Journal of Dynamical and Control Systems, Vol. 18, No. 3, July 2012, 355–378 (©2012)
DOI: 10.1007/s10883-012-9148-1

EXISTENCE AND CRITICAL SPEED OF TRAVELING WAVE FRONTS IN A MODIFIED VECTOR DISEASE MODEL WITH DISTRIBUTED DELAY

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ABSTRACT. In this paper, we consider a modified disease model with distributed delay. The existence of traveling wave fronts connecting the zero equilibrium and the positive equilibrium is established by using an iterative technique and a nonstandard ordering for the set of profiles of the corresponding wave system. We also study the critical wave speed and give a detailed analysis on its location and asymptotic behavior with respect to the time delay. Our work extends some previous results.

1. INTRODUCTION

Traveling wave solutions have been widely studied for nonlinear reaction-diffusion equations modeling a variety of physical and biological phenomena (see, e.g., [1–5]), for time-delayed reaction-diffusion equations (see, e.g., [6–9]), and for nonlocal delayed reaction-diffusion equations (see, e.g., [10–12]).

Recently, Zhang [13] studied the modified host-vector disease model

$$\begin{aligned}\frac{\partial u(t,x)}{\partial t} &= \frac{\partial^2 u(t,x)}{\partial x^2} - au(t,x) + b[1-u(t,x)]\\ &\quad\times \int_{-\infty}^{t}\int_{-\infty}^{\infty} F(t,s,x,y)u(s,y)\mathrm{d}y\mathrm{d}s + ru(t,x)[1-u(t,x)], \quad (1)\end{aligned}$$

where the function $u(t,x)$ denotes the normalized spatial density of infectious host at time $t > 0$ and spatial location $x \in \mathbb{R}$, $a > 0$ is the cure/recovery rate of the infected host, $b > 0$ is the host-vector contact rate, $F(t,s,x,y)$ is the convolution kernel, which is positive, continuous in its variables $(t,s) \in D := \{(t,s) : t \geq 0, -\infty < s \leq t\}$, and Borel measurable in its variables $x, y \in \mathbb{R}$, $r \geq 0$ denotes the susceptible-infected host

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2000 *Mathematics Subject Classification.* 34K10, 35K57, 74H20, 92D30.

Key words and phrases. Traveling wave fronts, upper and lower solution, iterative techniques, critical wave speed.

contact rate. For $F(t,s,x,y) = F(t-s, x-y)$ with

$$F(t,x) = \frac{t}{\tau^2\sqrt{4\pi t}\,\exp\left(\dfrac{t}{\tau} + \dfrac{x^2}{4t}\right)},$$

Zhang [13] proved the existence of traveling wave fronts by using the geometric singular perturbation theory. For $F(t,s,x,y) = \delta(x-y)G(t-s)$ with $\delta(x)$ being the Dirac δ-function and

$$G(t) = \frac{t}{\tau^2 \exp\left(t/\tau\right)},$$

Huang and Huo [14] proved the existence of traveling wave fronts by using the theory developed in [15]. They also showed that the critical wave speed decreases as the time delay increases.

If $r = 0$, then Eq. (1) reduces to

$$\begin{aligned}\frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2} - au(t,x) + b[1 - u(t,x)] \\ \times \int_{-\infty}^{t}\int_{-\infty}^{\infty} F(t,s,x,y)u(s,y)\mathrm{d}y\mathrm{d}s, \quad (2)\end{aligned}$$

which has been considered by Ruan and Xiao [16]. The global stability of steady states for model (2) was obtained in [16] when $x, y \in \Omega \subset \mathbb{R}$ and Ω is bounded. In the case $F(t,s,x,y) = \delta(x-y)G(t-s)$ and

$$G(t) = \frac{t}{\tau^2 \exp\left(t/\tau\right)},$$

Ruan and Xiao [16] also showed that for any $c_0 \geq 2\sqrt{b-a}$, there exists a small number $\tau_0 = \tau_0(c_0) > 0$ such that for any $\tau \in [0, \tau_0]$, the model (2) admits a traveling wave front connecting two equilibria 0 and $\dfrac{b-a}{b}$ with the wave speed $c = c(\tau)$ close to c_0. Lv and Wang [17] studied the existence, uniqueness and asymptotic behavior of traveling wave fronts for Eq. (2) with $F(t,s,x,y) = F(t-s, x-y)$ and

$$F(t,x) = \frac{1}{\tau\sqrt{4\pi t}\,\exp\left(\dfrac{t}{\tau} + \dfrac{x^2}{4t}\right)}.$$

For a large class of delayed reaction-diffusion equations, Lv and Wang [18] proved that the traveling wave fronts are exponentially stable to perturbations in some exponentially weighted L^∞ spaces and obtained the time decay rates by the weighted energy method, which is recently developed by Mei et al [19, 20].

If $F(t,s,x,y) = \delta(x-y)\delta(t-s-\tau)$, then (2) becomes the model

$$\frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2} - au(t,x) + b[1-u(t,x)]u(t-\tau,x). \tag{3}$$

Lin and Hong [21] showed that there exist a critical wave speed c^* and a delay parametric value $\tau^*(c)$ $(c > c^*)$ such that for any $\tau < \tau^*(c)$, Eq. (3) has a traveling wave front connecting the two equilibria 0 and $\dfrac{b-a}{b}$.

In [22] a time delay reaction diffusion equation with nonlocality for the population dynamics of single species was studied. For the critical wave speed, a detailed analysis was given on its location and asymptotic behavior with respect to the parameters of the diffusion rate and mature age, respectively. Recently, Wei et al [23] also gave a remark on critical wave speed for Nicholson's blowflies equation with diffusion.

In this paper, we consider model (1) with $F(t,s,x,y) = \delta(x-y)G(t-s)$ and

$$G(t) = \frac{t^n}{n!\ \tau^{n+1} \exp(t/\tau)} \qquad \text{or} \qquad G(t) = \frac{\sin(t/\tau) + \cos(t/\tau)}{\tau \exp(t/\tau)}.$$

Firstly, we prove the existence of traveling wave fronts by using the theory developed in [15]. Secondly, for the critical wave speed, we give a detailed analysis on its location and asymptotic behavior with respect to the time delay. Our work extends some previous results in [14].

The rest of this paper is organized as follows. In Sec. 2, we recall the main theorem from [15] that will be employed in this paper. Sec. 3 is devoted to the proof of the existence of traveling wave fronts of (1) with two different delay kernels by the theory developed in [15]. In Sec. 4, we give some analysis on the critical wave speed.

