International Journal of Bifurcation and Chaos, Vol. 23, No. 3 (2013) 1330007 (19 pages)
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DOI: 10.1142/S0218127413300073

SOME COMMON EXPRESSIONS AND NEW BIFURCATION PHENOMENA FOR NONLINEAR WAVES IN A GENERALIZED mKdV EQUATION*

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Received January 15, 2012

Using the bifurcation method of dynamical systems, we study nonlinear waves in the generalized mKdV equation $u_t + a(1 + bu^2)u^2u_x + u_{xxx} = 0$.

(i) We obtain four types of new expressions. The first type is composed of four common expressions of the symmetric solitary waves, the kink waves and the blow-up waves. The second type includes four common expressions of the anti-symmetric solitary waves, the kink waves and the blow-up waves. The third type is made of two trigonometric expressions of periodic-blow-up waves. The fourth type is composed of two fractional expressions of 1-blow-up waves.
(ii) We point out that there are two sets of kink waves which are called tall-kink waves and low-kink waves, respectively.
(iii) We reveal two kinds of new bifurcation phenomena. The first phenomenon is that the low-kink waves can be bifurcated from four types of nonlinear waves, the symmetric solitary waves, blow-up waves, tall-kink waves and anti-symmetric solitary waves. The second phenomenon is that the 1-blow-up waves can be bifurcated from the periodic-blow-up waves.

We also show that the common expressions include many results given by pioneers.

Keywords: Generalized mKdV equation; common expressions; solitary waves; kink waves; blow-up waves; new bifurcation phenomena.

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*Supported by the National Natural Science Foundation (No. 11241002, 11171115), Natural Science Foundation of Guangdong Province (No. S2012040007959) and the Fundamental Research Funds for the Central Universities (No. 2012ZM0057).
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1. Introduction

Many authors have been interested in the generalized mKdV equation

$$u_t + a(1 + bu^2)u^2 u_x + u_{xxx} = 0, \tag{1}$$

where a and b are real parameters and $ab \neq 0$. For example, Dey [1986, 1988] studied the exact Hamiltonian density and the conservation laws, and obtained four kink wave solutions

$$u_{\mathrm{a}}^{\pm} = \pm\sqrt{\frac{6c}{a}}\sqrt{1 + \tanh(\sqrt{c}\,\xi)}, \tag{2}$$

$$u_{\mathrm{b}}^{\pm} = \pm\sqrt{\frac{6c}{a}}\sqrt{1 - \tanh(\sqrt{c}\,\xi)}, \tag{3}$$

where $a > 0$, $c > 0$, $b = -\frac{5a}{48c}$ and

$$\xi = x - ct. \tag{4}$$

For $a > 0$, $c > 0$, $-\frac{5a}{48c} < b < 0$, Liu and Li [2002] gave two kink wave solutions

$$u_{\mathrm{c}}^{\pm} = \pm\sqrt{\frac{(5a + \Delta)(5a - 2\Delta)}{6ab[3\Delta + (2\Delta - 5a)\sinh^2(\eta_1\xi)]}} \times \sinh(\eta_1\xi) \tag{5}$$

where

$$\Delta = \sqrt{5a(5a + 36bc)} \tag{6}$$

and

$$\eta_1 = \sqrt{-\frac{\Delta(\Delta + 5a)}{180ab}}. \tag{7}$$

For $c > 0$, $ab > 0$ or $-\frac{5a}{48c} < b < 0$, Tang *et al.* [2002] obtained two solitary wave solutions

$$u_{\mathrm{d}}^{\pm} = \pm\sqrt{\frac{60c}{5a + \Omega\cosh(2\sqrt{c}\,\xi)}}, \tag{8}$$

where

$$\Omega = \sqrt{5a(5a + 48bc)}. \tag{9}$$

Zhang *et al.* [2002] gave two solitary wave solutions

$$u_{\mathrm{e}}^{\pm} = \pm\frac{\sqrt{60c}\,\mathrm{sech}\,(\sqrt{c}\,\xi)}{\sqrt{(5a - \Omega)\,\mathrm{sech}^2(\sqrt{c}\,\xi) + 2\Omega}}. \tag{10}$$

It is easy to verify that $u_{\mathrm{d}}^{\pm} = u_{\mathrm{e}}^{\pm}$. Li *et al.* [2003] obtained some complex solutions and some solitary wave solutions which are similar to the solutions above.

When $b = 0$, Eq. (1) becomes the mKdV equation

$$u_t + au^2 u_x + u_{xxx} = 0, \tag{11}$$

which has been studied by a number of authors, for instance, Fu *et al.* [2004a, 2004b], Gardner *et al.* [1995], Grimshaw *et al.* [2002], Gorsky and Himonas [2005], Kevrekidis *et al.* [2004], Kudryashov and Sinrlshchiov [2011], Lakshmanan and Tamizhmani [1985], Li *et al.* [2003], Liu and Yang [2002], Miura *et al.* [1968], Miura [1976], Smyth and Worthy [1995].

Recently, the bifurcation method of dynamical systems has been employed to study nonlinear waves successively (e.g. [Li & Chen, 2005a, 2005b; Li *et al.*, 2009a, 2009b; Li & Chen, 2010; Liu, 2010a, 2010b; Liu & Liang, 2011]). In this paper, we investigate nonlinear waves in Eq. (1) by using the bifurcation method mentioned above. We obtain four types of new expressions and reveal two kinds of new bifurcation phenomena which are introduced in the abstract above.

This paper is organized as follows. In Sec. 2, we state our results. In Sec. 3, we give derivations for our results and a brief conclusion is given in Sec. 4.

2. Main Results

For a given constant number $c > 0$, on a–b plane, let l_1 and l_2 represent the following two lines

$$l_1: \quad b = -\frac{5a}{48c}, \tag{12}$$

$$l_2: \quad b = -\frac{5a}{36c}. \tag{13}$$

Let A_i $(i = 1, 2, \ldots, 6)$ represent the regions surrounded by lines l_1, l_2 and the coordinate axes (see Fig. 1).

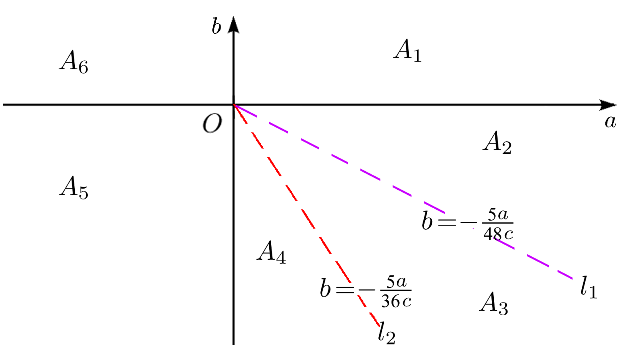

Fig. 1. The locations of the regions A_i $(i = 1, 2, \ldots, 6)$ and lines l_1, l_2 for given constant number $c > 0$.

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Also, let $\xi = x - ct$ be the intermediate variable, $\lambda \neq 0$, $\mu \neq 0$ be two arbitrary real numbers. Then our main results are listed in Propositions 1–3.

2.1. *The common explicit expressions of the symmetric solitary waves, kink waves and blow-up waves*

Proposition 1. (i) *For $ab \neq 0$, Eq. (1) has four real nonlinear wave solutions*

$$u_{\mathrm{f}}^{\pm} = \pm\sqrt{\frac{3600c\lambda}{300a\lambda + \Omega^2 e^{2\sqrt{c}\xi} + 900\lambda^2 e^{-2\sqrt{c}\xi}}}, \quad (14)$$

and

$$u_{\mathrm{g}}^{\pm} = \pm\sqrt{\frac{3600c\lambda}{300a\lambda + \Omega^2 e^{-2\sqrt{c}\xi} + 900\lambda^2 e^{2\sqrt{c}\xi}}}, \quad (15)$$

where Ω is given in (9). Corresponding to $\lambda > 0$ or $\lambda < 0$, these solutions have the following wave shapes and properties.

(1) *For the case of $\lambda > 0$, there are four properties as follows:*

$(1)_{\mathrm{a}}$ *If $\lambda > 0$ and $(a, b) \in l_1$, that is $a > 0$ and $b = -\frac{5a}{48c}$, then $u_{\mathrm{f}}^{\pm}$ and $u_{\mathrm{g}}^{\pm}$ become*

$$u_{\mathrm{f0}}^{\pm} = \pm\sqrt{\frac{12c}{a + 3\lambda e^{-2\sqrt{c}\xi}}} \quad (16)$$

and

$$u_{\mathrm{g0}}^{\pm} = \pm\sqrt{\frac{12c}{a + 3\lambda e^{2\sqrt{c}\xi}}}, \quad (17)$$

which represent four low-kink waves [refer to Fig. 2(d)]. Specially, if $a > 0$, $b = -\frac{5a}{48c}$ and $\lambda = \frac{a}{3}$, then $u_{\mathrm{f}}^{\pm}$ and $u_{\mathrm{g}}^{\pm}$ become $u_{\mathrm{a}}^{\pm}$ and $u_{\mathrm{b}}^{\pm}$ (see (2), (3)).

$(1)_{\mathrm{b}}$ *If (a, b) belongs to any one of the regions A_1, A_2, A_5 and $\lambda \neq \frac{\Omega}{30}$, then $u_{\mathrm{f}}^{\pm} \neq u_{\mathrm{g}}^{\pm}$ and they represent four symmetric solitary waves [refer to Figs. 2(a)–2(c)]. Specially, when $(a, b) \in A_2$ and $b \to -\frac{5a}{48c} + 0$, the four symmetric solitary waves become four low-kink waves with the expressions $u_{\mathrm{f0}}^{\pm}$ and $u_{\mathrm{g0}}^{\pm}$. For the varying process, see Fig. 2.*

$(1)_{\mathrm{c}}$ *If (a, b) belongs to one of the regions A_3, l_2, A_4, A_6 and $\lambda \neq \frac{\Omega}{30}$, then $u_{\mathrm{f}}^{\pm} \neq u_{\mathrm{g}}^{\pm}$ and they represent four 1-blow-up waves. Specially, when $a > 0$ and*

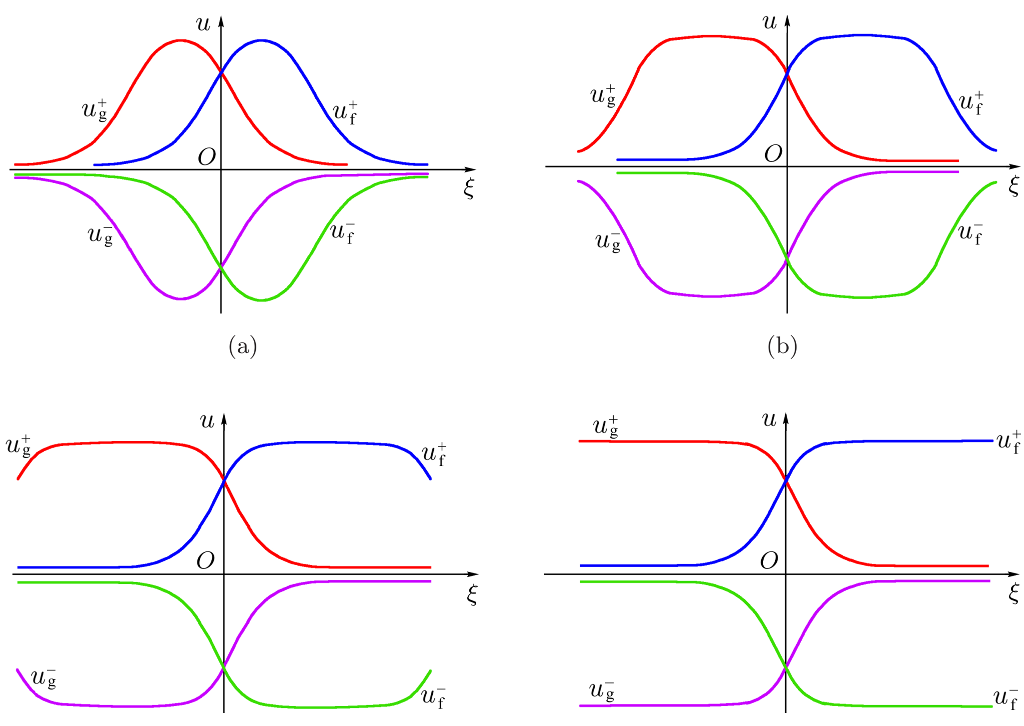

Fig. 2. (Four low-kink waves are bifurcated from four symmetric solitary waves.) The varying process for $u_{\mathrm{f}}^{\pm}$ and $u_{\mathrm{g}}^{\pm}$ when $\lambda > 0$, $\lambda \neq \frac{\Omega}{30}$, $(a, b) \in A_2$ and $b \to -\frac{5a}{48c} + 0$, where $a = 12$, $c = 2$, $\lambda = 4$, l_1: $b = -\frac{5a}{48c} = -\frac{5}{8}$, and (a) $b = -\frac{5}{8} + 10^{-2}$, (b) $b = -\frac{5}{8} + 10^{-5}$, (c) $b = -\frac{5}{8} + 10^{-8}$ and (d) $b = -\frac{5}{8} + 10^{-12}$.

