = \frac{6g + 12kc - 6c^2 + 2(1+b)c^3}{3(b-1)}, \tag{12}$$

$$a_1 = \frac{2(1+b)c^2 - 2c + 4k}{b}, \tag{13}$$

$$a_2 = 2c, \tag{14}$$

$$a_3 = \frac{2(1+b)}{3(2+b)}, \tag{15}$$

and

$$H(\varphi, y) = (\varphi - c)^{b-1}\left[a_0 + a_1(\varphi - c) + a_2(\varphi - c)^2 + a_3(\varphi - c)^3 - y^2\right]. \tag{16}$$

It is easy to check that

$$H(\varphi, y) = h \tag{17}$$

is the first integration for both systems (10) and (11). Therefore both systems (10) and (11) have the same topological phase portraits except the line $\varphi = c$. This implies that one can study the phase portraits of system (10) from that of system (11).

Now we are in a position to study the bifurcation phase portraits of system (11). Let

$$z = f(\varphi), \tag{18}$$

where

$$f(\varphi) = g + (2k - c)\varphi + \frac{1+b}{3}\varphi^3. \tag{19}$$

We have

$$f'(\varphi) = (1+b)\varphi^2 + 2k - c. \tag{20}$$

When $g = 0$, it follows

$$f\left(\pm\sqrt{\frac{c-2k}{1+b}}\right) = \mp\frac{2(c-2k)^{3/2}}{3\sqrt{1+b}}. \tag{21}$$

Let

$$\varphi^* = \sqrt{\frac{3(c-2k)}{1+b}}, \tag{22}$$

$$\varphi_0 = \sqrt{\frac{c-2k}{1+b}}, \tag{23}$$

and

$$g_0 = \frac{2(c-2k)^{3/2}}{3\sqrt{1+b}}. \tag{24}$$

We draw the graph of $z = f(\varphi)$ as Fig.1.

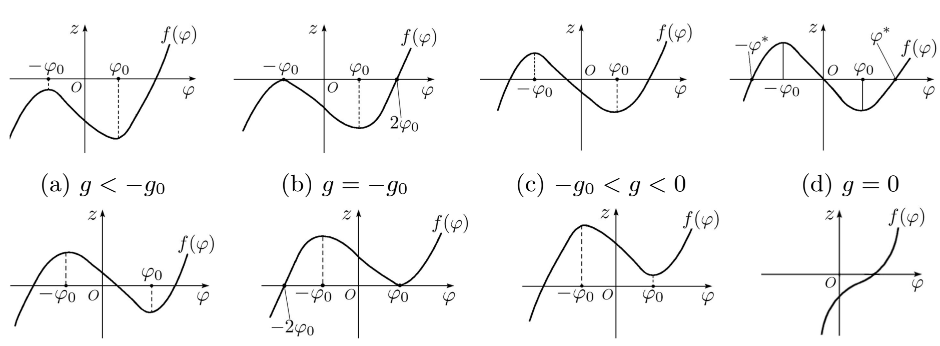


Fig.1 The graph of $z = f(\varphi)$, where (a)–(g) correspond to $c > 2k$.

From (11) and (19), it is seen that $(\widetilde{\varphi}, 0)$ is a singular point of system (11) if and only if $f(\widetilde{\varphi}) = 0$. At the singular point $(\widetilde{\varphi}, 0)$, it is easy to see that the linearized system of system (11) has the eigenvalues

$$\lambda_{\pm}(\widetilde{\varphi}, 0) = \pm\sqrt{(\widetilde{\varphi} - c)f'(\widetilde{\varphi})}. \tag{25}$$

From (17) and (25) we see that the singular point $(\widetilde{\varphi}, 0)$ is of the following properties:

(i) If $(\widetilde{\varphi} - c)f'(\widetilde{\varphi}) > 0$, then $(\widetilde{\varphi}, 0)$ is a saddle point of system (11).

(ii) If $(\widetilde{\varphi} - c)f'(\widetilde{\varphi}) = 0$, then $(\widetilde{\varphi}, 0)$ is a degenerate saddle point of system (11).

(iii) If $(\widetilde{\varphi} - c)f'(\widetilde{\varphi}) < 0$, then $(\widetilde{\varphi}, 0)$ is a center point of system (11).

Let

$$y_0 = \frac{2(1+b)c^3 - 6c^2 + 12kc + 6g}{3(b-1)}. \tag{26}$$

Similarly, it can be seen that if $y_0 > 0$, then $(c, -\sqrt{y_0})$ and $(c, \sqrt{y_0})$ are two saddle points of system (11). According to the analysis above and the values of $H(\varphi, y)$ at the singular points, we obtain four bifurcation curves

$$g_1(c) = \frac{2(c-2k)^{3/2}}{3\sqrt{1+b}}, \tag{27}$$

$$g_2(c) = \frac{-2b^2c^3 + 6bc\Delta_0^2 + 2(b-1)\sqrt{b(2+b)}\Delta_0^3}{3b^2(1+b)^2}, \tag{28}$$

$$g_3(c) = c^2 - 2kc - \frac{(1+b)c^3}{3}, \tag{29}$$

$$g_4(c) = \frac{-2(c-2k)^{3/2}}{3\sqrt{1+b}}, \tag{30}$$

where

$$\Delta_0 = \sqrt{(1+b)c - c^2 - 2(1+b)k}. \tag{31}$$

Solving $g_i(c) = g_j(c)$ ($i,\ j$ =1–4 and $i \neq j$) for c, we get the following seven values

$$c_1 = \frac{1 - \sqrt{1 - 8k(1+b)}}{2(1+b)}, \tag{32}$$

$$c_2 = \frac{1 + \sqrt{1 - 8k(1+b)}}{2(1+b)}, \tag{33}$$

$$c_3 = \frac{2\left(1 - \sqrt{1 - 2k - 2bk}\right)}{1+b}, \tag{34}$$

$$c_4 = \frac{2\left(1 + \sqrt{1 - 2k - 2bk}\right)}{1+b}, \tag{35}$$

$$c_5 = \frac{1}{2}\left(1 + b - \sqrt{(1+b)(1+b-8k)}\right), \tag{36}$$

$$c_6 = \frac{1}{2}\left(1 + b + \sqrt{(1+b)(1+b-8k)}\right), \tag{37}$$

and

$$c_7 = \frac{2(1+b) - 2\sqrt{(1+b)\left(1 + b - 2k(4 + 2b + b^2)\right)}}{4 + 2b + b^2}. \tag{38}$$

It is easy to see the following inequalities:

(i) When $k < 0$, it follows that

$$2k < c_5 < c_3 < c_7 < c_1 < c_2 < c_4 < c_6. \tag{39}$$

(ii) When $k = 0$, it follows that

$$0 = c_1 = c_3 = c_5 = c_7 < c_2 = \frac{1}{1+b} < c_4 = \frac{4}{1+b} < c_6 = 1 + b. \tag{40}$$

(iii) When $0 < k < \frac{1}{8(1+b)}$, it follows that

$$0 < 2k < c_5 < c_3 < c_1 < c_2 < c_4 < c_6. \tag{41}$$

(iv) When $k = \frac{1}{8(1+b)}$, it follows that

$$0 < 2k < c_5 < c_3 < c_2 = c_1 < c_4 < c_6. \tag{42}$$

(v) When $\frac{1}{8(1+b)} < k < \frac{1+b}{8}$, c_1, c_2 are complex and it follows that

$$0 < 2k < c_5 < c_3 < c_4 < c_6. \tag{43}$$

(vi) When $k = \frac{1+b}{8}$, c_1, c_2, c_3, c_4 are complex, and $c_5 = c_6 = (1 + b)/2$.
(vii) When $k > \frac{1+b}{8}$, c_i $(i = 1\text{–}6)$ are complex.
According to above discussions, we obtain the bifurcation phase portraits of system (11) as Fig.2–Fig.7.