2. Preliminaries

In [15] the nonlinear integro-differential equation with diffusion and the distributed delay of a general view was investigated. The existence theorem of traveling wave fronts was formulated. The traveling wave fronts can be estimated by upper solution and lower solution respectively. The theory developed in [15] is quite general and can be extended to coupled systems as well as scalar equations.

Consider the following reaction-diffusion equation with nonlocal delays:

$$\frac{\partial u(t,x)}{\partial t} = D\frac{\partial^2 u(t,x)}{\partial x^2} + f(u(t,x),(g*u)(t,x)), \tag{4}$$

where $t \geq 0$, $x \in \mathbb{R}$, $D > 0$, $f \in C(\mathbb{R}^2, \mathbb{R})$, and

$$(g*u)(t,x) = \int_{-\infty}^{t}\int_{-\infty}^{\infty} g(t-s,x-y)u(s,y)\mathrm{d}y\mathrm{d}s;$$

the kernel $g(t,x)$ is any integrable nonnegative function satisfying

$$g(t,x)=g(t,-x),$$
$$\int_0^\infty\int_{-\infty}^\infty g(s,y)\mathrm{d}y\mathrm{d}s=1;$$

and

$$\int_{-\infty}^\infty g(t,x)\mathrm{d}x$$

is uniformly convergent for $t\in[0,a]$, $a>0$. In other words, $\forall\,\varepsilon>0$, $\exists\,M>0$, such that

$$\int_M^\infty g(t,x)\mathrm{d}x<\varepsilon$$

for any $t\in[0,a]$.

Letting $u(t,x)=\varphi(\xi)$ with $\xi=x+ct$, then Eq. (4) becomes

$$-D\varphi''(\xi)+c\varphi'(\xi)=f(\varphi(\xi),(g*\varphi)(\xi)), \tag{5}$$

where

$$(g*\varphi)(\xi)=\int_0^\infty\int_{-\infty}^\infty g(s,y)\varphi(\xi-y-cs)\mathrm{d}y\mathrm{d}s.$$

Let

$$BC(\mathbb{R},\mathbb{R})=\left\{\varphi\in C(\mathbb{R},\mathbb{R}):\sup_{\xi\in\mathbb{R}}|\varphi(\xi)|<\infty\right\},$$

and

$$BC^2(\mathbb{R},\mathbb{R})=\{\varphi\in BC(\mathbb{R},\mathbb{R}):\varphi',\varphi''\in BC(\mathbb{R},\mathbb{R})\}.$$

A traveling wave front of (4) with wave speed $c>0$ is a function $\varphi(x+ct)$, $\varphi\in BC^2(\mathbb{R},\mathbb{R})$, satisfying (5) and the following boundary conditions:

$$\lim_{\xi\to-\infty}\varphi(\xi)=0,\quad \lim_{\xi\to\infty}\varphi(\xi)=K>0. \tag{6}$$

We will employ the following hypotheses:
(H_1) $f(u,u)=0$ when $u=0$ or K;
(H_2) There exists a constant $\beta>0$ such that

$$f(\phi(\xi),(g*\phi)(\xi))+\beta\phi(\xi)\geq f(\psi(\xi),(g*\psi)(\xi))+\beta\psi(\xi),$$

where $\phi,\psi\in C(\mathbb{R},\mathbb{R})$ satisfy $0\leq\psi(\xi)\leq\phi(\xi)\leq K$ for $\xi\in\mathbb{R}$.

Next we define upper solution and lower solutions for (5).

Definition 1. A continuous function φ is called an upper solution of (5) if φ' and φ'' exist almost everywhere and are essentially bounded on $\mathbb{R}$, and φ satisfies the inequality

$$-D\varphi''(\xi)+c\varphi'(\xi)\geq f(\varphi(\xi),(g*\varphi)(\xi)),\ \text{a.e. on }\mathbb{R}. \tag{7}$$

A lower solution of (5) is defined similarly by reversing the inequality sign in (7).

Let

$$BC[0,K] = \{\varphi \in BC(\mathbb{R},\mathbb{R}) : 0 \le \varphi(\xi) \le K\},$$
$$Y = \{\varphi \in BC(\mathbb{R},\mathbb{R}) : \varphi', \varphi'' \in L^\infty(\mathbb{R},\mathbb{R})\},$$

and

$$\Gamma = \left\{\varphi \in Y : \lim_{\xi\to-\infty} \varphi(\xi) \in [0,K), \lim_{\xi\to\infty} \varphi(\xi) = K; \varphi \text{ is nondecreasing in } \mathbb{R}\right\}.$$

Define an operator $F : BC[0,K] \to BC(\mathbb{R},\mathbb{R})$ by

$$(F\varphi)(\xi) = f(\varphi(\xi), (g * \varphi)(\xi)), \quad \xi \in \mathbb{R}.$$

Now, we rewrite Corollary 4.9 in [15] as follows.

Lemma 2 ([15]). *Assume that* (H_1) *and* (H_2) *hold, and there exists some* δ *such that* $f(u,u) \ne 0$ *for* $0 < \delta \le u < K$. *Also assume that* ϕ *and* ψ, *where* $\phi \in \Gamma$ *with* $\lim_{\xi\to-\infty} \phi(\xi) = 0$, *and* $\psi \in BC[0,K] \cap Y$ *with* $\delta \le \sup_{\xi\in\mathbb{R}} \psi(\xi)$, $\psi \le \phi$ *are upper and lower solutions of* (5), *respectively, then Eq.* (4) *has a traveling wave solution* φ *such that* (6) *holds.*

3. Existence of traveling wave fronts

In this section, we derive the sufficient condition for the existence of traveling wave fronts of Eq. (1) with two different delay kernels.

3.1. The case of $F(t,s,x,y) = \delta(x-y)\dfrac{(t-s)^n}{n!\ \tau^{n+1}}\mathrm{e}^{-\frac{t-s}{\tau}}$, with $\tau > 0$ and $n \in \mathbb{N}$.