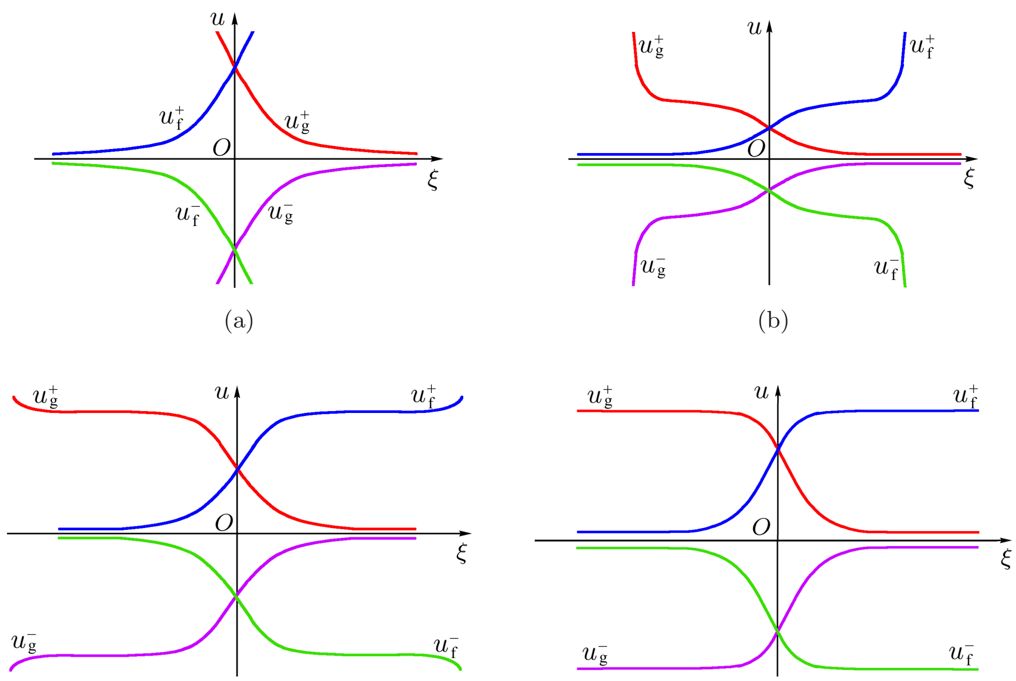

Fig. 3. (Four low-kink waves are bifurcated from four 1-blow-up waves.) The varying process for $u_{\rm f}^{\pm}$ and $u_{\rm g}^{\pm}$ when $\lambda > 0$, $a > 0$, $\lambda \neq \frac{\Omega}{30}$, $(a, b) \in A_3$ and $b \to -\frac{5a}{48c} - 0$, where $a = 12$, $c = 2$, $\lambda = 4$, l_1: $b = -\frac{5a}{48c} = -\frac{5}{8}$, and (a) $b = -\frac{5}{8} - 10^{-2}$, (b) $b = -\frac{5}{8} - 10^{-5}$, (c) $b = -\frac{5}{8} - 10^{-8}$ and (d) $b = -\frac{5}{8} - 10^{-12}$.

$b \to -\frac{5a}{48c} - 0$, the four 1-blow-up waves become four low-kink waves with the expressions $u_{\rm f0}^{\pm}$ and $u_{\rm g0}^{\pm}$. For the varying process, see Fig. 3.

$(1)_{\rm d}$ *If (a, b) belongs to one of A_1, A_2, A_5 and $\lambda = \frac{\Omega}{30}$, then $u_{\rm f}^{\pm} = u_{\rm g}^{\pm}$ and equal to the hyperbolic solitary wave solutions $u_{\rm d}^{\pm}$ (see (8)). When $(a, b) \in A_2$ and $b \to -\frac{5a}{48c} + 0$, $u_{\rm d}^{\pm}$ tend to the trivial solutions $u = \pm\sqrt{\frac{12c}{a}}$. The varying process for $u_{\rm d}^{\pm}$ is showed in Fig. 4.*

(2) *For the case of $\lambda < 0$, there are four properties as follows:*

$(2)_{\rm a}$ *If $(a, b) \in A_2$ or $(a, b) \in l_1$, then $u_{\rm f}^{\pm}$ and $u_{\rm g}^{\pm}$ are real solutions. For other cases, $u_{\rm f}^{\pm}$ and $u_{\rm g}^{\pm}$ are complex solutions.*

$(2)_{\rm b}$ *If $(a, b) \in l_1$, that is, $a > 0$, and $b = -\frac{5a}{48c}$, then $u_{\rm f}^{\pm}$ and $u_{\rm g}^{\pm}$ become $u_{\rm f0}^{\pm}$ and $u_{\rm g0}^{\pm}$ which represent four 1-blow-up waves [refer to Fig. 5(d)]. Specially, if $a > 0$, $b = -\frac{5a}{48c}$ and $\lambda = -\frac{a}{3}$, $u_{\rm f}^{\pm}$ and $u_{\rm g}^{\pm}$ respectively become the hyperbolic blow-up wave solutions*

$$u_{\rm f1}^{\pm} = \pm\sqrt{\frac{6c}{a}}\sqrt{1 + \coth(\sqrt{c}\,\xi)}, \tag{18}$$

and

$$u_{\rm g1}^{\pm} = \pm\sqrt{\frac{6c}{a}}\sqrt{1 - \coth(\sqrt{c}\,\xi)}. \tag{19}$$

$(2)_{\rm c}$ *If $(a, b) \in A_2$ and $\lambda \neq -\frac{\Omega}{30}$, then $u_{\rm f}^{\pm} \neq u_{\rm g}^{\pm}$ and they represent four 2-blow-up waves. When $b \to -\frac{5a}{48c} + 0$, $u_{\rm f}^{\pm}$ and $u_{\rm g}^{\pm}$ respectively become $u_{\rm f0}^{\pm}$ and $u_{\rm g0}^{\pm}$. The varying process for $u_{\rm f}^{\pm}$ and $u_{\rm g}^{\pm}$ is showed in Fig. 5.*

$(2)_{\rm d}$ *If $(a, b) \in A_2$ and $\lambda = -\frac{\Omega}{30}$, then $u_{\rm f}^{\pm} = u_{\rm g}^{\pm}$ and equal to the hyperbolic blow-up wave solutions*

$$u_{\rm fg}^{\pm} = \pm\sqrt{\frac{60c}{5a - \Omega\cosh(2\sqrt{c}\,\xi)}}. \tag{20}$$

When $b \to -\frac{5a}{48c} + 0$, $u_{\rm fg}^{\pm}$ tends to the trivial solution $u = \pm\sqrt{\frac{12c}{a}}$. The varying process for $u_{\rm fg}^{\pm}$ is displayed in Fig. 6.

2.2. *The common explicit expressions of kink waves, anti-symmetric solitary waves and blow-up waves*

From the expression of Δ in (6), we get the following lemma.

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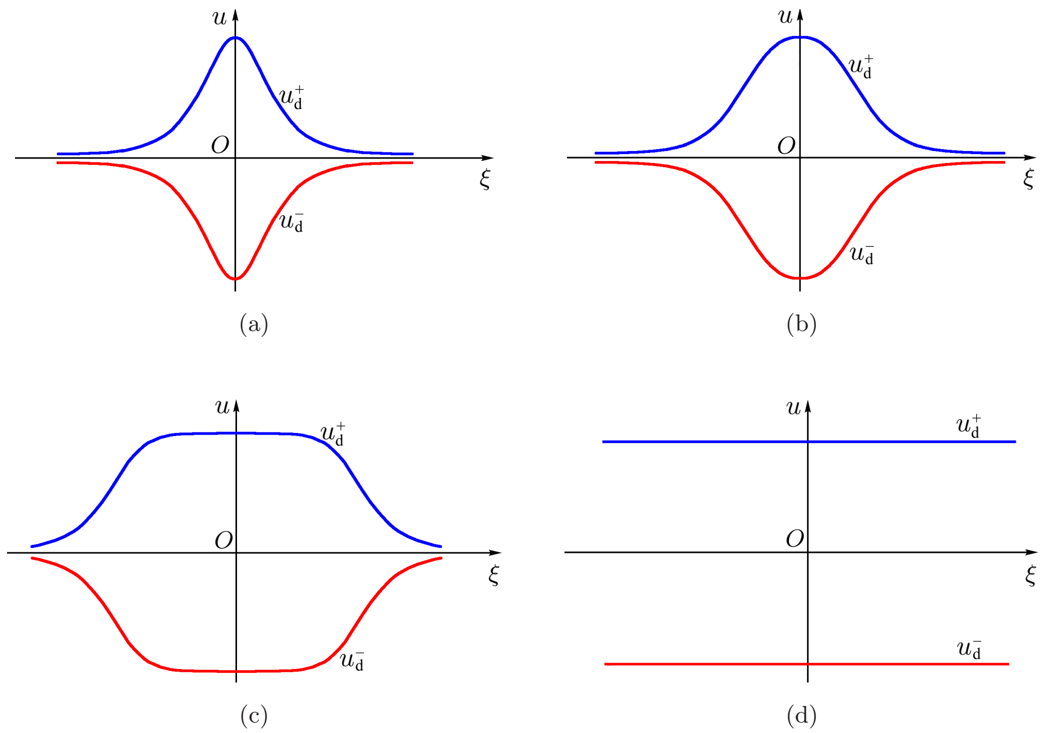

Fig. 4. (Two symmetric solitary waves become two trivial waves.) The varying process for the figures of $u_{\mathrm{d}}^{\pm}$ (that is, $u_{\mathrm{f}}^{\pm}$ and $u_{\mathrm{g}}^{\pm}$ with $\lambda = \frac{\Omega}{30}$) when $(a, b) \in A_2$ and $b \to -\frac{5a}{48c} + 0$, where $a = 12$, $c = 2$, l_1: $b = -\frac{5a}{48c} = -\frac{5}{8}$, and (a) $b = -\frac{5}{8} + 10^{-2}$, (b) $b = -\frac{5}{8} + 10^{-5}$, (c) $b = -\frac{5}{8} + 10^{-8}$ and (d) $b = -\frac{5}{8} + 10^{-12}$.