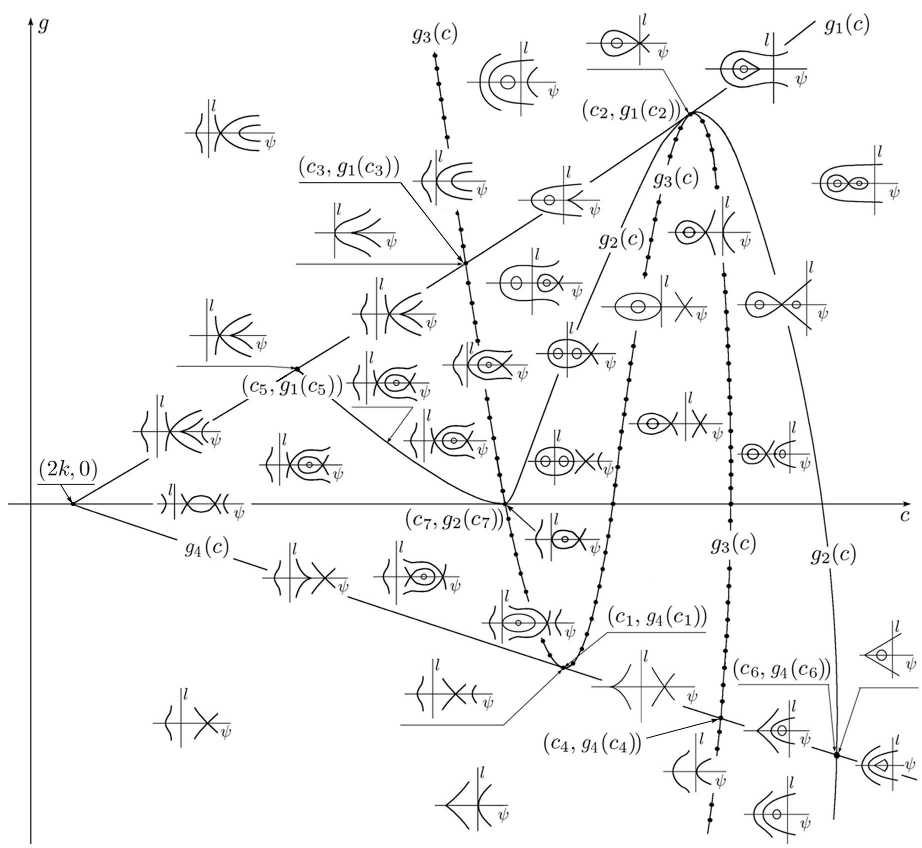


Fig.2 The bifurcation phase portraits of system (11) when $k < 0$.

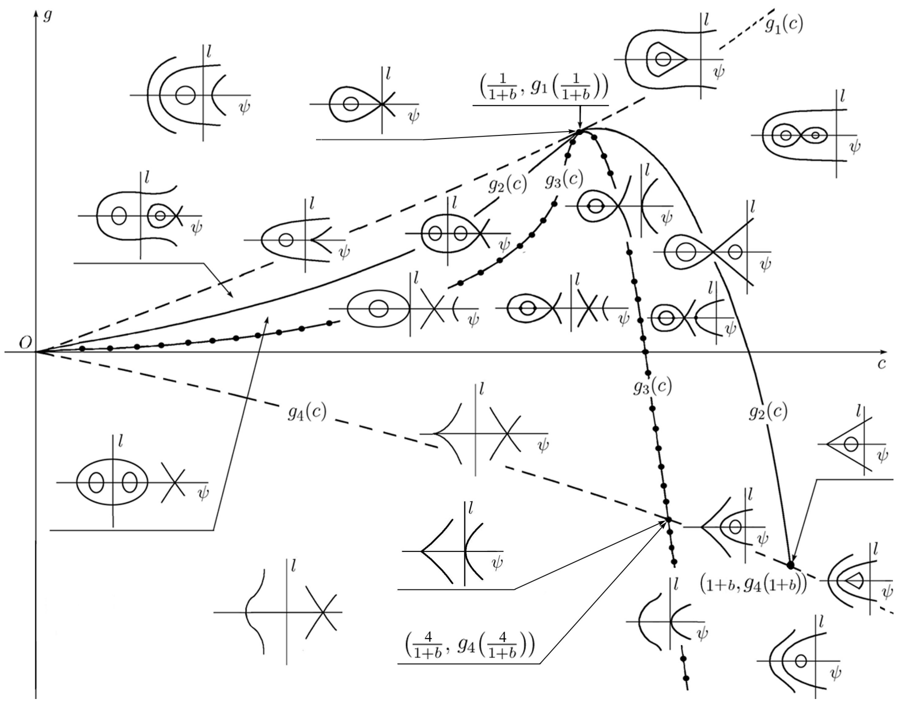


Fig.3 The bifurcation phase portraits of system (11) when $k = 0$.

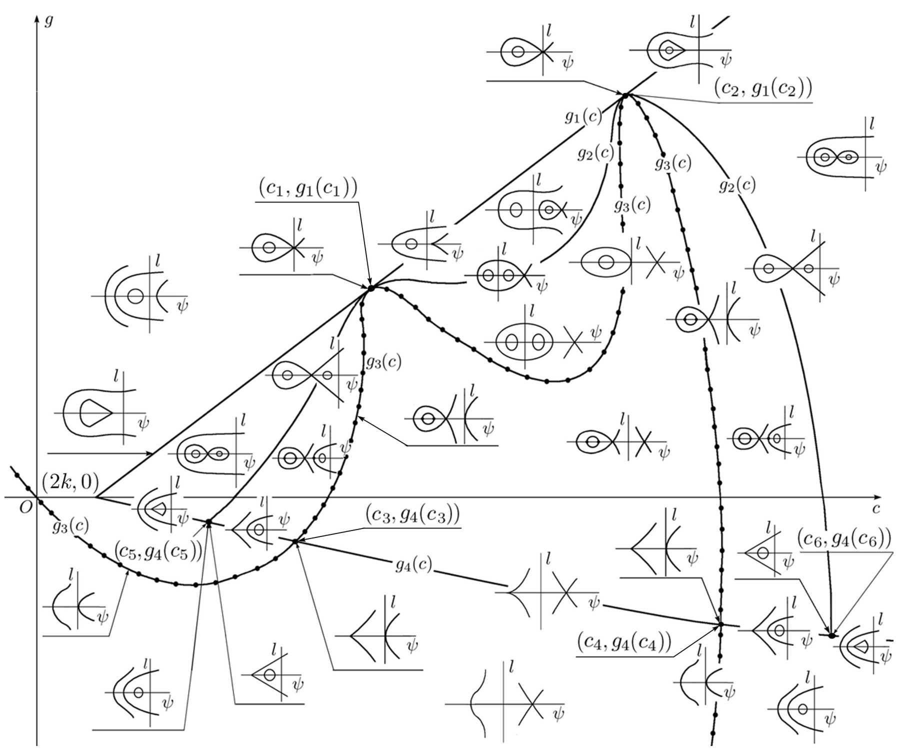


Fig.4 The bifurcation phase portraits of system (11) when $0 < k < \frac{1}{8(1+b)}$.