In this case, Eq. (1) becomes

$$\frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2} - au(t,x) + b[1-u(t,x)] \times \int_{-\infty}^{t} \frac{(t-s)^n}{n!\ \tau^{n+1}}\mathrm{e}^{-\frac{t-s}{\tau}} u(s,x)\mathrm{d}s + ru(t,x)[1-u(t,x)], \quad (8)$$

where $\tau > 0$ measures the delay. It is easy to know that Eq. (8) has two nonnegative equilibrium points 0 and $\dfrac{r+b-a}{r+b}$, if $r+b > a$. Converting of Eq. (8) into the traveling wave form with $u(t,x) = \varphi(\xi)$ and $\xi = x + ct$, we obtain

$$\varphi''(\xi) - c\varphi'(\xi) - a\varphi(\xi) + b(1-\varphi(\xi)) \int_0^\infty \frac{s^n}{n!\tau^{n+1}}\mathrm{e}^{-\frac{s}{\tau}}\varphi(\xi - cs)\mathrm{d}s + r\varphi(\xi)(1-\varphi(\xi)) = 0. \quad (9)$$

We seek a solution of this equation satisfying the asymptotic boundary conditions

$$\lim_{\xi\to-\infty} \varphi(\xi) = 0, \quad \lim_{\xi\to\infty} \varphi(\xi) = \frac{r+b-a}{r+b}. \quad (10)$$

For convenience, throughout this section, we set

$$K = \frac{r+b-a}{r+b} < 1,$$

and

$$f(\varphi(\xi), (g*\varphi)(\xi)) = -a\varphi(\xi) + b(1-\varphi(\xi))(g*\varphi)(\xi) + r\varphi(\xi)(1-\varphi(\xi)),$$

where

$$(g*\varphi)(\xi) = \int_0^\infty \frac{s^n}{n!\tau^{n+1}} \mathrm{e}^{-\frac{s}{\tau}} \varphi(\xi - cs)\mathrm{d}s.$$

In the following, Lemma 2 is applied to obtain the existence of traveling wave fronts for Eq. (8).

Obviously, the function $f(\varphi(\xi), (g*\varphi)(\xi))$ satisfies the hypothesis (H_1). Next, we prove that $f(\varphi(\xi), (g*\varphi)(\xi))$ satisfies (H_2).

Lemma 3. *The function $f(\varphi(\xi), (g*\varphi)(\xi))$ satisfies (H_2).*

Proof. Fix $\beta > r+b+a$. For any $\phi, \psi \in C(\mathbb{R}, \mathbb{R})$ and satisfying the inequalities $0 \le \psi(\xi) \le \phi(\xi) \le K < 1$, we have

$$\begin{aligned}
&f(\phi(\xi), (g*\phi)(\xi)) - f(\psi(\xi), (g*\psi)(\xi)) \\
&\quad = (-a\phi(\xi) + b(1-\phi(\xi))(g*\phi)(\xi) + r\phi(\xi)(1-\phi(\xi))) \\
&\qquad - (-a\psi(\xi) + b(1-\psi(\xi))(g*\psi)(\xi) + r\psi(\xi)(1-\psi(\xi))) \\
&\quad = (r-a)\left(\phi(\xi) - \psi(\xi)\right) + b\left((g*\phi)(\xi) - (g*\psi)(\xi)\right) \\
&\qquad - b\left(\phi(\xi)(g*\phi)(\xi) - \psi(\xi)(g*\psi)(\xi)\right) - r\left(\phi^2(\xi) - \psi^2(\xi)\right) \\
&\quad = (r-a)\left(\phi(\xi) - \psi(\xi)\right) + b(1-\phi(\xi))\left((g*\phi)(\xi) - (g*\psi)(\xi)\right) \\
&\qquad - b\left(\phi(\xi) - \psi(\xi)\right)(g*\psi)(\xi) - r\left(\phi(\xi) + \psi(\xi)\right)\left(\phi(\xi) - \psi(\xi)\right) \\
&\quad \ge (r-a)\left(\phi(\xi) - \psi(\xi)\right) - b\left(\phi(\xi) - \psi(\xi)\right) - 2r\left(\phi(\xi) - \psi(\xi)\right) \\
&\quad = -(r+b+a)\left(\phi(\xi) - \psi(\xi)\right) \\
&\quad > -\beta\left(\phi(\xi) - \psi(\xi)\right).
\end{aligned}$$

Therefore, $f(\varphi(\xi), (g*\varphi)(\xi))$ satisfies (H_2). □

Now, we construct an upper solution and a lower solution for Eq. (9) to satisfy the assumptions in Lemma 2. Introduce the notation

$$\Delta_1(c, \lambda) = \lambda^2 - c\lambda + (r-a) + \frac{b}{(1+\lambda c\tau)^{n+1}}. \tag{11}$$

Since

$$\Delta_1(0, \lambda) = \lambda^2 + r + b - a \neq 0,$$

we know $\Delta_1(0, \lambda) = 0$ has no real positive roots. By the continuity, for sufficiently small $c > 0$, $\Delta_1(c, \lambda) = 0$ still has no real positive roots. For any $c > 0$, we have

$$\Delta_1(c, 0) = r + b - a > 0,$$

$$\lim_{\lambda\to\infty} \Delta_1(c,\lambda) = \infty,$$

$$\frac{\partial}{\partial c}\Delta_1(c,\lambda) = -\lambda - \frac{(n+1)b\lambda\tau}{(1+\lambda c\tau)^{n+2}} < 0, \quad \lambda \in (0,\infty),$$

and

$$\frac{\partial^2}{\partial\lambda^2}\Delta_1(c,\lambda) = 2 + \frac{(n+1)(n+2)bc^2\tau^2}{(1+\lambda c\tau)^{n+3}} > 0.$$

So, it is easy to see that the equation $\Delta_1(c,\lambda) = 0$ has two real positive roots as c increases and the following result holds.

Lemma 4. *There exist $c_1^* > 0$ and $\lambda_1^* > 0$ such that*
(i)

$$\begin{aligned} \Delta_1(c_1^*,\lambda_1^*) &= 0, \\ \frac{\partial}{\partial\lambda}\Delta_1(c_1^*,\lambda)\Big|_{\lambda=\lambda_1^*} &= 0; \end{aligned} \tag{12}$$

(ii) *for $0 < c < c_1^*$ and $\lambda > 0$, we have $\Delta_1(c,\lambda) > 0$;*
(iii) *for $c > c_1^*$, the equation $\Delta_1(c,\lambda) = 0$ has two real positive roots, which are denoted by $0 < \lambda_1 < \lambda_2$, and we have*

$$\Delta_1(c,\lambda) \begin{cases} > 0, & \text{for} \quad 0 < \lambda < \lambda_1, \\ < 0, & \text{for} \quad \lambda_1 < \lambda < \lambda_2, \\ > 0, & \text{for} \quad \lambda > \lambda_2. \end{cases} \tag{13}$$

The $c_1^* > 0$ is called the critical wave speed. Now we set

$$\phi(\xi) = \min\{K, K\mathrm{e}^{\lambda_1\xi}\},$$

$$\psi(\xi) = \max\{0, K(1 - M\mathrm{e}^{\varepsilon\xi})\mathrm{e}^{\lambda_1\xi}\},$$

where $M > 1$ and $\varepsilon > 0$ are two constants to be determined later.