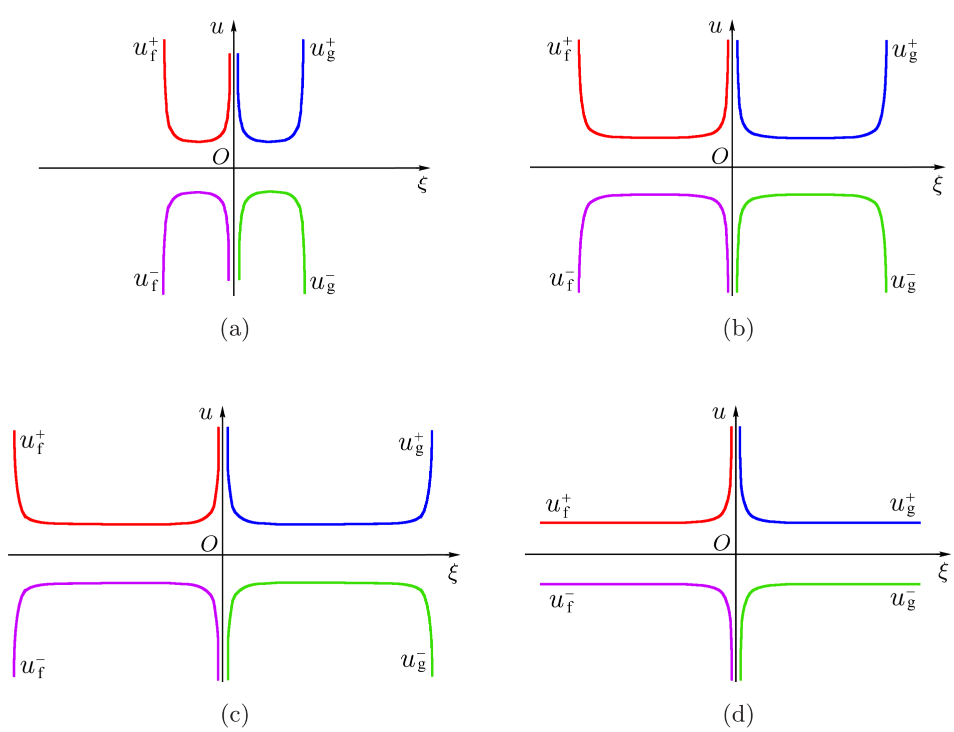

Fig. 5. (Four 2-blow-up waves become four 1-blow-up waves.) The varying process for $u_{\mathrm{f}}^{\pm}$ and $u_{\mathrm{g}}^{\pm}$ when $\lambda < 0$, $(a, b) \in A_2$ and $b \to -\frac{5a}{48c} + 0$, where $a = 12$, $c = 2$, $\lambda = -4$, l_1: $b = -\frac{5a}{48c} = -\frac{5}{8}$, and (a) $b = -\frac{5}{8} + 10^{-2}$, (b) $b = -\frac{5}{8} + 10^{-5}$, (c) $b = -\frac{5}{8} + 10^{-8}$ and (d) $b = -\frac{5}{8} + 10^{-12}$.

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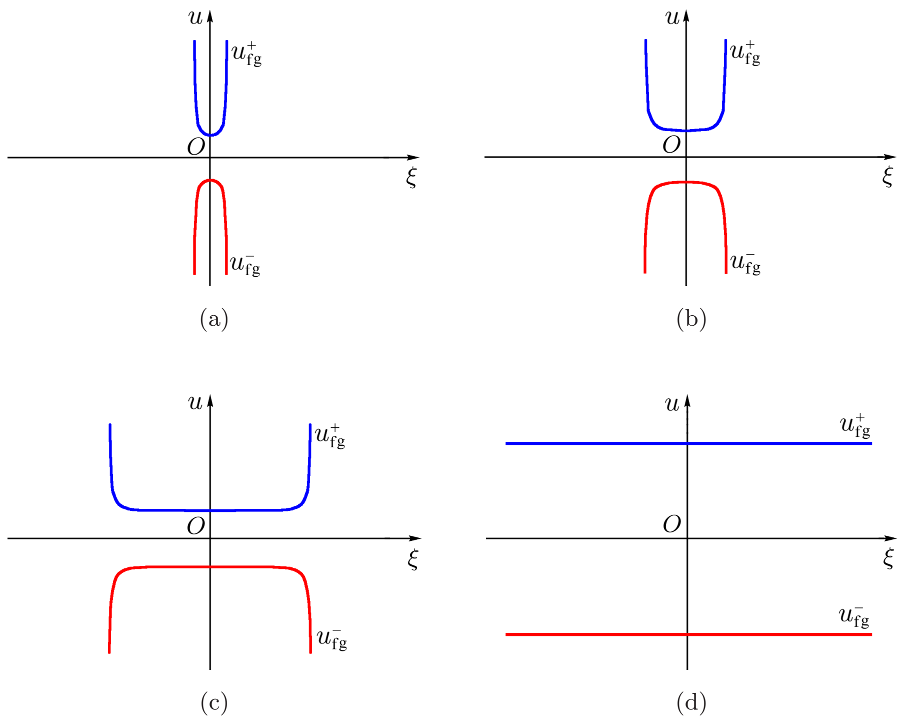

Fig. 6. (Two 2-blow-up waves become two trivial waves.) The varying process for $u_{fg}^{\pm}$ when $\lambda = -\frac{\Omega}{30}$, $(a,b) \in A_2$ and $b \to -\frac{5a}{48c} + 0$, where $a = 12$, $c = 2$, l_1: $b = -\frac{5a}{48c} = -\frac{5}{8}$, and (a) $b = -\frac{5}{8} + 10^{-2}$, (b) $b = -\frac{5}{8} + 10^{-5}$, (c) $b = -\frac{5}{8} + 10^{-8}$ and (d) $b = -\frac{5}{8} + 10^{-12}$.

Lemma 1. *If*

$$\mu_0 = \frac{5a - 2\Delta}{6ab}, \tag{21}$$

then it follows that

$$\mu_0 \begin{cases} > 0 & \text{for } (a,b) \in A_2, \\ = 0 & \text{for } (a,b) \in l_1, \\ < 0 & \text{for } (a,b) \in A_3. \end{cases} \tag{22}$$

Proposition 2. (i) *If (a,b) belongs to one of the regions A_2, A_3, l_1 and l_2, then Eq. (1) has four real nonlinear wave solutions*

$$u_h^{\pm} = \pm \frac{\alpha(6ab\mu + \delta\,e^{\eta_2\xi})}{\sqrt{36a^2b^2\mu^2 - 12ab\mu\omega\,e^{\eta_2\xi} + \delta^2\,e^{2\eta_2\xi}}}, \tag{23}$$

and

$$u_i^{\pm} = \pm \frac{\alpha(6ab\mu + \delta\,e^{-\eta_2\xi})}{\sqrt{36a^2b^2\mu^2 - 12ab\mu\omega\,e^{-\eta_2\xi} + \delta^2\,e^{-2\eta_2\xi}}}, \tag{24}$$

where Δ is given in (6), and

$$\alpha = \sqrt{-\frac{\Delta + 5a}{6ab}}, \tag{25}$$

$$\delta = 2\Delta - 5a, \tag{26}$$

$$\omega = 4\Delta + 5a, \tag{27}$$

$$\eta_2 = \sqrt{\frac{-\Delta(5a + \Delta)}{45ab}}. \tag{28}$$

Specially, if $(a,b) \in l_1$, that is, $b = -\frac{5a}{48c}$, then $u_h^{\pm}$ and $u_i^{\pm}$ respectively become

$$u_{h0}^{\mp} = \mp\sqrt{\frac{12\mu c}{48c\,e^{2\sqrt{c}\xi} + a\mu}}, \tag{29}$$

and

$$u_{i0}^{\mp} = \mp\sqrt{\frac{12\mu c}{48c\,e^{-2\sqrt{c}\xi} + a\mu}}. \tag{30}$$

(ii) *For other cases, $u_h^{\pm}$ and $u_i^{\pm}$ are complex solutions of Eq. (1).*

Corresponding to $\mu > 0$ or $\mu < 0$, these solutions have the following wave shapes and properties.

(1°) *For the case of $\mu > 0$, there are five properties as follows:*

$(1°)_a$ *If $(a,b) \in l_1$, then $u_{h0}^{\mp}$ and $u_{i0}^{\mp}$ represent four low-kink waves. u_{h0}^{+} and u_{i0}^{+} have asymptotic lines*

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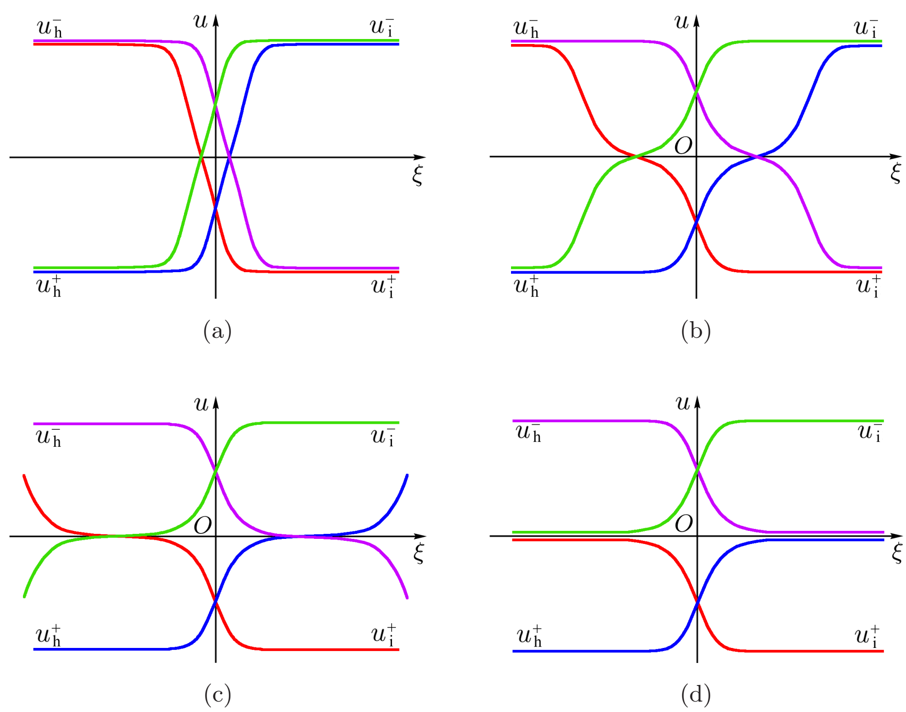

Fig. 7. (Four low-kink waves are bifurcated from four tall-kink waves.) The varying process for the figures of $u_h^{\pm}$ and $u_i^{\pm}$ when $\mu > 0$, $\mu \neq |\mu_0|$, $(a, b) \in A_2$ and $b \to -\frac{5a}{48c} + 0$, where $a = 12$, $c = 2$, $\mu = 4$, l_1: $b = -\frac{5a}{48c} = -\frac{5}{8}$, and (a) $b = -\frac{5}{8} + 10^{-2}$, (b) $b = -\frac{5}{8} + 10^{-5}$, (c) $b = -\frac{5}{8} + 10^{-8}$ and (d) $b = -\frac{5}{8} + 10^{-12}$.

$u = 0$ and $u = \sqrt{\frac{12c}{a}}$. u_{h0}^- and u_{i0}^- have asymptotic lines $u = 0$ and $u = -\sqrt{\frac{12c}{a}}$ [refer to Fig. 7(d) or Fig. 8(d)].

Specially, if $a > 0$, $b = -\frac{5a}{48c}$ and $\mu = \frac{48c}{a}$, then $u_h^{\pm} = u_b^{\mp}$ and $u_i^{\pm} = u_a^{\mp}$. For $u_a^{\mp}$ and $u_b^{\mp}$, see (2) and (3).

$(1°)_b$ *If $(a, b) \in A_2$ and $\mu \neq |\mu_0|$, then $u_h^{\pm} \neq u_i^{\pm}$ and they represent four tall-kink waves [see Figs. 7(a)–7(c)]. When $b \to -\frac{5a}{48c} + 0$, the four tall-kink waves become four low-kink waves with the expressions $u_{h0}^{\mp}$ and $u_{i0}^{\mp}$ [see Fig. 7(d)]. The varying process is displayed in Fig. 7.*

$(1°)_c$ *If $(a, b) \in A_3$ and $\mu \neq |\mu_0|$, then $u_h^{\pm} \neq u_i^{\pm}$ and they represent four anti-symmetric solitary waves with nonzero asymptotic lines $u = \pm\alpha$ [see Figs. 8(a)–8(c)].*

(i) *When $b \to -\frac{5a}{48c} - 0$, the four anti-symmetric solitary waves become four low-kink waves with the expressions $u_{h0}^{\pm}$ and $u_{i0}^{\pm}$. The varying process is displayed in Fig. 8.*

(ii) *When $b \to -\frac{5a}{36c} + 0$, the four anti-symmetric solitary waves become trivial waves $u = \pm\sqrt{\frac{6c}{a}}$.*

$(1°)_d$ *If $(a, b) \in A_2$ and $\mu = |\mu_0|$, then*

$$u_h^+ = u_i^- = u_c^+ \quad \text{and} \quad u_h^- = u_i^+ = u_c^- \tag{31}$$

which represent two tall-kink waves and tend to trivial waves $u = 0$ (see Fig. 9) when $b \to -\frac{5a}{48c} + 0$.