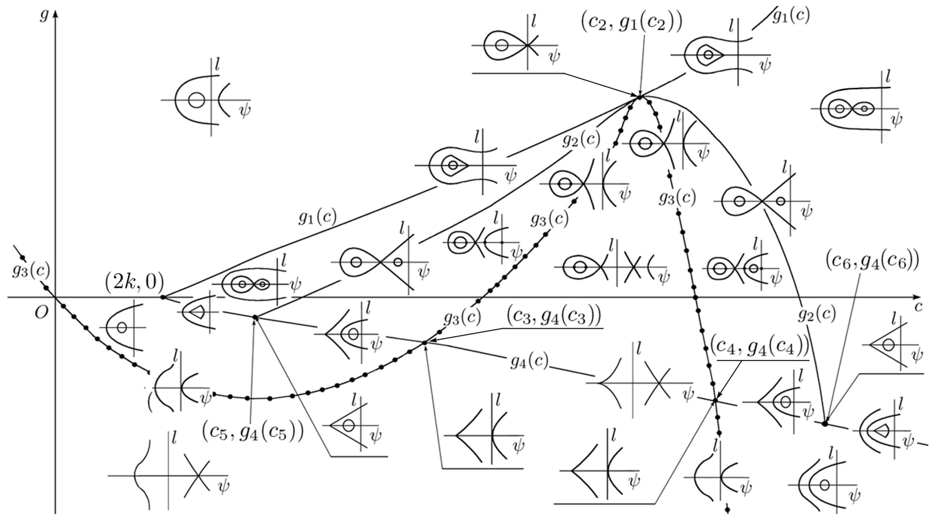


Fig.5 The bifurcation phase portraits of system (11) when $k = \frac{1}{8(1+b)}$.

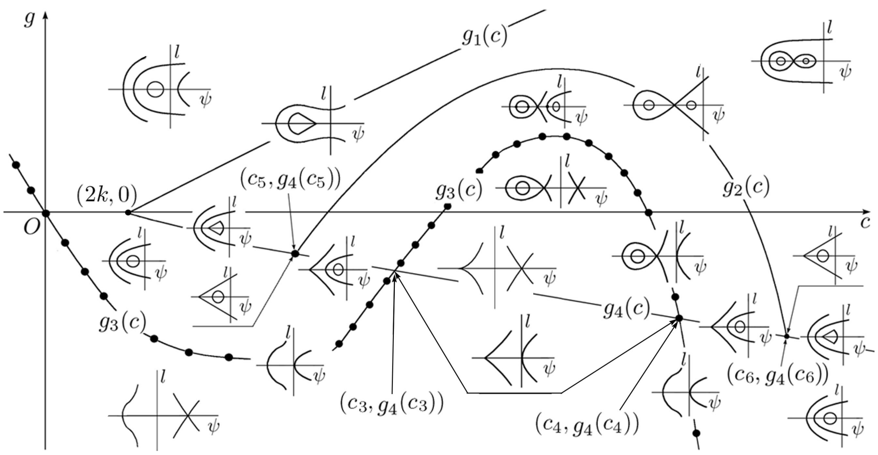
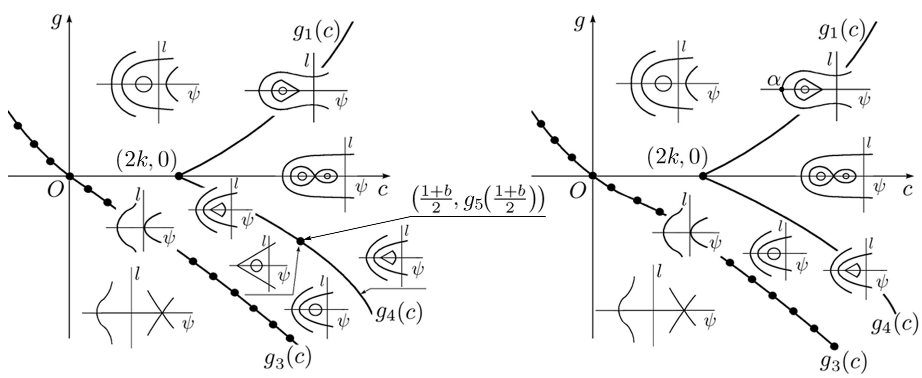


Fig.6 The bifurcation phase portraits of system (11) when $\frac{1}{8(1+b)} < k < \frac{1+b}{8}$.



$k = \frac{1+b}{8}$ $k > \frac{1+b}{8}$

Fig.7 The bifurcation phase portraits of system (11) when $k \geq \frac{1+b}{8}$.

3. **Smooth solitary wave solution.** For Eq.(1) with $k = 0$ and $c = \frac{2+b}{2}$, Wazwaz [22] got a smooth solitary wave solution

$$u_1^*(x,t) = -\frac{3(2+b)}{2(2+b)} \operatorname{sech}^2\left(x - \frac{2+b}{2}t\right). \tag{44}$$

In this section, we extend Wazwaz's result as follows.

Theorem 1. For Eq.(1), if the parameter k satisfies $-\infty < k < \frac{1+b}{8}$ and wave speed c satisfies $c_5 < c < c_6$ (see (36), (37) for c_5 and c_6), then there is a smooth solitary wave solution

$$u_1(x,t,c) = \frac{-bc + \Delta(c) - 3\Delta(c)\operatorname{sech}^2\eta(c)(x-ct)}{b(1+b)}, \tag{45}$$

where

$$\Delta(c) = \sqrt{b(2+b)\left[(1+b)c - c^2 - 2(1+b)k\right]}, \tag{46}$$

and

$$\eta(c) = \sqrt{\frac{\Delta(c)}{2b(2+b)}}. \tag{47}$$

The smooth solitary wave solution $u_1(x,t,c)$ is of the following properties:

(1°) When $c \to c_5 + 0$, $u_1(x,t,c)$ becomes trivial solution

$$u = \frac{\sqrt{(1+b)(1+b-8k)} - 1 - b}{2(1+b)}. \tag{48}$$

(2°) When $c \to c_6 - 0$, $u_1(x,t,c)$ becomes trivial solution

$$u = -\frac{\sqrt{(1+b)(1+b-8k)} + 1 + b}{2(1+b)}. \tag{49}$$

Proof. For given $c \in (c_5, c_6)$, from Fig.2–Fig.6 it is seen that on the bifurcation curve $g = g_2(c)$, there is a homoclinic orbit connecting with the saddle point $(\widetilde{\varphi}, 0)$ and passing through the point $(\widetilde{\varphi}_*, 0)$, where

$$\widetilde{\varphi} = \frac{-bc + \Delta(c)}{b(1+b)}, \tag{50}$$

and

$$\widetilde{\varphi}_* = -\frac{bc + 2\Delta(c)}{b(1+b)}. \tag{51}$$

On the φ–y plane, the homoclinic orbit is of the expression

$$y^2 = \frac{2(1+b)}{3(2+b)}(\widetilde{\varphi} - \varphi)^2(\varphi - \widetilde{\varphi}_*), \quad \text{where} \quad \widetilde{\varphi}_* \le \varphi < \widetilde{\varphi}. \tag{52}$$

Substituting (52) into the first equation of system (10) and integrating it, we have

$$\int_{\widetilde{\varphi}_*}^{\varphi} \frac{\mathrm{d}s}{(\widetilde{\varphi} - s)\sqrt{s - \widetilde{\varphi}_*}} = \sqrt{\frac{2(1+b)}{3(2+b)}}|\xi|, \quad \text{where} \quad \widetilde{\varphi}_* \le \varphi < \widetilde{\varphi}_0. \tag{53}$$

Completing the above integral and solving the equation for φ, it follows that

$$\varphi = \widetilde{\varphi}_* - (\widetilde{\varphi}_* - \widetilde{\varphi})\tanh^2(\eta(c)\xi) = \widetilde{\varphi} + (\widetilde{\varphi}_* - \widetilde{\varphi})\operatorname{sech}^2(\eta(c)\xi). \tag{54}$$

Noticing that the transformation $u = \varphi(\xi)$, from (54) we get the hyperbolic smooth solitary wave solution $u_1(x,t,c)$. From (46) it is seen that $\Delta(c) \to 0$ when $c \to c_5 + 0$ or $c \to c_6 - 0$. This implies that $u_1(x,t,c)$ is of the properties listed in Theorem 1. The proof is completed. $\square$

Remark 1. It is easy to see that when $k=0$, c_5 and c_6 become 0 and $1+b$ respectively. Clearly, it holds that $0<\frac{2+b}{2}<1+b$. On the other hand, when $k=0$ and $c=\frac{2+b}{2}$, $u_1(x,t,c)$ becomes $u_1^*(x,t)$. This implies that $u_1^*(x,t)$ is a special case of $u_1(x,t,c)$.