Lemma 5. (i) *$\phi(\xi)$ is nondecreasing in $\xi \in \mathbb{R}$ and satisfies the boundary condition* (10)*;*
(ii) *$0 \le \psi(\xi) \le \phi(\xi) \le K$, $\xi \in \mathbb{R}$.*

Proof. (i) The conclusion is obvious.
(ii) Obviously $0 < \phi(\xi) \le K$ and $\psi(\xi) \ge 0$, we only need to verify $\psi(\xi) \le \phi(\xi)$ for all $\xi \in \mathbb{R}$.

Let $\xi_0 = -\dfrac{1}{\varepsilon}\ln M < 0$. For $\xi > \xi_0$, we have

$$0 = \psi(\xi) < \phi(\xi).$$

For $\xi \le \xi_0$, we have

$$\psi(\xi) = K(1 - M\mathrm{e}^{\varepsilon\xi})\mathrm{e}^{\lambda_1\xi} < K\mathrm{e}^{\lambda_1\xi} = \phi(\xi).$$

Thus,

$$0 \le \psi(\xi) \le \phi(\xi) \le K$$

for all $\xi \in \mathbb{R}$. □

Lemma 6. *Assume that $c > c_1^*$, then $\phi(\xi)$ is an upper solution of Eq.* (9) *and $\phi(\xi) \in \Gamma$.*

Proof. $\phi(\xi) \in \Gamma$ is obvious.

(i) If $\xi > 0$, then

$$\begin{aligned} \phi(\xi) &= K, \\ \phi'(\xi) &= 0, \\ \phi''(\xi) &= 0, \\ 0 < \phi(\xi - cs) &\le K. \end{aligned}$$

Introduce the notation

$$\begin{aligned} h(\phi)(\xi) = {} & \phi''(\xi) - c\phi'(\xi) - a\phi(\xi) \\ & + b(1-\phi(\xi)) \int_0^\infty \frac{s^n}{n!\tau^{n+1}} \mathrm{e}^{-\frac{s}{\tau}} \phi(\xi - cs) ds + r\phi(\xi)(1-\phi(\xi)). \end{aligned} \tag{14}$$

It follows that

$$h(\phi)(\xi) \le -aK + bK(1-K) + rK(1-K) = 0.$$

(ii) If $\xi \le 0$, then

$$\begin{aligned} \phi(\xi) &= K\mathrm{e}^{\lambda_1 \xi}, \\ \phi'(\xi) &= K\lambda_1 \mathrm{e}^{\lambda_1 \xi}, \\ \phi''(\xi) &= K\lambda_1^2 \mathrm{e}^{\lambda_1 \xi}, \\ \phi(\xi - cs) &= K\mathrm{e}^{\lambda_1(\xi - cs)}. \end{aligned}$$

We have

$$\begin{aligned} \int_0^\infty \frac{s^n}{n!\ \tau^{n+1}} \mathrm{e}^{-\frac{s}{\tau}} \phi(\xi - cs) \mathrm{d}s &= K \int_0^\infty \frac{s^n}{n!\ \tau^{n+1}} \mathrm{e}^{-\frac{s}{\tau}} \mathrm{e}^{\lambda_1(\xi - cs)} \mathrm{d}s \\ &= \frac{K\mathrm{e}^{\lambda_1 \xi}}{(1+\lambda_1 c\tau)^{n+1}}. \end{aligned}$$

Then

$$\begin{aligned} h(\phi)(\xi) &= K\left(\lambda_1^2 \mathrm{e}^{\lambda_1 \xi} - c\lambda_1 \mathrm{e}^{\lambda_1 \xi} + (r-a)\mathrm{e}^{\lambda_1 \xi}\right) \\ &\qquad + bK(1 - K\mathrm{e}^{\lambda_1 \xi}) \frac{\mathrm{e}^{\lambda_1 \xi}}{(1+\lambda_1 c\tau)^{n+1}} - rK^2 \mathrm{e}^{2\lambda_1 \xi} \\ &\le K\left(\lambda_1^2 \mathrm{e}^{\lambda_1 \xi} - c\lambda_1 \mathrm{e}^{\lambda_1 \xi} + (r-a)\mathrm{e}^{\lambda_1 \xi} + \frac{b\mathrm{e}^{\lambda_1 \xi}}{(1+\lambda_1 c\tau)^{n+1}}\right) \\ &= K\mathrm{e}^{\lambda_1 \xi} \Delta_1(c, \lambda_1) = 0. \end{aligned}$$

Therefore, $\phi(\xi)$ is an upper solution of Eq. (9). ∎

Next, we need a lower solution. Recall that λ_1 and λ_2 are two real positive roots of the equation $\Delta_1(c,\lambda) = 0$. Now, let $\varepsilon > 0$ be sufficiently small such that $\varepsilon < \lambda_1$, $\lambda_1 < \lambda_1 + \varepsilon < \lambda_2$ (so that $\Delta_1(c, \lambda_1 + \varepsilon) < 0$). We have the following lemma.

Lemma 7. *If* $M > \max\left\{1, -\dfrac{K\left(4r(1+\lambda_1 c\tau)^{n+1} + b\right)}{(1+\lambda_1 c\tau)^{n+1}\Delta_1(c,\lambda_1+\varepsilon)}\right\}$, *then* $\psi(\xi)$ *is a lower solution of Eq.* (9).

Proof. Let $\xi_0 = -\dfrac{1}{\varepsilon}\ln M < 0$.

(i) If $\xi > \xi_0$, then

$$\begin{gathered}\psi(\xi) = \psi'(\xi) = \psi''(\xi) = 0,\\ \psi(\xi - cs) \geq 0.\end{gathered}$$

Introduce the notation

$$\begin{aligned}h(\psi)(\xi) &= \psi''(\xi) - c\psi'(\xi) - a\psi(\xi)\\ &\quad + b(1-\psi(\xi))\int_0^\infty \frac{s^n}{n!\tau^{n+1}}\mathrm{e}^{-\frac{s}{\tau}}\psi(\xi - cs)\mathrm{d}s + r\psi(\xi)(1-\psi(\xi)).\end{aligned} \tag{15}$$

Then we have

$$h(\psi)(\xi) = b\int_0^\infty \frac{s^n}{n!\tau^{n+1}}\mathrm{e}^{-\frac{s}{\tau}}\psi(\xi - cs)\mathrm{d}s \geq 0.$$