$(1°)_e$ *If $(a, b) \in A_3$ and $\mu = |\mu_0|$, then $u_h^{\pm} = u_i^{\pm} = u_j^{\mp}$ of form*

$$u_j^{\mp} = \mp\sqrt{\frac{2(5a - 2\Delta)}{(5a + 4\Delta) + (5a - 2\Delta)\cosh(\eta_2\xi)}} \times \alpha\cosh\left(\frac{\eta_2\xi}{2}\right), \tag{32}$$

which represent two anti-symmetric solitary waves and tend to the trivial wave $u = 0$ (see Fig. 10) when $b \to -\frac{5a}{48c} - 0$ and tend to $u = \pm\alpha$ when $b \to -\frac{5a}{36c} + 0$.

$(2°)$ *For the case of $\mu < 0$, there are four properties as follows:*

$(2°)_a$ *If $\mu < 0$ and $(a, b) \in l_1$ then $u_{h0}^{\pm}$ and $u_{i0}^{\pm}$ represent four 1-blow-up waves [refer to Fig. 11(d)]. Specially, if $\mu = -\frac{48c}{a}$, then $u_{h0}^{\pm}$ and $u_{i0}^{\pm}$ become the hyperbolic blow-up wave solutions $u_{f1}^{\pm}$ and $u_{g1}^{\pm}$ (see (18), (19)).*

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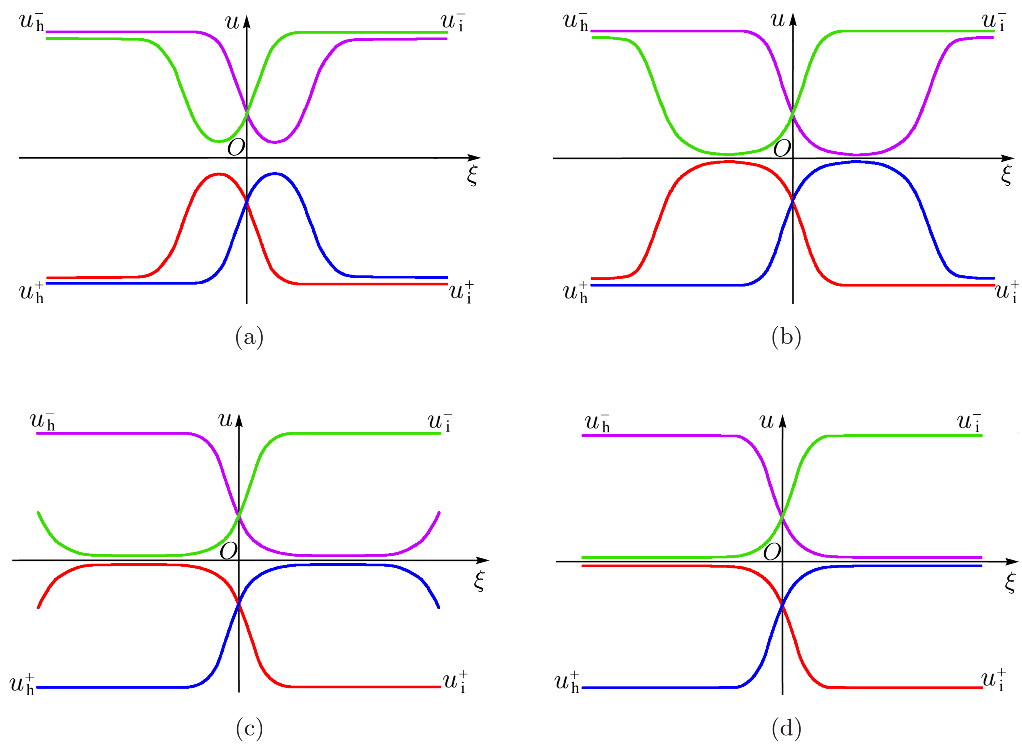

Fig. 8. (Four low-kink waves are bifurcated from four anti-symmetric solitary waves.) The varying process for the figures of $u_h^{\pm}$ and $u_i^{\pm}$ when $\mu > 0$, $\mu \neq |\mu_0|$, $(a, b) \in A_3$ and $b \to -\frac{5a}{48c} - 0$, where $a = 12$, $c = 2$, $\mu = 4$, l_1: $b = -\frac{5a}{48c} = -\frac{5}{8}$, and (a) $b = -\frac{5}{8} - 10^{-2}$, (b) $b = -\frac{5}{8} - 10^{-5}$, (c) $b = -\frac{5}{8} - 10^{-8}$ and (d) $b = -\frac{5}{8} - 10^{-12}$.

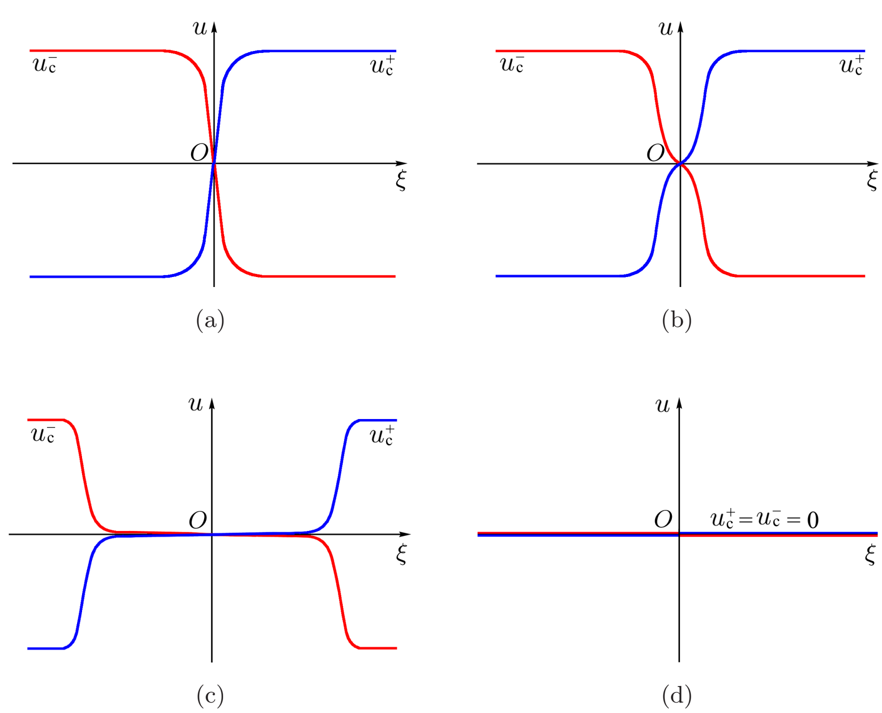

Fig. 9. (Two tall-kink waves become a trivial wave.) The varying process for the figures of $u_c^{\pm}$ when $(a, b) \in A_2$ and $b \to -\frac{5a}{48c} + 0$, where $a = 12$, $c = 2$, l_1: $b = -\frac{5a}{48c} = -\frac{5}{8}$, and (a) $b = -\frac{5}{8} + 10^{-2}$, (b) $b = -\frac{5}{8} + 10^{-5}$, (c) $b = -\frac{5}{8} + 10^{-8}$ and (d) $b = -\frac{5}{8} + 10^{-12}$.

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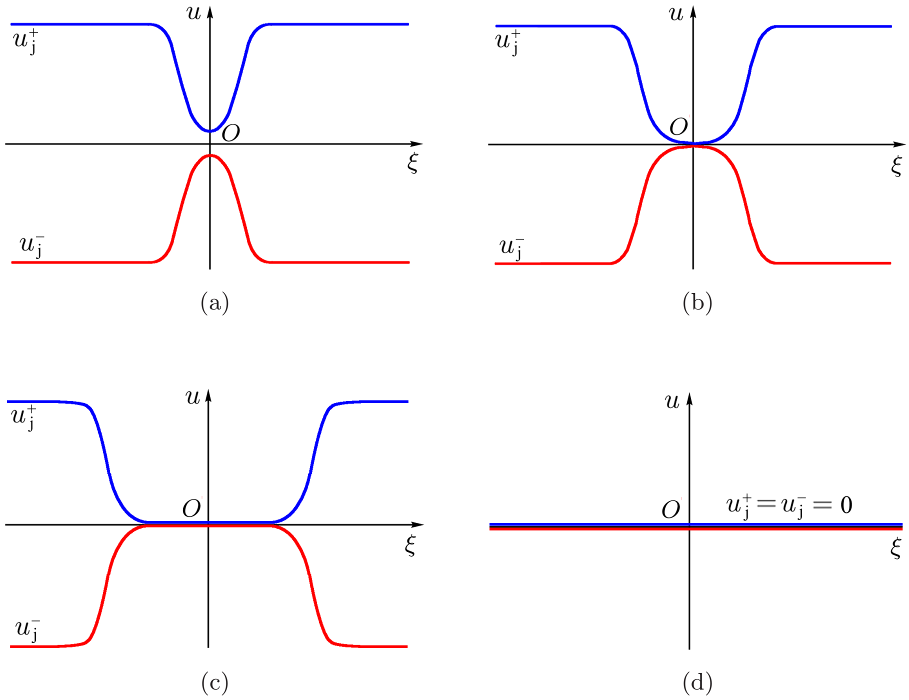

Fig. 10. (Two anti-symmetric solitary waves become a trivial wave.) The varying process for the figures of $u_j^{\pm}$ when $(a, b) \in A_3$ and $b \to -\frac{5a}{48c} - 0$, where $a = 12$, $c = 2$, $\mu = 4$, l_1: $b = -\frac{5a}{48c} = -\frac{5}{8}$, and (a) $b = -\frac{5}{8} - 10^{-2}$, (b) $b = -\frac{5}{8} - 10^{-5}$, (c) $b = -\frac{5}{8} - 10^{-8}$ and (d) $b = -\frac{5}{8} - 10^{-12}$.

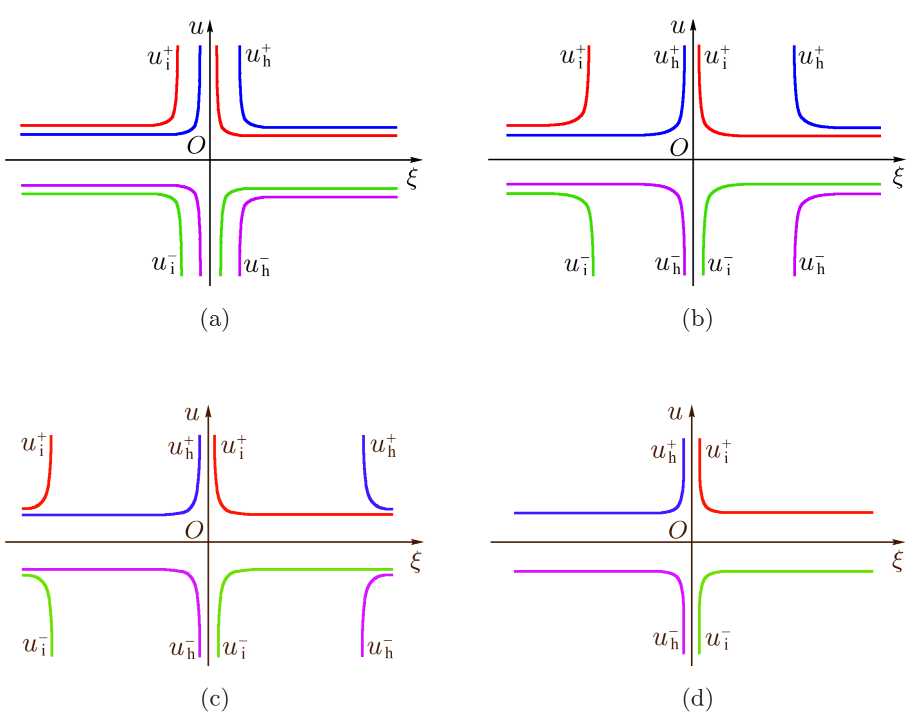

Fig. 11. (Two pairs of 1-blow-up waves are bifurcated from four pairs of 1-blow-up waves.) The varying process for the figures of $u_h^{\pm}$ and $u_i^{\pm}$ when $\mu < 0$, $\mu \neq -|\mu_0|$ and $b \to -\frac{5a}{48c}$, where $a = 12$, $c = 2$, l_1: $b = -\frac{5a}{48c} = -\frac{5}{8}$, and (a) $b = -\frac{5}{8} \pm 10^{-2}$, (b) $b = -\frac{5}{8} \pm 10^{-5}$, (c) $b = -\frac{5}{8} \pm 10^{-8}$ and (d) $b = -\frac{5}{8} \pm 10^{-12}$.