4. **Hyperbolic peakon wave solution.** In [22] Wazwaz conjectured that there is not peakon wave solution in Eq.(1). When $k=0$ and $c=\frac{2+b}{2}$, Liu [35] verified that there is peakon wave solution

$$u_2^*(x,t)=\frac{3(2+b)(2+3b+b^2)}{2(1+b)\left[\sqrt{3(2+b)}\cosh\frac{1}{2}\left(x-\frac{2+b}{2}t\right)+\sqrt{8+6b+b^2}\sinh\frac{1}{2}\left|x-\frac{2+b}{2}t\right|\right]^2}. \tag{55}$$

In this section, we extend the hyperbolic peakon wave solution as follows.

Theorem 2. In Eq.(1), assume that $-\infty<k<\frac{1+b}{8}$ and denote c_i (i=1–7) as (32)–(38). Consider the following five cases:

Case 1 $k<0$, $c_7<c<c_6$ and $c\neq c_2$.
Case 2 $k=0$, $0<c<1+b$ and $c\neq\frac{1}{1+b}$.
Case 3 $0<k<\frac{1}{8(1+b)}$, $c_5<c<c_6$, $c\neq c_1$ and $c\neq c_2$.
Case 4 $k=\frac{1}{8(1+b)}$, $c_5<c<c_6$ and $c\neq c_2$.
Case 5 $\frac{1}{8(1+b)}<k<\frac{1+b}{8}$ and $c_5<c<c_6$.

Under one of the five cases above, Eq.(1) has a hyperbolic peakon wave solution

$$u_2(x,t,c)=\frac{\Delta(c)-bc}{b(1+b)}+\frac{3\Delta(c)(2bc+b^2c-\Delta(c))}{b(1+b)\left(\sqrt{3\Delta(c)}\cosh\eta(c)(x-ct)+\Delta^*\sinh|\eta(c)(x-ct)|\right)^2}, \tag{56}$$

where $\Delta(c)$ and $\eta(c)$ are given in (46), (47) and

$$\Delta^*=\sqrt{2bc+b^2c+2\Delta(c)}\,. \tag{57}$$

Proof. Note that under one of the five cases and when $g=g_2(c)$, system (10) has three singular points $(\widetilde{\varphi},0)$, $(\varphi_+^\circ,0)$ and $(\varphi_-^\circ,0)$, where $\widetilde{\varphi}$ is given in (50) and

$$\varphi_\pm^\circ=\frac{1}{3}\left[-\widetilde{\varphi}\pm\sqrt{\frac{12(c-2k)}{1+b}-3\,\widetilde{\varphi}^{\,2}}\right]. \tag{58}$$

Now we introduce nine hypotheses below

($\mathbf{H}_1$) $k<0$ and $c\in(c_7,c_2)$.
($\mathbf{H}_2$) $k=0$ and $c\in\left(0,\frac{1}{1+b}\right)$.
($\mathbf{H}_3$) $0<k<\frac{1}{8(1+b)}$ and $c\in(c_1,c_2)$.
($\mathbf{H}_4$) $k<0$ and $c\in(c_2,c_6)$.
($\mathbf{H}_5$) $k=0$ and $c\in\left(\frac{1}{1+b},1+b\right)$.
($\mathbf{H}_6$) $0<k<\frac{1}{8(1+b)}$ and $c\in(c_5,c_1)$.
($\mathbf{H}_7$) $0<k<\frac{1}{8(1+b)}$ and $c\in(c_2,c_6)$.
($\mathbf{H}_8$) $k=\frac{1}{8(1+b)}$, $c\in(c_5,c_6)$ and $c\neq c_2$.
($\mathbf{H}_9$) $\frac{1}{8(1+b)}<k<\frac{1+b}{8}$ and $c\in(c_5,c_6)$.

Under one of (H_i) ($i=1,2,3$), it is easy to see the following inequalities

$$\varphi_-^\circ<c<\varphi_+^\circ<\widetilde{\varphi}. \tag{59}$$

Under one of (H_j) $(j = 4\text{–}9)$, it follows that

$$\varphi_-^\circ < \widetilde{\varphi} < \varphi_+^\circ < c. \tag{60}$$

Therefore, under one of the nine hypotheses, $(\varphi_+^\circ, 0)$ is a center point. The boundary of the closed orbits is a homoclinic orbit which passes $(c, 0)$ and connects with $(\widetilde{\varphi}, 0)$.

On the φ–y plane, the boundary is of expression

$$y = \pm\sqrt{\frac{2(1+b)}{3(2+b)}}(\widetilde{\varphi}-\varphi)\sqrt{\varphi-\widetilde{\varphi}_*}, \;\; c \le \varphi < \widetilde{\varphi}, \;\; \text{under one of } (\mathrm{H}_i)\ (i = 1-3), \tag{61}$$

or

$$y = \pm\sqrt{\frac{2(1+b)}{3(2+b)}}(\varphi-\widetilde{\varphi})\sqrt{\varphi-\widetilde{\varphi}_*}, \;\; \widetilde{\varphi} < \varphi \le c, \;\; \text{under one of } (\mathrm{H}_j)\ (j = 4-9). \tag{62}$$

Substituting the expressions of the boundary into $\frac{\mathrm{d}\varphi}{\mathrm{d}\xi} = y$ and integrating it along the boundary, it follows that

$$\int_\varphi^c \frac{\mathrm{d}s}{(s-\widetilde{\varphi})\sqrt{s-\widetilde{\varphi}_*}} = \sqrt{\frac{2(1+b)}{3(2+b)}}\,|\xi|. \tag{63}$$

Completing the above integral and solving the equation for φ, it follows that

$$\varphi = \frac{\Delta(c)-bc}{b(1+b)} + \frac{3\Delta(c)(2bc+b^2c-\Delta(c))}{b(1+b)\left(\sqrt{3\Delta(c)}\cosh\eta(c)\,\xi + \sqrt{2bc+b^2c+2\Delta(c)}\sinh|\eta(c)\,\xi|\right)^2}. \tag{64}$$

From the expression (64) and noting that $u = \varphi(\xi)$ with $\xi = x - ct$, we get the hyperbolic peakon wave solution $u_2(x, t, c)$ as (56). □

Remark 2. From (56) it is easy to see that $u_2(x, t, c)$ is of the following properties:

(1°) Suppose $k \le 0$. Then there is a bifurcation value c_2 in the interval (c_7, c_6). $u_2(x, t, c)$ represents an anti-peakon wave for $c \in (c_7, c_2)$, a peakon wave for $c \in (c_2, c_6)$. When $c \to c_2$, $u_2(x, t, c) \to c_2$.

(2°) Suppose $0 < k < \frac{1}{8(1+b)}$. Then there are two bifurcation values c_1 and c_2 in the interval (c_5, c_6). $u_2(x, t, c)$ represents a peakon wave for $c \in (c_5, c_1)$ or $c \in (c_2, c_6)$, an anti-peakon wave for $c \in (c_1, c_2)$. When $c \to c_1$ and c_2 respectively, $u_2(x, t, c) \to c_1$ and c_2.