(ii) If $\xi \leq \xi_0$, then we have

$$\begin{aligned}\psi(\xi) &= K(1 - M\mathrm{e}^{\varepsilon\xi})\mathrm{e}^{\lambda_1\xi},\\ \psi(\xi - cs) &= K(1 - M\mathrm{e}^{\varepsilon(\xi - cs)})\mathrm{e}^{\lambda_1(\xi - cs)},\\ \psi'(\xi) &= K[\lambda_1 - M(\lambda_1+\varepsilon)\mathrm{e}^{\varepsilon\xi}]\mathrm{e}^{\lambda_1\xi},\\ \psi''(\xi) &= K[\lambda_1^2 - M(\lambda_1+\varepsilon)^2\mathrm{e}^{\varepsilon\xi}]\mathrm{e}^{\lambda_1\xi}.\end{aligned}$$

Further,

$$\begin{aligned}
&\int_0^\infty \frac{s^n}{n!\tau^{n+1}} \mathrm{e}^{-\frac{s}{\tau}} \psi(\xi - cs)\mathrm{d}s \\
&\quad = K \int_0^\infty \frac{s^n}{n!\tau^{n+1}} \mathrm{e}^{-\frac{s}{\tau}} (1 - M\mathrm{e}^{\varepsilon(\xi - cs)})\mathrm{e}^{\lambda_1(\xi - cs)}\mathrm{d}s \\
&\quad = K\mathrm{e}^{\lambda_1 \xi} \int_0^\infty \frac{s^n}{n!\tau^{n+1}} \mathrm{e}^{-\frac{s}{\tau}} \mathrm{e}^{-\lambda_1 cs}\mathrm{d}s \\
&\qquad - KM\mathrm{e}^{(\lambda_1+\varepsilon)\xi} \int_0^\infty \frac{s^n}{n!\tau^{n+1}} \mathrm{e}^{-\frac{s}{\tau}} \mathrm{e}^{-(\lambda_1+\varepsilon) cs}\mathrm{d}s \\
&\quad = \frac{K\mathrm{e}^{\lambda_1\xi}}{(1+\lambda_1 c\tau)^{n+1}} - \frac{KM\mathrm{e}^{(\lambda_1+\varepsilon)\xi}}{(1+(\lambda_1+\varepsilon)c\tau)^{n+1}} \\
&\quad \geq 0,
\end{aligned}$$

and

$$\psi(\xi) \int_0^\infty \frac{s^n}{n!\tau^{n+1}} \mathrm{e}^{-\frac{s}{\tau}} \psi(\xi - cs)\mathrm{d}s \leq \phi(\xi) \int_0^\infty \frac{s^n}{n!\tau^{n+1}} \mathrm{e}^{-\frac{s}{\tau}} \psi(\xi - cs)\mathrm{d}s.$$

Hence, it follows that

$$\begin{aligned}
h(\psi)(\xi) \geq{}& K\left(\lambda_1^2 - M(\lambda_1+\varepsilon)^2\mathrm{e}^{\varepsilon\xi}\right)\mathrm{e}^{\lambda_1\xi} - cK\left(\lambda_1 - M(\lambda_1+\varepsilon)\mathrm{e}^{\varepsilon\xi}\right)\mathrm{e}^{\lambda_1\xi} \\
&+ K(r-a)(1-M\mathrm{e}^{\varepsilon\xi})\mathrm{e}^{\lambda_1\xi} - rK^2(1-M\mathrm{e}^{\varepsilon\xi})^2\mathrm{e}^{2\lambda_1\xi} \\
&+ bK(1-K\mathrm{e}^{\lambda_1\xi})\left(\frac{\mathrm{e}^{\lambda_1\xi}}{(1+\lambda_1c\tau)^{n+1}} - \frac{M\mathrm{e}^{(\lambda_1+\varepsilon)\xi}}{(1+(\lambda_1+\varepsilon)c\tau)^{n+1}}\right) \\
\geq{}& K\mathrm{e}^{\lambda_1\xi}\left(\lambda_1^2 - c\lambda_1 + (r-a) + \frac{b}{(1+\lambda_1c\tau)^{n+1}}\right) - KM\mathrm{e}^{(\lambda_1+\varepsilon)\xi} \\
&\times\left((\lambda_1+\varepsilon)^2 - c(\lambda_1+\varepsilon) + (r-a) + \frac{b}{(1+(\lambda_1+\varepsilon)c\tau)^{n+1}}\right) \\
&- \frac{bK^2\mathrm{e}^{2\lambda_1\xi}}{(1+\lambda_1c\tau)^{n+1}} + \frac{bK^2M\mathrm{e}^{\lambda_1\xi}\mathrm{e}^{(\lambda_1+\varepsilon)\xi}}{[1+(\lambda_1+\varepsilon)c\tau]^{n+1}} - rK^2(1-M\mathrm{e}^{\varepsilon\xi})^2\mathrm{e}^{2\lambda_1\xi} \\
\geq{}& K\mathrm{e}^{\lambda_1\xi}\Delta_1(c,\lambda_1) - KM\mathrm{e}^{(\lambda_1+\varepsilon)\xi}\Delta_1(c,\lambda_1+\varepsilon) - \frac{bK^2\mathrm{e}^{2\lambda_1\xi}}{(1+\lambda_1c\tau)^{n+1}} \\
&- rK^2(1-M\mathrm{e}^{\varepsilon\xi})^2\mathrm{e}^{2\lambda_1\xi}.
\end{aligned}$$

Since $\xi \leq \xi_0 < 0$ and $\varepsilon < \lambda_1$, we have $\mathrm{e}^{\lambda_1\xi} < \mathrm{e}^{\varepsilon\xi}$ and

$$(1-M\mathrm{e}^{\varepsilon\xi})^2 < (1+M\mathrm{e}^{\varepsilon\xi})^2 \leq (1+M\mathrm{e}^{\varepsilon\xi_0})^2 \leq (1+1)^2 = 4.$$

Therefore, we obtain

$$h(\psi)(\xi) \geq K\mathrm{e}^{(\lambda_1+\varepsilon)\xi}\left(-M\Delta_1(c,\lambda_1+\varepsilon) - \frac{bK}{(1+\lambda_1c\tau)^{n+1}} - 4rK\right) \geq 0.$$

Therefore, $\psi(\xi)$ is a lower solution of Eq. (9). □

Furthermore, we have

$$\lim_{\xi\to-\infty}\phi(\xi)=0,$$

$$\sup_{\xi\in\mathbb{R}}\psi(\xi)=\frac{K\varepsilon}{\lambda_1+\varepsilon}\left(\frac{\lambda_1}{M(\lambda_1+\varepsilon)}\right)^{\frac{\lambda_1}{\varepsilon}}\geq\delta>0,$$

and

$$f(u,u)\neq 0\quad\text{ for }\quad u\in[\delta,K)\,.$$

From what has been discussed above and by Lemma 2, we can obtain the following result.