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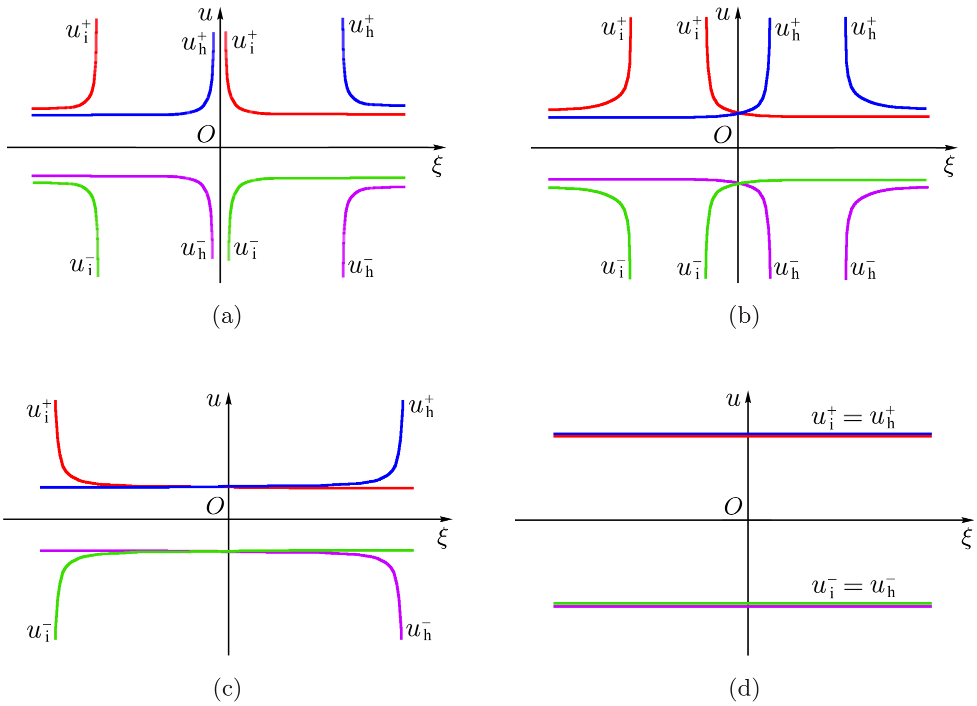

Fig. 12. (Four pairs of 1-blow-up waves become two trivial waves.) The varying process for $u_h^{\pm}$ and $u_i^{\pm}$ when $\mu < 0$, $\mu \neq -|\mu_0|$ and $b \to -\frac{5a}{36c} + 0$, where $a = 12$, $c = 2$, l_1: $b = -\frac{5a}{36c} = -\frac{5}{6}$, and (a) $b = -\frac{5}{6} + 10^{-2}$, (b) $b = -\frac{5}{6} + 10^{-5}$, (c) $b = -\frac{5}{6} + 10^{-8}$ and (d) $b = -\frac{5}{6} + 10^{-12}$.

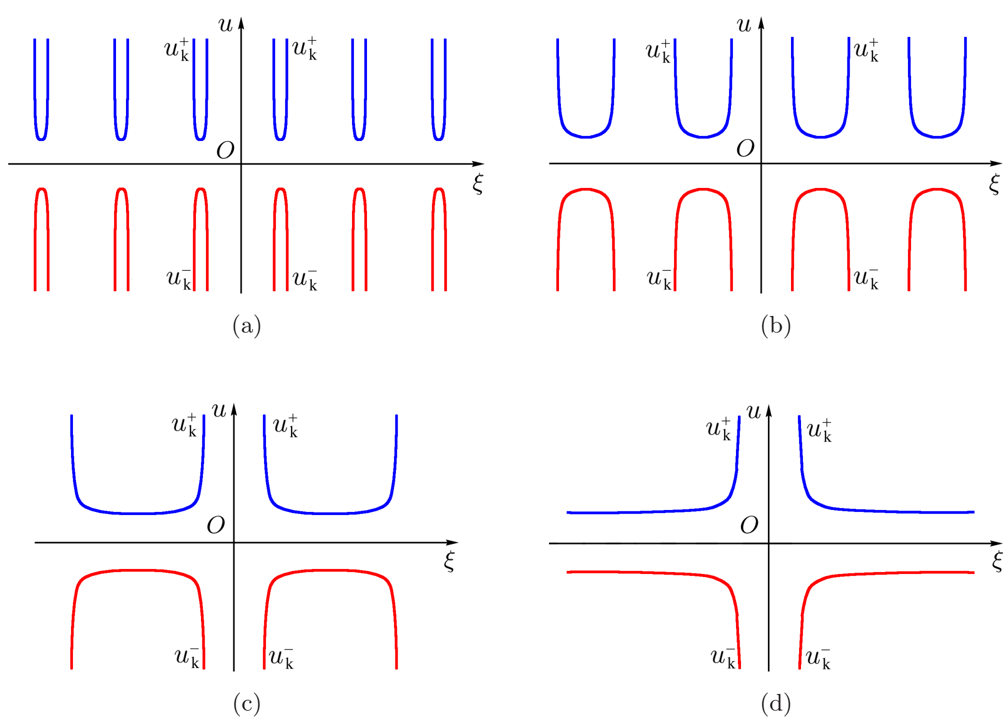

Fig. 13. (The 1-blow-up waves are bifurcated from the periodic-blow-up waves.) The varying process for the figures of $u_k^{\pm}$ when $b \to -\frac{5a}{36c} + 0$, where $a = 12$, $c = 2$, l_1: $b = -\frac{5a}{36c} = -\frac{5}{6}$, and (a) $b = -\frac{5}{6} + 10^{-2}$, (b) $b = -\frac{5}{6} + 10^{-5}$, (c) $b = -\frac{5}{6} + 10^{-8}$ and (d) $b = -\frac{5}{6} + 10^{-12}$.

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$(2°)_{\rm b}$ *If (a, b) belongs to one of A_2, A_3 and $\mu < 0$, $\mu \neq -|\mu_0|$, then $u_{\rm h}^{\pm} \neq u_{\rm i}^{\pm}$ and they represent four pairs of 1-blow-up waves. When $b \to -\frac{5a}{48c}$, the four pairs of 1-blow-up waves become two pairs of 1-blow-up waves with the expressions $u_{\rm h0}^{\pm}$ and $u_{\rm i0}^{\pm}$. For the varying process, see Fig. 11. When $b \to -\frac{5a}{36c} + 0$, the four pairs of 1-blow-up waves become the trivial waves $u = \pm\alpha$ (see Fig. 12).*

$(2°)_{\rm c}$ *If $(a, b) \in A_2$ and $\mu = -|\mu_0|$, then $u_{\rm h}^{\pm} = u_{\rm i}^{\pm} = u_{\rm j}^{\pm}$ (see (32)) which represent two pairs of 1-blow-up waves.*

$(2°)_{\rm d}$ *If $(a, b) \in A_3$ and $\mu = -|\mu_0|$, then $u_{\rm h}^{+} = u_{\rm i}^{-} = u_{\rm c}^{-}$ and $u_{\rm h}^{-} = u_{\rm i}^{+} = u_{\rm c}^{+}$ which represent two pairs of 1-blow-up waves.*

2.3. *The periodic blow-up waves and fractional blow-up waves*

Proposition 3

(i) *If (a, b) belongs to one of A_2, A_3 and l_1, then Eq. (1) has two periodic blow-up wave solutions*

$$u_{\rm k}^{\pm} = \pm\sqrt{\frac{(\Delta - 5a)(2\Delta + 5a)[1 - \cos(\eta_3\xi)]}{6ab[5a - 4\Delta - (5a + 2\Delta)\cos(\eta_3\xi)]}}, \tag{33}$$

where Δ is given in (6) and

$$\eta_3 = \sqrt{\frac{\Delta(-5a + \Delta)}{45ab}}. \tag{34}$$

(ii) *If $(a, b) \in l_2$, that is, $b = -\frac{5a}{36c}$, then Eq. (1) has two fractional 1-blow-up wave solutions*

$$u_{\rm l}^{\pm} = \pm\frac{\sqrt{6}\,c\xi}{\sqrt{a(-3 + c\xi^2)}}. \tag{35}$$

(iii) *If $(a, b) \in l_2$, then $u_{\rm k}^{\pm}$ are not defined. If (a, b) belongs to one of A_1, A_4, A_5 and A_6, then $u_{\rm k}^{\pm}$ are complex solutions of Eq. (1).*

(iv) *When $a > 0$ and $b \to -\frac{5a}{36c} + 0$, $u_{\rm k}^{\pm}$ tend to $u_{\rm l}^{\pm}$. The varying process is displayed in Fig. 13.*

3. The Derivations of Main Results

To derive our results, substituting $u = \varphi(\xi)$ with $\xi = x - c\,t$ into Eq. (1), it follows that

$$-c\varphi'(\xi) + a(1 + b\varphi^2(\xi))\varphi^2(\xi)\varphi'(\xi) + \varphi'''(\xi) = 0. \tag{36}$$

Integrating (12) once, we get the following equation

$$\varphi''(\xi) - c\varphi(\xi) + \frac{a}{3}\varphi^3(\xi) + \frac{ab}{5}\varphi^5(\xi) = 0. \tag{37}$$

From (13) we obtain planar system

$$\frac{d\varphi}{d\xi} = y, \quad \frac{dy}{d\xi} = c\,\varphi - \frac{a}{3}\varphi^3 - \frac{ab}{5}\varphi^5, \tag{38}$$

with the first integral

$$H(\varphi, y) = h, \tag{39}$$

where h is the integral constant and

$$H(\varphi, y) = y^2 - c\varphi^2 + \frac{a}{6}\varphi^4 + \frac{ab}{15}\varphi^6. \tag{40}$$

Let α be in (25) and

$$\beta = \sqrt{\frac{\Delta - 5a}{6ab}}, \tag{41}$$

where Δ is given in (6). According to the qualitative theory, we obtain the bifurcation phase portraits of the planar system (38) as in Fig. 14.

Next, using the information given by Fig. 14, we give derivations to Proposition 1–3 respectively.

3.1. *The derivations to Proposition 1*

In the first integral (39), letting $h = H(0, 0)$, it follows that

$$y_{\pm} = \pm\sqrt{\varphi^2\left(c - \frac{a}{6}\varphi^2 - \frac{ab}{15}\varphi^4\right)}. \tag{42}$$

Substituting (42) into the first equation of (38) and integrating it, we have

$$\int_p^{\varphi} \frac{ds}{\sqrt{s^2\left(c - \frac{a}{6}s^2 - \frac{ab}{15}s^4\right)}} = \xi, \tag{43}$$

where p is an arbitrary constant number.

Completing the integral above and solving the equation for φ, it follows that

$$\varphi = \pm\sqrt{\frac{720c\lambda\,e^{2\sqrt{c}\xi}}{60a\lambda\,e^{2\sqrt{c}\xi} + a(5a + 48bc)e^{4\sqrt{c}\xi} + 180\lambda^2}}, \tag{44}$$

where $\lambda = \lambda(p)$ is an arbitrary real number.