(3°) Suppose $k = \frac{1}{8(1+b)}$. Then there is a bifurcation value c_2 in the interval (c_5, c_6). $u_2(x, t, c)$ represents a peakon wave for $c \in (c_5, c_6)$ and $c \ne c_2$. When $c \to c_2$, $u_2(x, t, c) \to c_2$.

(4°) Suppose $\frac{1}{8(1+b)} < k < \frac{1+b}{8}$. Then there is not bifurcation value in the interval (c_5, c_6). $u_2(x, t, c)$ represents a peakon wave for $c \in (c_5, c_6)$.

(5°) When $k = 0$ and $c = \frac{2+b}{2}$, $u_2(x, t, c)$ becomes $u_2^*(x, t)$.

For the limit functions of $u_2(x, t, c)$, when $c \to c_7{+}0$, $c_5{+}0$ and $c_6{-}0$ respectively, we will make discussions in Section 6.

5. **Fractional peakon wave solutions.** In [35] Liu obtained a fractional peakon wave solution

$$u_3^*(x, t) = \frac{6(2+b)}{\left(\sqrt{6} + \sqrt{1+b}\ |x-(1+b)t\,|\right)^2} - 1. \tag{65}$$

In this section, we extend the fractional peakon wave solutions as follows.

Theorem 3. (i) If $-\infty < k \leq 0$, then Eq.(1) has a fractional peakon wave solution

$$u_3(x,t) = -\frac{\delta+1+b}{2(1+b)} + \frac{6(2+b)(1+b+\delta)}{(1+b)\left(2\sqrt{3}+\sqrt{1+b+\delta}\,|x-c_6t|\right)^2}, \tag{66}$$

where

$$\delta = \sqrt{(1+b)\,(1+b-8k)}. \tag{67}$$

(ii) If $0 < k < \frac{1+b}{8}$, then Eq.(1) has two fractional peakon wave solutions $u_3(x,t)$ and

$$u_4(x,t) = \frac{\delta-1-b}{2(1+b)} + \frac{6(2+b)(1+b-\delta)}{(1+b)\left(2\sqrt{3}+\sqrt{1+b-\delta}\,|x-c_5t|\right)^2}. \tag{68}$$

(iii) If $k = \frac{1+b}{8}$, then Eq.(1) has a unique peakon wave solution

$$u_5(x,t) = -\frac{1}{2} + \frac{6(2+b)}{\left(2\sqrt{3}+\sqrt{1+b}\,\left|x-\frac{1+b}{2}t\right|\right)^2}. \tag{69}$$

Proof. Firstly, we derive the expression of $u_3(x,t)$. From the bifurcation phase portraits, it is seen that when $-\infty < k < \frac{1+b}{8}$, $c = c_6$ and $g = g_4(c)$, system (10) has two singular points $(p_1, 0)$ and $(p_2, 0)$, where

$$p_1 = -\frac{1+b+\delta}{2(1+b)}, \tag{70}$$

and

$$p_2 = \frac{1+b+\delta}{1+b}. \tag{71}$$

The singular point $(p_1, 0)$ is a degenerate saddle point, and the singular point $(p_2, 0)$ is a center point.

The boundary of the closed orbits surrounding the center point $(p_2, 0)$ is of expression

$$y^2 = \frac{2(1+b)}{3(2+b)}(\varphi-p_1)^3, \quad p_1 < \varphi < c_6. \tag{72}$$

Substituting the expression of the boundary into $\frac{d\varphi}{d\xi} = y$ and integrating it along the boundary, it follows that

$$\int_{\varphi}^{c_6} \frac{ds}{(s-p_1)^{\frac{3}{2}}} = \sqrt{\frac{2(1+b)}{3(2+b)}}\,|\xi|. \tag{73}$$

Completing the integral above and solving the equation for φ, we get

$$\varphi = -\frac{\delta+1+b}{2(1+b)} + \frac{6(2+b)(1+b+\delta)}{(1+b)\left(2\sqrt{3}+\sqrt{1+b+\delta}\,|\xi|\right)^2}. \tag{74}$$

Noticing that $u = \varphi(\xi)$, we obtain the fractional peakon wave solution $u_3(x,t)$ as (66).

Secondly, we derive the expression of $u_4(x,t)$. Similarly, from the bifurcation phase portraits, we see that when $0 < k < \frac{1+b}{8}$, $c = c_5$ and $g = g_4(c)$, system (10) has two singular points $(q_1, 0)$ and $(q_2, 0)$, where

$$q_1 = \frac{\delta-1-b}{2(1+b)}, \tag{75}$$

and

$$q_2 = \frac{1+b-\delta}{1+b}. \tag{76}$$

The singular point $(q_1, 0)$ is a degenerate saddle point, and $(q_2, 0)$ is a center point. The boundary of the closed orbits surrounding the center point $(q_2, 0)$ possesses expression

$$y^2 = \frac{2(1+b)}{3(2+b)}(\varphi - q_1)^3, \quad q_1 < \varphi < c_5. \tag{77}$$

Substituting the expression of the boundary into $\frac{\mathrm{d}\varphi}{\mathrm{d}\xi} = y$ and integrating it along the boundary, it follows that

$$\int_{\varphi}^{c_5} \frac{\mathrm{d}s}{(s-q_1)^{\frac{3}{2}}} = \sqrt{\frac{2(1+b)}{3(2+b)}}\,|\xi|. \tag{78}$$

Completing the integral above and solving the equation for φ, we have

$$\varphi = \frac{\delta - 1 - b}{2(1+b)} + \frac{6(2+b)(1+b-\delta)}{(1+b)\left(2\sqrt{3} + \sqrt{1+b-\delta}\ |\xi|\right)^2}. \tag{79}$$

From $u = \varphi(\xi)$, we obtain the fractional peakon wave solution $u_4(x,t)$ as (68).

Thirdly, we show the uniqueness of peakon wave solution when $k = \frac{1+b}{8}$. Note that when $k = \frac{1+b}{8}$, it follows that $c_5 = c_6 = \frac{1+b}{2}$. Therefore, there is not hyperbolic peakon wave solution. On the other hand, $u_3(x,t)$ and $u_4(x,t)$ become $u_5(x,t)$. That is, $u_5(x,t)$ is a unique peakon wave solution when $k = \frac{1+b}{8}$. □

Remark 3. Obviously, when $k = 0$, it follows that $c_6 = \delta = 1+b$, $1+b+\delta = 2(1+b)$. Thus $u_3(x,t)$ becomes $u_3^*(x,t)$ when $k = 0$. This implies that $u_3^*(x,t)$ is a special case of $u_3(x,t)$.

6. **The relationships among the solitary wave solutions.** In this section, we point out the relationships among $u_1(x,t,c)$, $u_2(x,t,c)$, $u_3(x,t)$ and $u_4(x,t)$ as follows.

Theorem 4. Among $u_1(x,t,c)$, $u_2(x,t,c)$, $u_3(x,t)$ and $u_4(x,t)$ there are the following relationships:

(1°) When $k < 0$ and $c \to c_7 + 0$, $u_2(x,t,c)$ becomes the smooth solitary wave solution $u_1(x,t,c_7)$ (about $u_1(x,t,c)$ and $u_2(x,t,c)$ see (45) and (56)).

(2°) When $-\infty < k < \frac{1+b}{8}$ and $c \to c_6 - 0$, the hyperbolic peakon wave solution $u_2(x,t,c)$ becomes the fractional peakon wave solution $u_3(x,t)$ (about $u_3(x,t)$ see (66)).

(3°) When $0 < k < \frac{1+b}{8}$ and $c \to c_5 + 0$, the hyperbolic peakon wave solution $u_2(x,t,c)$ becomes the fractional peakon wave solution $u_4(x,t)$ (about $u_4(x,t)$ see (68)).