Theorem 8. *For every $c>c_1^*$ and $\tau>0$, Eq.* (8) *always has a traveling wave front with speed c connecting the zero equilibria and the positive equilibria.*

3.2. **The case of** $F(t,s,x,y)=\delta(x-y)\frac{1}{\tau}\left(\sin\frac{t-s}{\tau}+\cos\frac{t-s}{\tau}\right)\mathrm{e}^{-\frac{t-s}{\tau}},\tau>0$**.** In this case, Eq. (1) becomes

$$\begin{aligned}\frac{\partial u(t,x)}{\partial t}&=\frac{\partial^2u(t,x)}{\partial x^2}-au(t,x)+b[1-u(t,x)]\\&\times\int_{-\infty}^{t}\frac{1}{\tau}\left(\sin\frac{t-s}{\tau}+\cos\frac{t-s}{\tau}\right)\mathrm{e}^{-\frac{t-s}{\tau}}u(s,x)\mathrm{d}s+ru(t,x)[1-u(t,x)].\end{aligned}\tag{16}$$

It is easy to know that Eq. (16) has two nonnegative equilibrium points 0 and K. Converting of Eq. (16) into the traveling wave form, we obtain

$$\begin{aligned}&\varphi''(\xi)-c\varphi'(\xi)-a\varphi(\xi)+b(1-\varphi(\xi))\\&\quad\times\int_0^{\infty}\frac{1}{\tau}\left(\sin\frac{s}{\tau}+\cos\frac{s}{\tau}\right)\mathrm{e}^{-\frac{s}{\tau}}\varphi(\xi-cs)\mathrm{d}s+r\varphi(\xi)(1-\varphi(\xi))=0.\end{aligned}\tag{17}$$

We seek a solution of this equation satisfying the asymptotic boundary conditions

$$\lim_{\xi\to-\infty}\varphi(\xi)=0,\quad\lim_{\xi\to\infty}\varphi(\xi)=K.\tag{18}$$

Let

$$f(\varphi(\xi),(g*\varphi)(\xi))=-a\varphi(\xi)+b(1-\varphi(\xi))(g*\varphi)(\xi)+r\varphi(\xi)(1-\varphi(\xi)),$$

where

$$(g*\varphi)(\xi)=\int_0^{\infty}\frac{1}{\tau}\left(\sin\frac{s}{\tau}+\cos\frac{s}{\tau}\right)\mathrm{e}^{-\frac{s}{\tau}}\varphi(\xi-cs)\mathrm{d}s.$$

We have the following lemma.

Lemma 9. *$f(\varphi(\xi),(g*\varphi)(\xi))$ satisfies (H_2).*

Proof. The proof is similar to that of Lemma 3, so we omit the details here. □

Now, we construct an upper solution and a lower solution for Eq. (17) to satisfy the assumptions in Lemma 2. Introduce the notation

$$\Delta_2(c,\lambda)=\lambda^2-c\lambda+(r-a)+\frac{b(2+\lambda c\tau)}{1+(1+\lambda c\tau)^2}. \tag{19}$$

Then

$$\Delta_2(0,\lambda)=\lambda^2+r+b-a\neq 0.$$

For all $c>0$, we have

$$\Delta_2(c,0)=r+b-a>0,$$

$$\lim_{\lambda\to\infty}\Delta_2(c,\lambda)=\infty,$$

$$\frac{\partial}{\partial c}\Delta_2(c,\lambda)=-\lambda-\frac{b\lambda\tau((\lambda c\tau)^2+4\lambda c\tau+2)}{(1+(1+\lambda c\tau)^2)^2}<0,\quad \lambda\in(0,\infty),$$

and

$$\frac{\partial^2}{\partial\lambda^2}\Delta_2(c,\lambda)=2+\frac{2b\lambda c^3\tau^3((\lambda c\tau)^2+6\lambda c\tau+6)}{(1+(1+\lambda c\tau)^2)^3}>0.$$

So we have the following result, which is similar to Lemma 4.

Lemma 10. *There exist $c_2^*>0$ and $\lambda_2^*>0$ such that*
(i)

$$\begin{aligned}\Delta_2(c_2^*,\lambda_2^*)&=0,\\ \frac{\partial}{\partial\lambda}\Delta_2(c_2^*,\lambda)\Big|_{\lambda=\lambda_2^*}&=0;\end{aligned} \tag{20}$$

(ii) *for $0<c<c_2^*$ and $\lambda>0$, we have $\Delta_2(c,\lambda)>0$;*
(iii) *for $c>c_2^*$, the equation $\Delta_2(c,\lambda)=0$ has two real positive roots λ_1, λ_2, such that $0<\lambda_1<\lambda_2$ and*

$$\Delta_2(c,\lambda)\begin{cases}>0, & \text{for}\quad 0<\lambda<\lambda_1,\\ <0, & \text{for}\quad \lambda_1<\lambda<\lambda_2,\\ >0, & \text{for}\quad \lambda>\lambda_2.\end{cases} \tag{21}$$

$c_2^*>0$ is the critical wave speed. Now we set

$$\phi(\xi)=\min\{K,K\mathrm{e}^{\lambda_1\xi}\},$$

$$\psi(\xi)=\max\{0,K(1-M\mathrm{e}^{\varepsilon\xi})\mathrm{e}^{\lambda_1\xi}\}.$$

Lemma 11. *Assume that $c>c_2^*$. Then $\phi(\xi)$ is an upper solution of Eq.* (17) *and $\phi(\xi)\in\Gamma$.*

Proof. $\phi(\xi) \in \Gamma$ is obvious.