Note that if $u = \varphi(x - ct)$ is a solution of Eq. (1), then so is $u = \varphi(ct - x)$. Therefore,

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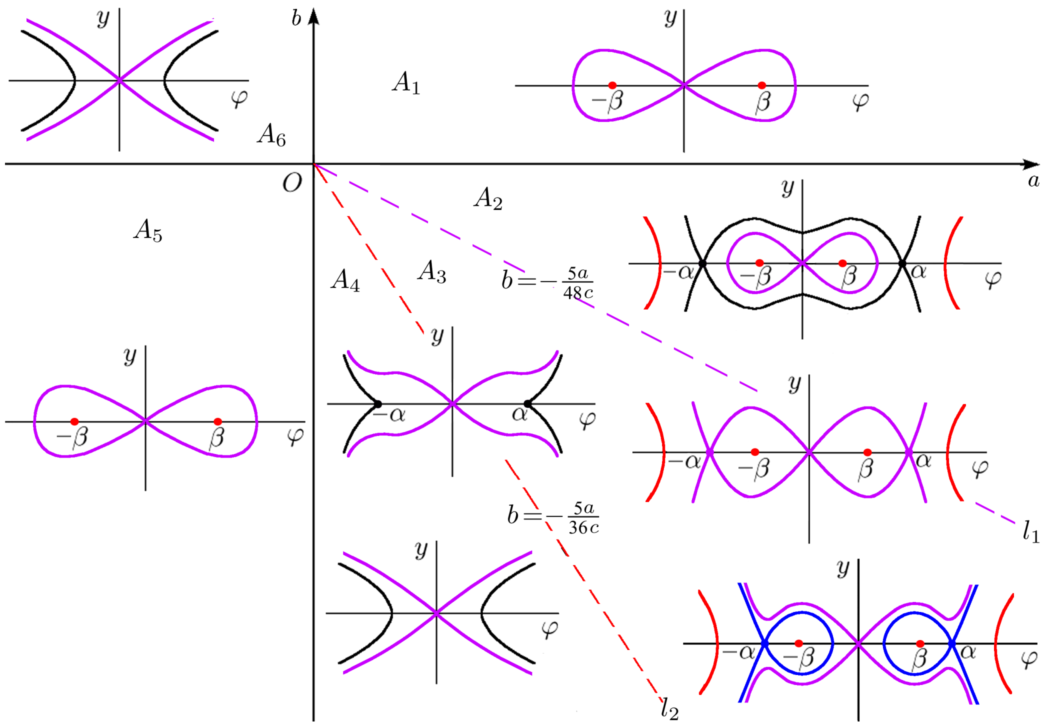

Fig. 14. The bifurcation phase portraits of the system (38) for given wave speed $c > 0$.

from (44) we obtain the solutions $u_f^\pm$ and $u_g^\pm$ as (14) and (15).

In (14) and (15) letting $b = -\frac{5a}{48c}$, we get (16) and (17). Furthermore, in (14) and (15) letting $b = -\frac{5a}{48c}$ and $\lambda = \frac{a}{3}$, it follows that

$$\begin{aligned} u_f^\pm &= \pm\sqrt{\frac{12c}{a(1+e^{-2\sqrt{c}\xi})}} \\ &= \pm\sqrt{\frac{12ce^{\sqrt{c}\xi}}{a(e^{\sqrt{c}\xi}+e^{-\sqrt{c}\xi})}} \\ &= \pm\sqrt{\frac{6c}{a}\left(1+\frac{e^{\sqrt{c}\xi}-e^{-\sqrt{c}\xi}}{e^{\sqrt{c}\xi}+e^{-\sqrt{c}\xi}}\right)} \\ &= \pm\sqrt{\frac{6c}{a}\left(1+\tanh(\sqrt{c}\xi)\right)} \\ &= u_a^\pm \quad (\text{see } (2)), \end{aligned} \tag{45}$$

and

$$\begin{aligned} u_g^\pm &= \pm\sqrt{\frac{12c}{a(1+e^{2\sqrt{c}\xi})}} \\ &= \pm\sqrt{\frac{12ce^{-\sqrt{c}\xi}}{a(e^{\sqrt{c}\xi}+e^{-\sqrt{c}\xi})}} \\ &= \pm\sqrt{\frac{6c}{a}\left(1-\frac{e^{\sqrt{c}\xi}-e^{-\sqrt{c}\xi}}{e^{\sqrt{c}\xi}+e^{-\sqrt{c}\xi}}\right)} \\ &= \pm\sqrt{\frac{6c}{a}\left(1-\tanh(\sqrt{c}\xi)\right)} \\ &= u_b^\pm \quad (\text{see } (3)). \end{aligned} \tag{46}$$

From (14)–(17) and (45) and (46), we get properties $(1)_a$–$(1)_c$ of Proposition 1.

When $\lambda = \frac{\Omega}{30}$, via (14) and (15) it follows that

$$\begin{aligned} u_f^\pm &= u_g^\pm \\ &= \sqrt{\frac{120c}{10a+\Omega(e^{2\sqrt{c}\xi}+e^{-2\sqrt{c}\xi})}} \\ &= \sqrt{\frac{60c}{5a+\Omega\cosh(2\sqrt{c}\xi)}} \\ &= u_d^\pm \quad (\text{see } (8)), \end{aligned} \tag{47}$$

which is property $(1)_d$ of Proposition 1.

When $\lambda < 0$, via (14) and (15), it follows that

$$u_f^\pm = \sqrt{\frac{3600c|\lambda|}{300a|\lambda| - \Omega^2 e^{2\sqrt{c}\xi} - 900\lambda^2 e^{-2\sqrt{c}\xi}}}, \tag{48}$$

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and

$$u_{\rm g}^{\pm} = \sqrt{\frac{3600c|\lambda|}{300a|\lambda| - \Omega^2 e^{-2\sqrt{c}\xi} - 900\lambda^2 e^{2\sqrt{c}\xi}}}. \quad (49)$$

Furthermore, when $b = -\frac{5a}{48c}$ and $\lambda = -\frac{a}{3}$, we have

$$\begin{aligned} u_{\rm f}^{\pm} &= \pm\sqrt{\frac{12c}{a(1+e^{-2\sqrt{c}\xi})}} \\ &= \pm\sqrt{\frac{12c\, e^{\sqrt{c}\xi}}{a(e^{\sqrt{c}\xi} - e^{-\sqrt{c}\xi})}} \\ &= \pm\sqrt{\frac{6c}{a}\left(1 + \frac{e^{\sqrt{c}\xi} + e^{-\sqrt{c}\xi}}{e^{\sqrt{c}\xi} - e^{-\sqrt{c}\xi}}\right)} \\ &= \pm\sqrt{\frac{6c}{a}(1+\coth(\sqrt{c}\xi))} \\ &= u_{\rm f1}^{\pm} \quad \text{(see (18))}, \end{aligned} \quad (50)$$

and

$$\begin{aligned} u_{\rm g}^{\pm} &= \pm\sqrt{\frac{12c}{a(1-e^{2\sqrt{c}\xi})}} \\ &= \pm\sqrt{\frac{12c e^{-\sqrt{c}\xi}}{a(e^{-\sqrt{c}\xi} - e^{\sqrt{c}\xi})}} \\ &= \pm\sqrt{\frac{6c}{a}\left(1 - \frac{e^{\sqrt{c}\xi} + e^{-\sqrt{c}\xi}}{e^{\sqrt{c}\xi} - e^{-\sqrt{c}\xi}}\right)} \\ &= \pm\sqrt{\frac{6c}{a}(1-\coth c\xi(\sqrt{c}\xi))} \\ &= u_{\rm g1}^{\pm} \quad \text{(see (19))}. \end{aligned} \quad (51)$$

Via (48)–(51), we get properties $(2)_{\rm a}$–$(2)_{\rm c}$ of Proposition 1.

When $\lambda = -\frac{\Omega}{30}$, from (48) and (49) we have

$$\begin{aligned} u_{\rm f}^{\pm} &= u_{\rm g}^{\pm} \\ &= \sqrt{\frac{120c}{10a - \Omega(e^{2\sqrt{c}\xi} + e^{-2\sqrt{c}\xi})}} \\ &= \sqrt{\frac{60c}{5a - \Omega\cosh(2\sqrt{c}\xi)}} \\ &= u_{\rm fg}^{\pm} \quad \text{(see (20))}. \end{aligned} \quad (52)$$

Via (52) we obtain property $(2)_{\rm d}$ of the Proposition 1. Hereto, we have completed the derivations for Proposition 1.

3.2. *The derivations to Proposition 2*

In the first integral (39), letting $h = H(\alpha, 0)$, it follows that

$$y = \pm\sqrt{\frac{-ab}{15}}\sqrt{(\alpha^2 - \varphi^2)^2(\varphi^2 + \mu_0)}, \quad (53)$$

where μ_0 and α are given in (21) and (25), respectively. Substituting (53) into the first equation of (38) and integrating it, we get

$$\int_q^{\varphi} \frac{ds}{\sqrt{(\alpha^2 - s^2)^2(s^2 - \mu_0)}} = \sqrt{\frac{-ab}{15}}\xi, \quad (54)$$

where q is an arbitrary constant number.

Completing the integral above and solving the equation for φ, it follows that

$$\varphi = \pm\left(\alpha^2 - \frac{4a_1\mu\, e^{\eta_2\xi}}{(b_1^2 - 4a_1) + \mu^2 e^{2\eta_2\xi} - 2b_1\mu\, e^{\eta_2\xi}}\right)^{\frac{1}{2}}, \quad (55)$$

where α, η_2 are given in (25) and (28) respectively, and $\mu = \mu(q)$ is an arbitrary constant number, and

$$a_1 = \frac{\Delta(\Delta + 5a)}{12a^2b^2}, \quad (56)$$

and

$$b_1 = \frac{4\Delta + 5a}{6ab}. \quad (57)$$

Similar to the derivations for $u_{\rm f}^{\pm}$ and $u_{\rm g}^{\pm}$, we get $u_{\rm h}^{\pm}$ and $u_{\rm i}^{\pm}$ (see (23), (24)) from (55).

When $b = -\frac{5a}{48c}$, it follows that $\alpha = 2\sqrt{\frac{3c}{a}}$, $\delta = 0$, $\omega = 15a$ and $\eta_2 = 2\sqrt{c}$. Therefore, from (23) and (24) it is seen that $u_{\rm h}^{\pm}$ and $u_{\rm i}^{\pm}$ respectively become $u_{\rm h0}^{\mp}$ and $u_{\rm i0}^{\mp}$ (see (29), (30)) when $b = -\frac{5a}{48c}$. Specially, when $a > 0$, $b = -\frac{5a}{48c}$ and $\mu = \frac{48c}{a}$, we have

$$\begin{aligned} u_{\rm h}^{\pm} &= \mp\sqrt{\frac{12\,c}{a(e^{2\sqrt{c}\,\xi} + 1)}} \\ &= \mp\sqrt{\frac{6\,c}{a}(1 - \tanh(\sqrt{c}\,\xi))} \\ &= u_{\rm b}^{\mp} \quad \text{(see (3))}, \end{aligned} \quad (58)$$

and

$$u_i^{\pm} = \mp\sqrt{\frac{12c}{a(e^{-2\sqrt{c}\xi}+1)}} = \mp\sqrt{\frac{6c}{a}(1+\tanh(\sqrt{c}\xi))} = u_a^{\mp} \quad (\text{see } (2)). \tag{59}$$

For other cases of μ, we derive the properties as follows:

(i) When $\mu \neq \pm|\mu_0|$, from (23) and (24) and (29) and (30), we obtain properties $(1°)_a$–$(1°)_e$ and $(2°)_a$ and $(2°)_b$.