Proof. Firstly, When $k < 0$ and $c \to c_7 + 0$, it follows that $\Delta^* \to 0$ and $2bc + b^2c - \Delta(c) \to -3\Delta(c_7)$. Hence,

$$\begin{aligned}
&\lim_{c\to c_7+0} u_2(x,t,c)\\
=&\lim_{c\to c_7+0} \frac{\Delta(c) - bc}{b(1+b)} + \frac{3\Delta(c)(2bc + b^2c - \Delta(c))}{b(1+b)\left(\sqrt{3\Delta(c)}\cosh\eta(c)\,(x-ct) + \Delta^*\sinh|\eta(c)(x-ct)|\right)^2}\\
=&\frac{\Delta(c_7) - bc_7}{b(1+b)} - \frac{9\Delta^2(c_7)}{b(1+b)3\Delta(c_7)\cosh^2\eta(c_7)\,(x-c_7t)}\\
=&\frac{\Delta(c_7) - bc_7 - 3\Delta(c_7)\,\mathrm{sech}^2\,\eta(c_7)\,(x-c_7t)}{b(1+b)} = u_1(x,t,c_7)
\end{aligned}$$

Secondly, when $-\infty < k < \frac{1+b}{8}$ and $c \to c_6 - 0$, it follows that $\Delta(c) \to 0$, $\eta(c) \to 0$, $2bc + b^2c - \Delta(c) \to \frac{1}{2}b(2+b)(1+b+\delta)$ and $\Delta^* \to \frac{1}{\sqrt{2}}\sqrt{b(2+b)(1+b+\delta)}$. If let

$$w(x,t,c) = \frac{3\Delta(c)(2bc + b^2c - \Delta(c))}{b(1+b)\left(\sqrt{3\Delta(c)}\cosh\eta(c)\,(x-ct) + \Delta^*\sinh|\eta(c)\,(x-ct)|\right)^2}, \quad (80)$$

then

$$\lim_{c\to c_6-0} w(x,t,c) = \lim_{c\to c_6-0} \frac{3\Delta(c)(2bc + b^2c - \Delta(c))}{b(1+b)\left(\sqrt{3\Delta(c)} + \Delta^*\eta(c)|x-ct| + o(\sqrt{\Delta(c)})\right)^2}$$

$$= \frac{6(2+b)(1+b+\delta)}{b(1+b)\left(2\sqrt{3} + \sqrt{1+b+\delta}\,|x-c_6t|\right)^2}. \quad (81)$$

Further

$$\lim_{c\to c_6-0} u_2(x,t,c) = \lim_{c\to c_6-0}\left(\frac{\Delta(c)-bc}{b(1+b)} + w(x,t,c)\right)$$

$$= -\frac{1+b+\delta}{2(1+b)} + \frac{6(2+b)(1+b+\delta)}{(1+b)\left(2\sqrt{3} + \sqrt{1+b+\delta}\,|x-c_6t|\right)^2} = u_3(x,t). \quad (82)$$

Thirdly, when $0 < k < \frac{1+b}{8}$ and $c \to c_5 + 0$, it follows that $\Delta(c) \to 0$, $\eta(c) \to 0$, $2bc + b^2c - \Delta(c) \to \frac{1}{2}b(2+b)(1+b-\delta)$ and $\Delta^* \to \frac{1}{\sqrt{2}}\sqrt{b(2+b)(1+b-\delta)}$. Further, it follows that

$$\lim_{c\to c_5+0} w(x,t,c) = \lim_{c\to c_5+0} \frac{3\Delta(c)(2bc + b^2c - \Delta(c))}{b(1+b)\left(\sqrt{3\Delta(c)} + \Delta^*\eta(c)|x-ct| + o(\sqrt{\Delta(c)})\right)^2}$$

$$= \frac{6(2+b)(1+b-\delta)}{b(1+b)\left(2\sqrt{3} + \sqrt{1+b-\delta}\,|x-c_5t|\right)^2}. \quad (83)$$

Hence,

$$\lim_{c\to c_5+0} u_2(x,t,c) = \lim_{c\to c_5+0}\left(\frac{\Delta(c)-bc}{b(1+b)} + w(x,t,c)\right)$$

$$= \frac{\delta-1-b}{2(1+b)} + \frac{6(2+b)(1+b-\delta)}{(1+b)\left(2\sqrt{3} + \sqrt{1+b-\delta}\,|x-c_5t|\right)^2} = u_4(x,t). \quad (84)$$

This completes the proof of Theorem 4. □

7. **Blow-up solutions.**

When $k = 0$ and $c = \frac{2+b}{2}$, Wazwaz [22] showed that Eq.(1) has a blow-up solution

$$u_4^*(x,t) = \frac{3(2+b)}{2(b+1)}\operatorname{csch}^2\frac{1}{2}\left(x - \frac{2+b}{2}t\right). \quad (85)$$

When $k = 0$ and $c = \frac{1}{1+b}$, $1+b$ respectively, Liu [35] confirmed that Eq.(1) has two blow-up solutions

$$u_5^*(x,t) = \frac{1}{(1+b)^2}\left[1 + b + 3(2+b)\operatorname{csch}^2\frac{1}{\sqrt{2(1+b)}}\left(x - \frac{t}{1+b}\right)\right], \quad (86)$$

and

$$u_6^*(x,t) = \frac{6(2+b)}{(1+b)[x-(1+b)t]^2} - 1. \quad (87)$$

In this section, we extend the blow-up solutions as follows.

Theorem 5. Consider Eq.(1).

(i) If $-\infty < k \leq 0$, then there are two blow-up solutions which are hyperbolic blow-up solution

$$u_6(x,t,c) = \frac{-bc + \Delta(c) + 3\Delta(c)\operatorname{csch}^2(\eta(c)(x-ct))}{b(1+b)} \quad \text{for} \quad c \in (c_5, c_6), \tag{88}$$

and fractional blow-up solution

$$u_7(x,t) = -\frac{\delta+1+b}{2(1+b)} + \frac{6(2+b)}{(1+b)(x-c_6 t)^2}. \tag{89}$$

(ii) If $0 < k < \frac{1+b}{8}$, then there are three blow-up solutions which are hyperbolic blow-up solution $u_6(x,t,c)$ ($c \in (c_5, c_6)$), fractional blow-up solutions $u_7(x,t)$ and

$$u_8(x,t) = \frac{\delta-1-b}{2(1+b)} + \frac{6(2+b)}{(1+b)(x-c_5 t)^2}. \tag{90}$$

(iii) If $k = \frac{1+b}{8}$, then there is unique blow-up solution

$$u_9(x,t) = -\frac{1}{2} + \frac{6(2+b)}{(1+b)\left(x - \frac{1+b}{2}t\right)^2}. \tag{91}$$

About c_5, c_6, $\Delta(c)$, $\eta(c)$ and δ, see (36), (37), (46), (47) and (67).

Among these solutions, there are the following relationships:

(1°) When $k=0$ and $c = \frac{2+b}{2}$, $u_6(x,t,c)$ becomes $u_4^*(x,t)$.
(2°) When $k=0$ and $c = \frac{1}{1+b}$, $u_6(x,t,c)$ becomes $u_5^*(x,t)$.
(3°) When $k=0$ and $c = 1+b$, $u_7(x,t)$ becomes $u_6^*(x,t)$.
(4°) When $-\infty < k < \frac{1+b}{8}$ and $c \to c_6 - 0$, $u_5(x,t,c)$ becomes $u_7(x,t)$.
(5°) When $0 < k < \frac{1+b}{8}$ and $c \to c_5 + 0$, $u_5(x,t,c)$ becomes $u_8(x,t)$.