(i) If $\xi > 0$, then

$$\begin{aligned} \phi(\xi) &= K, \\ \phi'(\xi) &= 0, \\ \phi''(\xi) &= 0, \\ 0 < \phi(\xi - cs) &\leq K. \end{aligned}$$

Introduce the notation

$$\begin{aligned} h(\phi)(\xi) = {} & \phi''(\xi) - c\phi'(\xi) - a\phi(\xi) + b(1 - \phi(\xi)) \\ & \times \int_0^\infty \frac{1}{\tau}\left(\sin\frac{s}{\tau} + \cos\frac{s}{\tau}\right) \mathrm{e}^{-\frac{s}{\tau}} \phi(\xi - cs)\mathrm{d}s + r\phi(\xi)(1 - \phi(\xi)). \end{aligned} \tag{22}$$

It follows that

$$h(\phi)(\xi) \leq -aK + bK(1 - K) + rK(1 - K) = 0.$$

(ii) If $\xi \leq 0$, then

$$\begin{aligned} \phi(\xi) &= K\mathrm{e}^{\lambda_1 \xi}, \\ \phi'(\xi) &= K\lambda_1 \mathrm{e}^{\lambda_1 \xi}, \\ \phi''(\xi) &= K\lambda_1^2 \mathrm{e}^{\lambda_1 \xi}, \\ \phi(\xi - cs) &= K\mathrm{e}^{\lambda_1(\xi - cs)}. \end{aligned}$$

We have

$$\begin{aligned} & \int_0^\infty \frac{1}{\tau}\left(\sin\frac{s}{\tau} + \cos\frac{s}{\tau}\right) \mathrm{e}^{-\frac{s}{\tau}} \phi(\xi - cs)\mathrm{d}s \\ & \quad = K\int_0^\infty \frac{1}{\tau}\left(\sin\frac{s}{\tau} + \cos\frac{s}{\tau}\right) \mathrm{e}^{-\frac{s}{\tau}} \mathrm{e}^{\lambda_1(\xi - cs)}\mathrm{d}s \\ & \quad = \frac{K\mathrm{e}^{\lambda_1 \xi}(2 + \lambda_1 c\tau)}{1 + (1 + \lambda_1 c\tau)^2}. \end{aligned}$$

Then

$$\begin{aligned} h(\phi)(\xi) &= K\left(\lambda_1^2 \mathrm{e}^{\lambda_1 \xi} - c\lambda_1 \mathrm{e}^{\lambda_1 \xi} + (r - a)\mathrm{e}^{\lambda_1 \xi}\right) \\ & \qquad + bK(1 - K\mathrm{e}^{\lambda_1 \xi}) \frac{\mathrm{e}^{\lambda_1 \xi}(2 + \lambda_1 c\tau)}{1 + (1 + \lambda_1 c\tau)^2} - rK^2 \mathrm{e}^{2\lambda_1 \xi} \\ & \leq K\left(\lambda_1^2 \mathrm{e}^{\lambda_1 \xi} - c\lambda_1 \mathrm{e}^{\lambda_1 \xi} + (r - a)\mathrm{e}^{\lambda_1 \xi} + \frac{b\mathrm{e}^{\lambda_1 \xi}(2 + \lambda_1 c\tau)}{1 + (1 + \lambda_1 c\tau)^2}\right) \\ & = K\mathrm{e}^{\lambda_1 \xi} \Delta_2(c, \lambda_1) \\ & = 0. \end{aligned}$$

Therefore, $\phi(\xi)$ is an upper solution of Eq. (17). $\square$

Next, we need a lower solution. Let $\varepsilon > 0$ be sufficiently small such that $\varepsilon < \lambda_1$, $\lambda_1 < \lambda_1 + \varepsilon < \lambda_2$. Then we have the following lemma.

Lemma 12. *If* $M > \max\left\{1, -\dfrac{K\left(4r + 4r(1+\lambda_1 c\tau)^2 + b(2+\lambda_1 c\tau)\right)}{\Delta_2(c, \lambda_1+\varepsilon)\left(1 + (1+\lambda_1 c\tau)^2\right)}\right\}$, *then* $\psi(\xi)$ *is a lower solution of Eq.* (17).

Proof. Let $\xi_0 = -\dfrac{1}{\varepsilon}\ln M < 0$.

(i) If $\xi > \xi_0$, then

$$\psi(\xi) = \psi'(\xi) = \psi''(\xi) = 0,$$
$$\psi(\xi - cs) \geq 0.$$

Introduce the notation

$$\begin{aligned} h(\psi)(\xi) = {}& \psi''(\xi) - c\psi'(\xi) - a\psi(\xi) + b(1-\psi(\xi)) \\ & \times \int_0^\infty \frac{1}{\tau}\left(\sin\frac{s}{\tau} + \cos\frac{s}{\tau}\right) \mathrm{e}^{-\frac{s}{\tau}} \psi(\xi - cs)\mathrm{d}s + r\psi(\xi)(1-\psi(\xi)). \end{aligned} \tag{23}$$

Then we have

$$h(\psi)(\xi) = b\int_0^\infty \frac{1}{\tau}\left(\sin\frac{s}{\tau} + \cos\frac{s}{\tau}\right) \mathrm{e}^{-\frac{s}{\tau}} \psi(\xi - cs)\mathrm{d}s \geq 0.$$

(ii) If $\xi \leq \xi_0$, then we have

$$\begin{aligned} \psi(\xi) &= K(1 - M\mathrm{e}^{\varepsilon\xi})\mathrm{e}^{\lambda_1\xi}, \\ \psi(\xi - cs) &= K(1 - M\mathrm{e}^{\varepsilon(\xi - cs)})\mathrm{e}^{\lambda_1(\xi - cs)}, \\ \psi'(\xi) &= K[\lambda_1 - M(\lambda_1+\varepsilon)\mathrm{e}^{\varepsilon\xi}]\mathrm{e}^{\lambda_1\xi}, \\ \psi''(\xi) &= K[\lambda_1^2 - M(\lambda_1+\varepsilon)^2\mathrm{e}^{\varepsilon\xi}]\mathrm{e}^{\lambda_1\xi}. \end{aligned}$$

Further,

$$\begin{aligned} &\int_0^\infty \frac{1}{\tau}\left(\sin\frac{s}{\tau} + \cos\frac{s}{\tau}\right) \mathrm{e}^{-\frac{s}{\tau}} \psi(\xi - cs)\mathrm{d}s \\ &\quad = K\int_0^\infty \frac{1}{\tau}\left(\sin\frac{s}{\tau} + \cos\frac{s}{\tau}\right) \mathrm{e}^{-\frac{s}{\tau}} (1 - M\mathrm{e}^{\varepsilon(\xi - cs)})\mathrm{e}^{\lambda_1(\xi - cs)}\mathrm{d}s \\ &\quad = K\mathrm{e}^{\lambda_1\xi}\int_0^\infty \frac{1}{\tau}\left(\sin\frac{s}{\tau} + \cos\frac{s}{\tau}\right) \mathrm{e}^{-\frac{s}{\tau}}\mathrm{e}^{-\lambda_1 cs}\mathrm{d}s \\ &\qquad - KM\mathrm{e}^{(\lambda_1+\varepsilon)\xi}\int_0^\infty \frac{1}{\tau}\left(\sin\frac{s}{\tau} + \cos\frac{s}{\tau}\right) \mathrm{e}^{-\frac{s}{\tau}}\mathrm{e}^{-(\lambda_1+\varepsilon)cs}\mathrm{d}s \\ &= \frac{K\mathrm{e}^{\lambda_1\xi}(2 + \lambda_1 c\tau)}{1 + (1+\lambda_1 c\tau)^2} - \frac{KM\mathrm{e}^{(\lambda_1+\varepsilon)\xi}(2 + (\lambda_1+\varepsilon)c\tau)}{1 + (1 + (\lambda_1+\varepsilon)c\tau)^2} \\ &\geq 0, \end{aligned}$$