(ii) When $(a,b) \in A_2$ and $\mu = |\mu_0| = \frac{5a-2\Delta}{6ab}$, we have

$$\begin{aligned}
u_h^{\pm} &= \pm\frac{\alpha[6ab\mu + (2\Delta - 5a)e^{\eta_2\xi}]}{\sqrt{36a^2b^2\mu^2 - 12ab\mu(4\Delta+5a)e^{\eta_2\xi} + (5a-2\Delta)^2 e^{2\eta_2\xi}}} \\
&= \pm\frac{\alpha(2\Delta-5a)(e^{\eta_2\xi}-1)}{\sqrt{(5a-2\Delta)^2 - 2(5a-2\Delta)(4\Delta+5a)e^{\eta_2\xi} + (5a-2\Delta)^2 e^{2\eta_2\xi}}} \\
&= \pm\frac{\alpha\sqrt{2\Delta-5a}(e^{\eta_2\xi/2} - e^{-\eta_2\xi/2})}{\sqrt{(2\Delta-5a)(e^{-\eta_2\xi}+e^{\eta_2\xi}) + 2(4\Delta+5a)}} \\
&= \pm\frac{2\alpha\sqrt{2\Delta-5a}\sinh\left(\frac{\eta_2\xi}{2}\right)}{\sqrt{2(2\Delta-5a)\cosh(\eta_2\xi) + 8\Delta + 10a}} \\
&= \pm\frac{2\alpha\sqrt{2\Delta-5a}\sinh\left(\frac{\eta_2\xi}{2}\right)}{\sqrt{4(2\Delta-5a)\sinh^2\left(\frac{\eta_2\xi}{2}\right) + 12\Delta}} \\
&= \pm\sqrt{\frac{(5a+\Delta)(5a-2\Delta)}{6ab[3\Delta + (2\Delta-5a)\sinh^2(\eta_1\xi)]}}\sinh(\eta_1\xi) \\
&= u_c^{\pm} \quad (\text{see } (5)),
\end{aligned} \tag{60}$$

and

$$\begin{aligned}
u_i^{\pm} &= \pm\frac{\alpha(6ab\mu + \delta e^{-\eta_2\xi})}{\sqrt{36a^2b^2\mu^2 - 12ab\mu\omega\, e^{-\eta_2\xi} + \delta^2 e^{-2\eta_2\xi}}} \\
&= \pm\frac{\alpha(2\Delta-5a)(e^{-\eta_2\xi}-1)}{\sqrt{(2\Delta-5a)^2 + 2(2\Delta-5a)(4\Delta+5a)e^{-\eta_2\xi} + (2\Delta-5a)^2 e^{-2\eta_2\xi}}} \\
&= \pm\frac{\alpha\sqrt{2\Delta-5a}(e^{-\eta_2\xi/2} - e^{\eta_2\xi/2})}{\sqrt{(2\Delta-5a)(e^{\eta_2\xi}+e^{-\eta_2\xi}) + 2(4\Delta+5a)}} \\
&= \mp\frac{2\alpha\sqrt{2\Delta-5a}\sinh\left(\frac{\eta_2\xi}{2}\right)}{\sqrt{2(2\Delta-5a)\cosh(\eta_2\xi) + 8\Delta + 10a}} \\
&= \mp\sqrt{\frac{(5a+\Delta)(5a-2\Delta)}{6ab[3\Delta + (2\Delta-5a)\sinh^2(\eta_1\xi)]}}\sinh(\eta_1\xi) \\
&= u_c^{\mp} \quad (\text{see } (5)).
\end{aligned} \tag{61}$$

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(iii) When $(a,b) \in A_3$ and $\mu = |\mu_0| = \frac{2\Delta-5a}{6ab} > 0$ (see Lemma 1) which implies $5a - 2\Delta > 0$, we have

$$\begin{aligned} u_{\mathrm{h}}^{\pm} &= \pm\frac{\alpha(2\Delta-5a)(1+\mathrm{e}^{\eta_2\xi})}{\sqrt{(5a-2\Delta)^2+2(5a-2\Delta)(4\Delta+5a)\mathrm{e}^{\eta_2\xi}+(5a-2\Delta)^2\mathrm{e}^{2\eta_2\xi}}} \\ &= \mp\frac{\alpha(5a-2\Delta)(1+\mathrm{e}^{-\eta_2\xi})}{\sqrt{(5a-2\Delta)}\sqrt{(5a-2\Delta)(1+\mathrm{e}^{2\eta_2\xi})+2(4\Delta+5a)\mathrm{e}^{\eta_2\xi}}} \\ &= \mp\frac{\alpha\sqrt{2(5a-2\Delta)}\cosh\left(\frac{\eta_2\xi}{2}\right)}{\sqrt{(5a-2\Delta)\cosh(\eta_2\xi)+4\Delta+5a}} \\ &= u_{\mathrm{j}}^{\mp} \quad \text{(see (32))}, \end{aligned} \tag{62}$$

and

$$\begin{aligned} u_{\mathrm{i}}^{\pm} &= \pm\frac{\alpha(2\Delta-5a)(1+\mathrm{e}^{-\eta_2\xi})}{\sqrt{(5a-2\Delta)^2+2(5a-2\Delta)(4\Delta+5a)\mathrm{e}^{-\eta_2\xi}+(5a-2\Delta)^2\mathrm{e}^{-2\eta_2\xi}}} \\ &= \mp\frac{\alpha\sqrt{2(5a-2\Delta)}\cosh\left(\frac{\eta_2\xi}{2}\right)}{\sqrt{(5a-2\Delta)\cosh(\eta_2\xi)+(4\Delta+5a)}} \\ &= \mp\sqrt{\frac{2(5a-2\Delta)}{(5a+4\Delta)+(5a-2\Delta)\cosh(\eta_2\xi)}}\,\alpha\cosh\left(\frac{\eta_2\xi}{2}\right) \\ &= u_{\mathrm{j}}^{\mp} \quad \text{(see (32))}. \end{aligned} \tag{63}$$

(iv) When $(a,b) \in A_2$ and $\mu = -|\mu_0| = \frac{2\Delta-5a}{6ab} < 0$ (see (22)) which implies $2\Delta - 5a > 0$, we have

$$\begin{aligned} u_{\mathrm{h}}^{\pm} &= \pm\frac{\alpha(6ab\mu+\delta\,\mathrm{e}^{\eta_2\xi})}{\sqrt{36a^2b^2\mu^2-12ab\mu\omega\,\mathrm{e}^{\eta_2\xi}+\delta^2\,\mathrm{e}^{2\eta_2\xi}}} \\ &= \pm\frac{\alpha(2\Delta-5a)(1+\mathrm{e}^{\eta_2\xi})}{\sqrt{(2\Delta-5a)^2-2(2\Delta-5a)(4\Delta+5a)\mathrm{e}^{\eta_2\xi}+(2\Delta-5a)^2\mathrm{e}^{2\eta_2\xi}}} \\ &= \pm\frac{\alpha\sqrt{2(2\Delta-5a)}\cosh\left(\frac{\eta_2\xi}{2}\right)}{\sqrt{(2\Delta-5a)\cosh(\eta_2\xi)-(4\Delta+5a)}} \\ &= \pm\sqrt{\frac{2(5a-2\Delta)}{(5a+4\Delta)+(5a-2\Delta)\cosh(\eta_2\xi)}}\,\alpha\cosh\left(\frac{\eta_2\xi}{2}\right) \\ &= u_{\mathrm{j}}^{\pm} \quad \text{(see (32))}, \end{aligned} \tag{64}$$

and

$$\begin{aligned} u_{\mathrm{i}}^{\pm} &= \pm\frac{\alpha(6ab\mu+\delta\,\mathrm{e}^{-\eta_2\xi})}{\sqrt{36a^2b^2\mu^2-12ab\mu\omega\,\mathrm{e}^{-\eta_2\xi}+\delta^2\mathrm{e}^{-2\eta_2\xi}}} \\ &= \pm\frac{\alpha(2\Delta-5a)(1+\mathrm{e}^{-\eta_2\xi})}{\sqrt{(2\Delta-5a)^2-2(2\Delta-5a)(4\Delta+5a)\,\mathrm{e}^{-\eta_2\xi}+(2\Delta-5a)^2\,\mathrm{e}^{-2\eta_2\xi}}} \end{aligned}$$

$$= \pm \frac{\alpha\sqrt{2(2\Delta - 5a)}\cosh\left(\frac{\eta_2\xi}{2}\right)}{\sqrt{(2\Delta - 5a)\cosh(\eta_2\xi) - (4\Delta + 5a)}}$$

$$= u_{\mathrm{j}}^{\pm} \quad \text{(see (32))}. \tag{65}$$

(v) When $(a, b) \in A_3$ and $\mu = -|\mu_0| = \frac{5a-2\Delta}{6ab} < 0$ (see (21) and (22)) which implies $5a - 2\Delta > 0$, we have

$$u_{\mathrm{h}}^{\pm} = \pm \frac{\alpha[6ab\mu + (2\Delta - 5a)\mathrm{e}^{\eta_2\xi}]}{\sqrt{36a^2b^2\mu^2 - 12ab\mu(4\Delta + 5a)\mathrm{e}^{\eta_2\xi} + (2\Delta - 5a)^2\mathrm{e}^{2\eta_2\xi}}}$$

$$= \pm \frac{\alpha(5a - 2\Delta)(1 - \mathrm{e}^{\eta_2\xi})}{\sqrt{(5a - 2\Delta)^2 - 2(5a - 2\Delta)(4\Delta + 5a)\mathrm{e}^{\eta_2\xi} + (5a - 2\Delta)^2\mathrm{e}^{2\eta_2\xi}}}$$

$$= \mp \frac{2\alpha\sqrt{5a - 2\Delta}\sinh\left(\frac{\eta_2\xi}{2}\right)}{\sqrt{(5a - 2\Delta)(\mathrm{e}^{-\eta_2\xi} + \mathrm{e}^{\eta_2\xi}) - 2(4\Delta + 5a)}}$$

$$= \mp \frac{2\alpha\sqrt{5a - 2\Delta}\sinh\left(\frac{\eta_2\xi}{2}\right)}{\sqrt{2(5a - 2\Delta)\cosh(\eta_2\xi) - 2(4\Delta + 5a)}}$$

$$= \mp \frac{2\alpha\sqrt{5a - 2\Delta}\sinh\left(\frac{\eta_2\xi}{2}\right)}{\sqrt{2(5a - 2\Delta) - 2(4\Delta + 5a) + 4(5a - 2\Delta)\sinh^2\left(\frac{\eta_2\xi}{2}\right)}}$$

$$= \mp \sqrt{\frac{5a - 2\Delta}{-3\Delta + (5a - 2\Delta)\sinh^2(\eta_1\xi)}}\,\alpha\sinh(\eta_1\xi)$$

$$= \mp \sqrt{\frac{(5a + \Delta)(5a - 2\Delta)}{6ab[3\Delta + (2\Delta - 5a)\sinh^2(\eta_1\xi)]}}\sinh(\eta_1\xi)$$

$$= u_{\mathrm{c}}^{\mp} \quad \text{(see (5))}, \tag{66}$$

and

$$u_{\mathrm{i}}^{\pm} = \pm \frac{\alpha[6ab\mu + (2\Delta - 5a)\mathrm{e}^{-\eta_2\xi}]}{\sqrt{36a^2b^2\mu^2 - 12ab\mu(4\Delta + 5a)\mathrm{e}^{-\eta_2\xi} + (2\Delta - 5a)^2\mathrm{e}^{-2\eta_2\xi}}}$$

$$= \pm \frac{\alpha(5a - 2\Delta)(1 - \mathrm{e}^{-\eta_2\xi})}{\sqrt{(5a - 2\Delta)^2 - 2(5a - 2\Delta)(4\Delta + 5a)\mathrm{e}^{-\eta_2\xi} + (2\Delta - 5a)^2\mathrm{e}^{-2\eta_2\xi}}}$$

$$= \pm \frac{2\alpha\sqrt{5a - 2\Delta}\sinh\left(\frac{\eta_2\xi}{2}\right)}{\sqrt{(5a - 2\Delta)(\mathrm{e}^{\eta_2\xi} + \mathrm{e}^{-\eta_2\xi}) - 2(4\Delta + 5a)}}$$

$$= \pm \frac{2\alpha\sqrt{5a - 2\Delta}\sinh\left(\frac{\eta_2\xi}{2}\right)}{\sqrt{2(5a - 2\Delta)\cosh(\eta_2\xi) - 2(4\Delta + 5a)}}$$

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$$= \pm \frac{2\alpha\sqrt{5a-2\Delta}\sinh(\eta_1\xi)}{\sqrt{2(5a-2\Delta) - 2(4\Delta+5a) + 4(5a-2\Delta)\sinh^2\left(\frac{\eta_2\xi}{2}\right)}}$$

$$= \pm\sqrt{\frac{(5a-2\Delta)(5a+\Delta)}{6ab[3\Delta + (2\Delta-5a)\sinh^2(\eta_1\xi)]}}\sinh(\eta_1\xi) = u_{\rm c}^{\pm} \quad \text{(see (5))}. \tag{67}$$

Hereto, we have finished the derivations for Proposition 2.