Proof. Firstly, we derive the hyperbolic blow-up solution $u_6(x,t,c)$. From the bifurcation phase portraits it is seen that when $-\infty < k < \frac{1+b}{8}$ and $c_5 < c < c_6$, on $g = g_2(c)$ there are two heteroclicic orbits connecting with $(\widetilde{\varphi}, 0)$ and going to infinity. On φ–y plane their expressions are

$$y = \pm(\varphi - \widetilde{\varphi})\sqrt{\frac{2(1+b)(\varphi - \widetilde{\varphi}_*)}{3(2+b)}}, \quad \text{for} \quad \widetilde{\varphi} < \varphi. \tag{92}$$

Substituting the expressions into $\frac{d\varphi}{d\xi} = y$ and integrating it along these two heteroclinic orbits respectively, it follows that

$$\int_{\varphi}^{\infty} \frac{d\varphi}{(s-\widetilde{\varphi})\sqrt{s-\widetilde{\varphi}_*}} = \sqrt{\frac{2(1+b)}{3(2+b)}}\,|\xi|. \tag{93}$$

Completing the integral and solving the equation for φ, it follows that

$$\varphi = \frac{\Delta(c) - bc + 3\Delta(c)\operatorname{csch}^2 \eta(c)\xi}{b(1+b)}. \tag{94}$$

From (94) and noting that $u = \varphi(\xi)$, we get the blow-up solution $u_6(x,t,c)$ as (88).

Secondly, we derive the fractional blow-up solution $u_7(x,t)$. Note that when $-\infty < k < \frac{1+b}{8}$, $c = c_6$ and $g = g_4(c_6)$, there are two heteroclinic orbits connecting with $(p_1, 0)$ and going to infinity. On φ–y plane, their expressions are

$$y = \pm\sqrt{\frac{2(1+b)}{3(2+b)}}\,(\varphi - p_1)^{3/2}, \quad \text{for} \quad \varphi > p_1. \tag{95}$$

Substituting the expressions into $\frac{\mathrm{d}\varphi}{\mathrm{d}\xi} = y$ and integrating it along these two heteroclinic orbits, we have

$$\int_{\varphi}^{\infty} (s - p_1)^{-3/2}\,\mathrm{d}s = \sqrt{\frac{2(1+b)}{3(2+b)}}\,|\xi|. \tag{96}$$

Finishing the integral and solving the equation for φ, we get

$$\varphi = -\frac{1+b+\delta}{2(1+b)} + \frac{6(2+b)}{(1+b)\,\xi^2}. \tag{97}$$

From (97) and noting that $u = \varphi(\xi)$, we obtain the fractional blow-up solution $u_7(x,t)$ as (89).

Thirdly, we derive the fractional blow-up solution $u_8(x,t)$. Note that when $0 < k < \frac{1+b}{8}$, $c = c_5$ and $g = g_4(c_5)$, there are two heteroclinic orbits connecting with $(q_1, 0)$ and going to infinity. On φ–y plane, these two orbits possess expressions

$$y = \pm\sqrt{\frac{2(1+b)}{3(2+b)}}\,(\varphi - q_1)^{3/2}, \quad \text{for} \quad \varphi > q_1. \tag{98}$$

Substituting the expressions into $\frac{\mathrm{d}\varphi}{\mathrm{d}\xi} = y$ and integrating it along these two orbits, it follows that

$$\int_{\varphi}^{\infty} (s - q_1)^{-3/2}\,\mathrm{d}s = \sqrt{\frac{2(1+b)}{3(2+b)}}\,|\xi|. \tag{99}$$

Completing the integral and solving the equation for φ, we get

$$\varphi = \frac{\delta - 1 - b}{2(1+b)} + \frac{6(2+b)}{(1+b)\,\xi^2}. \tag{100}$$

Via (100) and noting that $u = \varphi(\xi)$, we obtain the fractional blow-up solution $u_8(x,t)$ as (90).

Now, let us to show the uniqueness of the blow-up solution when $k = \frac{1+b}{8}$. Note that when $k = \frac{1+b}{8}$, it follows that $c_5 = c_6 = \frac{1+b}{2}$. Consequently, there is not hyperbolic blow-up solution. While $u_7(x,t)$ and $u_8(x,t)$ become $u_9(x,t)$. That is, $u_9(x,t)$ is unique blow-up solution when $k = \frac{1+b}{8}$.

Finally, let us check the relationships among these blow-up solutions.

(1°) When $k = 0$ and $c = \frac{2+b}{2}$, it follows $\Delta(c) = \frac{b(2+b)}{2}$, $\Delta(c) - bc = 0$ and $\eta(c) = \frac{1}{2}$. Therefore $u_6(x,t,c)$ becomes $u_4^*(x,t)$.

(2°) When $k = 0$ and $c = \frac{1}{1+b}$, it follows that $\Delta(c) = \frac{b(2+b)}{1+b}$, $\eta(c) = \sqrt{\frac{1}{2(1+b)}}$, $\frac{\Delta(c)-bc}{2(1+b)} = b$ and $\frac{3\Delta(c)}{b(1+b)} = \frac{3(2+b)}{(1+b)^2}$. Hence, $u_6(x,t,c)$ becomes $u_5^*(x,t)$.

(3°) When $k = 0$, it follow $c_6 = 1+b$ and $\delta = 1+b$. Consequently $u_7(x,t)$ becomes $u_6^*(x,t)$.

(4°) When $-\infty < k < \frac{1+b}{8}$ and $c \to c_6 - 0$, it follows that $\Delta(c) \to 0$ and $\eta(c) \to 0$. Further we have

$$\lim_{c \to c_6 - 0} \Delta(c)\,\mathrm{csch}^2 \eta(c)(x - ct) = \lim_{c \to c_6 - 0} \frac{\Delta(c)}{\sinh^2 \eta(c)(x - ct)}$$

$$= \lim_{c\to c_6-0} \frac{\Delta(c)}{\left(\sqrt{\frac{\Delta(c)}{2b(2+b)}}(x-ct)+o(\sqrt{\Delta(c)})\right)^2}$$

$$= \frac{2b(2+b)}{(x-c_6t)^2}. \tag{101}$$

Thus it follows that $\lim\limits_{c\to c_6-0} u_6(x,t,c) = u_7(x,t)$.

(5°) When $0 < k < \frac{1+b}{8}$ and $c \to c_5 + 0$, it follows that $\Delta(c) \to 0$ and $\eta(c) \to 0$. Therefore we have

$$\lim_{c\to c_5+0} \Delta(c)\operatorname{csch}^2\eta(c)(x-ct) = \lim_{c\to c_5+0} \frac{\Delta(c)}{\sinh^2(\eta(c)(x-ct))}$$

$$= \lim_{c\to c_5+0} \frac{\Delta(c)}{\left(\sqrt{\frac{\Delta(c)}{2b(2+b)}}(x-ct)+o(\sqrt{\Delta(c)})\right)^2}$$

$$= \frac{2b(2+b)}{(x-c_5t)^2}. \tag{102}$$

Further it follows that $\lim\limits_{c\to c_5+0} u_6(x,t,c) = u_8(x,t)$. This completes the proof of Theorem 4. □

8. **Conclusion.**

In this paper, we have investigated the explicit nonlinear wave solutions of Eq.(1). Five types of explicit nonlinear wave solutions have been presented. These solutions are hyperbolic smooth solitary wave solution $u_1(x,t,c)$ (see (45)), hyperbolic peakon wave solution $u_2(x,t,c)$ (see (56)), fractional peakon wave solutions $u_3(x,t)$, $u_4(x,t)$ and $u_5(x,t)$ (see (66), (68) and (69)), hyperbolic blow-up solution $u_6(x,t,c)$ (see (88)), fractional blow-up solutions $u_7(x,t)$, $u_8(x,t)$ and $u_9(x,t)$, (see (89)–(91)). The relationships among these solutions also have been revealed (see Theorem 4). Our results imply that for given $b > 1$, when $k > \frac{1+b}{8}$, Eq.(1) has no peakon wave. We have employed the software Mathematica to check the correctness of these solutions. For example, the commands for $u_1(x,t,c)$ are as follows:

$\Delta = \sqrt{\mathrm{b\ (2+b)\ ((1+b)\ c - c^2 - 2\ (1+b)\ k)}}$

$\eta = \sqrt{\frac{\Delta}{\mathrm{2\ b\ (2+b)}}}$

$\xi = \mathrm{x - c\ t}$

$\mathrm{u1} = \frac{\mathrm{-b\ c+\Delta-3\ \Delta\ Sech[\eta\ \xi]^2}}{\mathrm{b\ (1+b)}}$

$\mathrm{u = u1}$

Simplify[D[u, t]+2 k D[u, x]−D[u, x, x, t]+(1+b) u²D[u, x]−
b D[u, x] D[u, x, x]−u D[u, x, x, x]]

For checking other solutions, the commands are similar. That is, only need to change u.

Comparing our work with that of predecessors, we have extended some results obtained in [20]–[35].

REFERENCES

[1] A. Degasperis, D. D. Holm and A. N. W. Hone, *A new integrable equation with peakon solutions*, Theoret. and Math. Phys., **133** (2002), 1463–1474.

[2] A. Degasperis, D. D. Holm and A. N. W. Hone, *Integrable and non-integrable equations with peakons*, Nonlinear physics: Theory and experiment, **II** (2002), 37–43. World Sci Publishing, River Edge, NJ, 2003.

[3] R. Camassa and D. D. Holm, *An integrable shallow water equation with peaked solitons*, Phys. Rev. Lett., **71** (1993), 1661–1664.

[4] F. Cooper and H. Shepard, *Solitons in the Camassa-Holm shallow water equation*, Phys. Lett. A, **194** (1994), 246–250.

[5] A. Constantin, *Soliton interactions for the Camassa-Holm equation*, Exposition. Math., **15** (1997), 251–264.

[6] A. Constantin and W. A. Strauss, *Stability of Peakons*, Comm. Pure Appl. Math., **53** (2000), 603–610.

[7] J. P. Boyd, *Peakons and coshoidal waves: travelling wave solutions of the Camassa-Holm equation*, Appl. Math. Comput., **81** (1997), 173–187.

[8] J. Lenells, *The scattering approach for the Camassa-Holm equation*, J. Non. Math. Phys., **9** (2002), 389–393.

[9] R. S. Johnson, *Camassa-Holm, Korteweg-de Vries and rlated models for water waves*, J. Fluid Mech., **455** (2002), 63–82.

[10] E. G. Reyes, *Geometric integrability of the Camassa-Holm equation*, Lett. Math. Phys., **59** (2002), 117–131.

[11] Z. R. Liu, R. Q. Wand and Z. J. Jing, *Peaked wave solutions of Camassa-Holm equation*, Chaos Solitons Fract., **19** (2004), 77–92.

[12] Z. R. Liu, A. M. Kayed and C. Chen, *Periodic waves and their limits for the Camassa-Holm equation*, Int. J. Bifurcat. Chaos, **16** (2006), 2261–2274.

[13] A. Degasperis and M. Procesi, *Asymptotic integrability*, in "Symmetry and Perturbation Theory" (eds. A.Degasperis and G.Gaeta), World Sci Publishing, (1999), 23–37.

[14] H. Lundmark and J. Szmigielski, *Multi-peakon solutions of the Degasperis-Procesi equation*, Inverse Probl., **19** (2003), 1241–1245.

[15] H. Lundmark and J. Szmigielski, *Degasperis-Procesi peakons and the discrete cubic string*, Int. Math. Res. Pap., **2** (2005), 53–116.

[16] C. Chen and M. Y. Tang, *A new type of bounded waves for Degasperis-Procesi equations*, Chaos Soliton Fract., **27** (2006), 698–704.

[17] P. Guha, *Euler-Poincare formalism of (two component) Degasperis-Procesi and Holm-Staley type systems*, J. Non. Math. Phys., **14** (2007), 390–421.

[18] D. D. Holm and M. F. Staley, *Nonlinear balance and exchange of stability in dynamics of solitons, peakons ramps/cliffs and leftons in a $1+1$ nolinear evolutionary PDEs*, Phys. Lett. A., **308** (2003), 437–444.

[19] B. L. Guo and Z. R. Liu, *Periodic cusp wave solutions and single-solitons for the b-equation*, Chaos Soliton Fract., **23** (2005), 1451–1463.

[20] Z. R. Liu and T. F. Qian, *Peakons and their bifurcation in a generalized Camassa-Holm equation*, Int. J. Bifurcat. Chaos, **11** (2001), 781–792.

[21] A. M. Wazwaz, *Solitary wave solutions for modified forms of Degasperis-Procesi and Camassa-Holm equations*, Phys. Lett. A, **352** (2006), 500–504.

[22] A. M. Wazwaz, *New solitary wave solutions to the modified forms of Degasperis-Procesi and Camassa-Holm equations*, Appl. Math. Comput., **186** (2007), 130–141.

[23] L. X. Tian and X. Y. Song, *New peaked solitary wave solutions of the generalized Camassa-Holm equation*, Chaos Soliton Fract., **21** (2004), 621–637.

[24] J. W. Shen and W. Xu, *Bifurcations of smooth and non-smooth travelling wave solutions in the generalized Camassa-Holm equation*, Chaos Soliton Fract., **26** (2005), 1149–1162.

[25] S. A. Khuri, *New ansatz for obtaining wave solutions of the generalized Camassa-Holm equation*, Chaos Soliton Fract., **25** (2005), 705–710.

[26] Z. R. Liu and Z. Y. Ouyang, *A note on solitary waves for modified forms of Camassa-Holm and Degasperis-Procesi equations*, Phys. Lett. A, **366** (2007), 377–381.

[27] B. He, W. G. Rui and C. Chen, *Exact travelling wave solutions for a generalized Camassa-Holm equation using the integral bifurcation method*, Appl. Math. Comput., **206** (2008), 141–149.

[28] Z. R. Liu and B. L. Guo, *Periodic blow-up solutions and their limit forms for the generalized Camassa-Holm equation*, Prog. Nat. Sci., **18** (2008), 259–266.

[29] L. J. Zhang, Q. C. Li and X. W. Huo, *Bifurcations of smooth and nonsmooth travelling wave solutions in a generalized Degasperis-Procesi equation*, J. Comput. Appl. Math., **205** (2007), 174–185.

[30] Q. D. Wang and M. Y. Tang, *New exact solutions for two nonlinear equations*, Phys. Lett. A, **372** (2008), 2995–3000.

[31] E. Yomba, *The sub-ODE method for finding exact travelling wave solutions of generalized nonlinear Camassa-Holm, and generalized nonlinear Schrödinger equations*, Phys. Lett. A, **372** (2008), 215–222.

[32] E. Yomba, *A generalized auxiliary equation method and its application to nonlinear Klein-Gordon and generalized nonlinear Camassa-Holm equations*, Phys. Lett. A, **372** (2008), 1048–1060.

[33] B. He, W. G. Rui and S. L. Li, *Bounded travelling wave solutions for a modified form of generalized Degasperis-Procesi equation*, Appl. Math. Comput., **206** (2008), 113–123.

[34] Z. R. Liu and J. Pan, *Coexistence of multifarious explicit nonlinear wave solutions for modified forms of Camassa-Holm and Degaperis-Procesi equations*, Int. J. Bifurcat. Chaos, **19** (2009), 2267–2282.

[35] R. Liu, *Several new types of solitary wave solutions for the generalized Camassa-Holm-Degasperis-Procesi equation*, Commun. Pur. Appl. Anal., **9** (2010), 77–90.

Received January 2011; revised July 2012.

E-mail address: scliurui@scut.edu.cn