3.3. *The derivations to Proposition 3*

Firstly, when (a,b) belongs to one of A_2, A_3 and l_1, in the first integral (39), let $h = H(\beta, 0)$. Thus we have

$$y = \pm\sqrt{\frac{-ab}{15}}\sqrt{(\varphi^2-\beta^2)^2(\varphi^2+\gamma)}, \tag{68}$$

where

$$\gamma = \frac{2\Delta+5a}{5ab}. \tag{69}$$

Similarly, we get

$$\varphi = \pm\sqrt{\frac{\beta^2\gamma[1-\cos(\eta_2\xi)]}{\sigma-\gamma\cos(\eta_2\xi)}}, \tag{70}$$

where

$$\sigma = \frac{5a-4\Delta}{6ab}. \tag{71}$$

From (70) we get $u_{\rm k}^{\pm}$ as (33).

Secondly, when $b = -\frac{5a}{36c}$, it follows that

$$\alpha = \beta = \sqrt{\frac{6c}{a}} \quad \text{(see (25) and (41))}. \tag{72}$$

In the first integral (39), letting $b = -\frac{5a}{36c}$ and $h = H\left(\sqrt{\frac{6c}{a}}, 0\right)$, we have

$$y = \pm\frac{a}{6\sqrt{3c}}\left(\varphi^2 - \frac{6c}{a}\right)^{3/2}. \tag{73}$$

Similarly, we have

$$\varphi = \frac{\sqrt{6}c\xi}{\sqrt{a(-3+c\xi^2)}}, \tag{74}$$

which yields $u_{\rm l}^{\pm}$ as (35).

Thirdly, from (33) it is easy to see that property (iii) is true.

Finally, we derive property (iv). Note that

$$\lim_{b\to -\frac{5a}{36c}+0}\Delta = \lim_{b\to -\frac{5a}{36c}+0}\sqrt{5a(5a+36bc)} = 0, \tag{75}$$

and

$$\cos(\eta_3\xi) = 1 - \frac{\eta_3^2\xi^2}{2} + \frac{\eta_3^4\xi^4}{4!} + \cdots = 1 - \frac{\Delta(\Delta-5a)\xi^2}{90ab} + O(\Delta^2). \tag{76}$$

Thus we have

$$u_{\rm k}^{\pm} = \pm\sqrt{\frac{(\Delta-5a)(2\Delta+5a)[1-\cos(\eta_3\xi)]}{6ab[5a-4\Delta-(5a+2\Delta)\cos(\eta_3\xi)]}} = \pm\sqrt{\frac{\frac{(\Delta-5a)^2(2\Delta+5a)\Delta\xi^2}{(90ab)+O(\Delta^2)}}{6ab\left[\frac{-6\Delta+(5a+2\Delta)(\Delta-5a)\Delta\xi^2}{(90ab)+O(\Delta^2)}\right]}}$$

$$= \pm\sqrt{\frac{(\Delta-5a)^2(2\Delta+5a)\xi^2+O(\Delta)}{6ab[-540ab+(5a+2\Delta)(\Delta-5a)\xi^2+O(\Delta)]}}. \tag{77}$$

Furthermore, we get

$$\lim_{b\to -\frac{5a}{36c}+0} u_{\rm k}^{\pm} = \lim_{b\to -\frac{5a}{36c}+0} \pm\sqrt{\frac{(\Delta-5a)^2(2\Delta+5a)\xi^2+O(\Delta)}{6ab[-540ab+(5a+2\Delta)(\Delta-5a)\xi^2+O(\Delta)]}}$$

$$= \sqrt{\frac{6\times 125c^2a^3\xi^2}{-5a^2(75a^2-25a^2c^2\xi^2)}} = \frac{\sqrt{6}c\xi}{\sqrt{a(-3+c\xi^2)}} = u_{\rm l}^{\pm} \quad \text{(see (35))}. \tag{78}$$

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Hereto, we have completed the derivations for our main results.

4. Conclusions

In this paper, we have investigated the explicit expressions of the nonlinear waves and their bifurcations in Eq. (1).

Firstly, we obtained four types of new expressions. The first type includes four common explicit expressions of the symmetric solitary waves, the low-kink waves and the blow-up waves. For the first type of common explicit expressions, see (14) and (15). For the symmetric solitary waves, see Figs. 2(a)–2(c). For the low-kink waves, see Fig. 2(d) or 3(d). For the blow-up waves, see Figs. 3(a)–3(c) or 5(a)–5(d). The second type is composed of four common explicit expressions of the tall-kink waves, the low-kink waves, the anti-symmetric solitary waves and the blow-up waves. For the second type of common explicit expressions, see (23) and (24). For the tall-kink waves, see Figs. 7(a)–7(c). For the low-kink waves, see Fig. 7(d) or 8(d). For the anti-symmetric solitary waves, see Figs. 8(a)–8(c). For the blow-up waves, see Figs. 11(a)–11(d) or 12(a)–12(c). The third type is made of two trigonometric expressions of the periodic-blow-up waves. For the trigonometric expressions, see (33). For the periodic-blow-up waves, see Figs. 13(a)–13(c). The fourth type is composed of two fractional expressions of the 1-blow-up waves. For the fractional expressions, see (35). For the 1-blow-up waves, see Fig. 13(d).

Secondly, we revealed two kinds of new bifurcation phenomena. The first phenomenon is that the low-kink waves can be bifurcated from four types of nonlinear waves, the symmetric solitary waves (see Fig. 2), the blow-up waves (see Fig. 3), the tall-kink waves (see Fig. 7), and the anti-symmetric solitary waves (see Fig. 8). The second phenomenon is that the 1-blow-up waves can be bifurcated from the periodic-blow-up waves (see Fig. 13).

Thirdly, we have shown that many previous results are some special cases. For instance, $u_{\rm a}^{\pm}$ and $u_{\rm b}^{\pm}$ are included in $u_{\rm f}^{\pm}$, $u_{\rm g}^{\pm}$, $u_{\rm h}^{\pm}$ and $u_{\rm i}^{\pm}$ (see (2), (3), (14), (15), (23), (24), $(1)_{\rm a}$ and $(1^{\circ})_{\rm a}$). $u_{\rm c}^{\pm}$ are included in $u_{\rm h}^{\pm}$, $u_{\rm i}^{\pm}$ (see (23), (24) and (31)). $u_{\rm d}^{\pm}$ and $u_{\rm e}^{\pm}$ are included in $u_{\rm f}^{\pm}$, $u_{\rm g}^{\pm}$ (see (8), (9), (14), (15) and $(1)_{\rm d}$).

Finally, we have pointed that the nonlinear wave solutions given in the literature cannot bifurcate out a nontrivial solution (see Figs. 4 and 9). We have also verified and confirmed these solutions by using the software Mathematica. For example, for $u_{\rm l}^{+}$, the orders are as follows:

$$a = 12;$$

$$c = 2;$$

$$b = -\frac{5a}{36c};$$

$$\xi = x - ct;$$

$$u = \frac{\sqrt{6}c\xi}{\sqrt{a(c\xi^2 - 3)}};$$

Simplify $[D[u,t]+a(1+bu^2)u^2D[u,x]+D[u,\{x,3\}]]$.

References

Dey, B. [1986] "Domain wall solutions of KdV like equations with higher order nonlinearity," *J. Phys. A: Math. Gen.* **19**, L9–L12.

Dey, B. [1988] *KdV Like Equations with Domain Wall Solutions and Their Hamiltonians*, *Solitons*, Springer Series on Nonlinear Dynamics (Springer, Berlin-NY), pp. 188–194.

Fu, Z. T., Zhang, L., Liu, S. D. *et al.* [2004a] "Fractional transformation and new solutions to mKdV equation," *Phys. Lett. A* **325**, 363–369.

Fu, Z. T., Liu, S. D. & Liu, S. K. [2004b] "New solutions to mKdV equation," *Phys. Lett. A* **326**, 364–374.

Gardner, L. R. T., Gardner, G. A. & Geyikli, T. [1995] "Solitary wave solutions of the mKdV equation," *Comput. Meth. Appl. Mech. Engrg.* **124**, 321–333.

Gorsky, J. & Himonas, A. [2005] "Construction of non-analytic solutions for the generalized KdV equation," *J. Math. Anal. Appl.* **303**, 522–529.

Grimshaw, R., Pelinovsky, D., Pelinovsky, E. *et al.* [2002] "Generation of large-amplitude solitons in the extended Korteweg–de Vries equation," *Chaos* **12**, 1070–1076.

Kevrekidis, P. G., Khare, A. & Saxena, A. *et al.* [2004] "On some classes of mKdV periodic solutions," *J. Phys. A: Math. Gen.* **37**, 10959–10965.

Kudryashov, N. A. & Sinrlshchiov, D. I. [2011] "A note on the lie symmetry analysis and exact solutions for the extended mKdV equation," *Acta. Appl. Math.* **113**, 41–44.

Lakshmanan, M. & Tamizhmani, K. M. [1985] "Lie–Backlund symmetries of certain nonlinear evolution equations under perturbation around their equations," *J. Math. Phys.* **26**, 1189–1200.

Li, B., Chen, Y. & Zhang, H. Q. [2003] "Explicit exact solutions for compound KdV-type and compound KdV–Burgers-type equations with nonlinear terms of any order," *Chaos Solit. Fract.* **15**, 647–654.

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Li, J. B. & Chen, G. R. [2005a] "Bifurcations of traveling wave solutions for four classes of nonlinear wave equations," *Int. J. Bifurcation and Chaos* **15**, 3973–3998.

Li, J. B. & Chen, G. R. [2005b] "Bifurcations of traveling wave and breather solutions of a general class of nonlinear wave equations," *Int. J. Bifurcation and Chaos* **15**, 2913–2926.

Li, J. B., Zhao, X. H. & Chen, G. R. [2009a] "Breaking wave solutions to the second class of singular nonlinear traveling wave equations," *Int. J. Bifurcation and Chaos* **19**, 1289–1306.

Li, J. B., Zhang, Y. & Chen, G. R. [2009b] "Exact solutions and their dynamics of traveling waves in three typical nonlinear wave equations," *Int. J. Bifurcation and Chaos* **19**, 2249–2266.

Li, J. B. & Chen, G. R. [2010] "On nonlinear wave equations with breaking loop-solutions," *Int. J. Bifurcation and Chaos* **20**, 519–537.

Liu, R. [2010a] "Some new results on explicit traveling wave solutions of $k(m, n)$ equation," *Discr. Cont. Dyn. Syst. B* **13**, 633–646.

Liu, R. [2010b] "Coexistence of multifarious exact nonlinear wave solutions for generalized b-equation," *Int. J. Bifurcation and Chaos* **20**, 3193–3208.

Liu, Z. R. & Li, J. B. [2002] "Bifurcations of solitary waves and domain wall waves for KdV-like equation with higher order nonlinearity," *Int. J. Bifurcation and Chaos* **12**, 397–407.

Liu, Z. R. & Yang, C. X. [2002] "The application of bifurcation method to a higher order KdV equation," *J. Math. Anal. Appl.* **275**, 1–12.

Liu, Z. R. & Liang, Y. [2011] "The explicit nonlinear wave solutions and their bifurcations of the generalized Camassa–Holm equation," *Int. J. Bifurcation and Chaos* **21**, 3119–3136.

Miura, R. M., Gardner, C. S. & Kruskal, M. D. [1968] "Korteweg–de Vries equation and generalizations. II. Existence of conservation laws and constants of motion," *J. Math. Phys.* **9**, 1204–1209.

Miura, R. M. [1976] "The Korteweg–de Vries equation: A survey of results," *SIAM Rev.* **18**, 412–459.

Smyth, N. F. & Worthy, A. L. [1995] "Solitary wave evolution for mKdV equations," *Wave Motion* **21**, 263–275.

Tang, M. Y., Wang, R. Q. & Jing, Z. J. [2002] "Solitary waves and their bifurcations of KdV like equation with higher order nonlinearity," *Sci. China* (*Ser. A*) **45**, 1255–1267.

Zhang, W. G., Chang, Q. S. & Jiang, B. G. [2002] "Explicit exact solitary-wave solutions for compound KdV-type and compound KdV–Burgers-type equations with nonlinear terms of any order," *Chaos Solit. Fract.* **13**, 311–319.