and

$$\psi(\xi)\int_0^\infty \frac{1}{\tau}\left(\sin\frac{s}{\tau}+\cos\frac{s}{\tau}\right)\mathrm{e}^{-\frac{s}{\tau}}\psi(\xi-cs)\mathrm{d}s$$
$$\leq \phi(\xi)\int_0^\infty \frac{1}{\tau}\left(\sin\frac{s}{\tau}+\cos\frac{s}{\tau}\right)\mathrm{e}^{-\frac{s}{\tau}}\psi(\xi-cs)\mathrm{d}s.$$

Hence, it follows that

$$\begin{aligned}
h(\psi)(\xi) \geq\ & K\left(\lambda_1^2 - M(\lambda_1+\varepsilon)^2\mathrm{e}^{\varepsilon\xi}\right)\mathrm{e}^{\lambda_1\xi} - cK\left(\lambda_1 - M(\lambda_1+\varepsilon)\mathrm{e}^{\varepsilon\xi}\right)\mathrm{e}^{\lambda_1\xi}\\
&+ K(r-a)(1-M\mathrm{e}^{\varepsilon\xi})\mathrm{e}^{\lambda_1\xi} - rK^2(1-M\mathrm{e}^{\varepsilon\xi})^2\mathrm{e}^{2\lambda_1\xi}\\
&+ bK(1-K\mathrm{e}^{\lambda_1\xi})\left(\frac{\mathrm{e}^{\lambda_1\xi}(2+\lambda_1 c\tau)}{1+(1+\lambda_1 c\tau)^2} - \frac{M\mathrm{e}^{(\lambda_1+\varepsilon)\xi}(2+(\lambda_1+\varepsilon)c\tau)}{1+(1+(\lambda_1+\varepsilon)c\tau)^2}\right)\\
\geq\ & K\mathrm{e}^{\lambda_1\xi}\left(\lambda_1^2 - c\lambda_1 + (r-a) + \frac{b(2+\lambda_1 c\tau)}{1+(1+\lambda_1 c\tau)^2}\right) - KM\mathrm{e}^{(\lambda_1+\varepsilon)\xi}\\
&\times\left((\lambda_1+\varepsilon)^2 - c(\lambda_1+\varepsilon) + (r-a) + \frac{b(2+(\lambda_1+\varepsilon)c\tau)}{1+(1+(\lambda_1+\varepsilon)c\tau)^2}\right)\\
&- \frac{bK^2\mathrm{e}^{2\lambda_1\xi}(2+\lambda_1 c\tau)}{1+(1+\lambda_1 c\tau)^2} + \frac{bK^2M\mathrm{e}^{\lambda_1\xi}\mathrm{e}^{(\lambda_1+\varepsilon)\xi}(2+(\lambda_1+\varepsilon)c\tau)}{1+(1+(\lambda_1+\varepsilon)c\tau)^2}\\
&- rK^2(1-M\mathrm{e}^{\varepsilon\xi})^2\mathrm{e}^{2\lambda_1\xi}\\
\geq\ & K\mathrm{e}^{\lambda_1\xi}\Delta_2(c,\lambda_1) - KM\mathrm{e}^{(\lambda_1+\varepsilon)\xi}\Delta_2(c,\lambda_1+\varepsilon)\\
&- \frac{bK^2\mathrm{e}^{2\lambda_1\xi}(2+\lambda_1 c\tau)}{1+(1+\lambda_1 c\tau)^2} - rK^2(1-M\mathrm{e}^{\varepsilon\xi})^2\mathrm{e}^{2\lambda_1\xi}.
\end{aligned}$$

Since $\xi \leq \xi_0 < 0$ and $\varepsilon < \lambda_1$, we have $\mathrm{e}^{\lambda_1\xi} < \mathrm{e}^{\varepsilon\xi}$ and

$$(1-M\mathrm{e}^{\varepsilon\xi})^2 < (1+M\mathrm{e}^{\varepsilon\xi})^2 \leq (1+M\mathrm{e}^{\varepsilon\xi_0})^2 \leq (1+1)^2 = 4.$$

Therefore, we obtain

$$h(\psi)(\xi) \geq K\mathrm{e}^{(\lambda_1+\varepsilon)\xi}\left(-M\Delta_2(c,\lambda_1+\varepsilon) - \frac{bK(2+\lambda_1 c\tau)}{1+(1+\lambda_1 c\tau)^2} - 4rK\right) \geq 0.$$

Therefore, $\psi(\xi)$ is a lower solution of Eq. (17). □

Furthermore, we have

$$\lim_{\xi\to-\infty}\phi(\xi) = 0,$$

$$\sup_{\xi\in\mathbb{R}}\psi(\xi) \geq \delta > 0,$$

and

$$f(u,u) \neq 0 \quad \text{for} \quad u \in [\delta, K).$$

From what has been discussed above and by Lemma 2, we can obtain the following result.

Theorem 13. *For every $c > c_2^*$ and $\tau > 0$, Eq.* (16) *always has a traveling wave front with speed c connecting the zero equilibria and the positive equilibria.*

4. Analysis on the critical wave speed

In this section, for the critical wave speed, we give a detailed analysis on its location and asymptotic behavior with respect to the time delay τ. Our main result is as follows.

Theorem 14. *Consider Eq.* (8).

(1)*If $a > r$, then the critical wave speed c_1^* satisfies:*

(i) *Upper and lower bounds of c_1^*:*

$$0 \leq c_1^* \leq \min\left\{2\sqrt{r+b-a}, \sqrt{\frac{1}{\tau}\left(\left(\frac{b}{a-r}\right)^{\frac{1}{n+1}}-1\right)}\right\}. \tag{24}$$

(ii) *Asymptotic behavior of c_1^* with respect to the time delay τ:*
Let τ be free, and the other parameters a, b, r and n be fixed, then

$$\lim_{\tau\to 0^+} c_1^* = 2\sqrt{r+b-a}, \tag{25}$$

$$\lim_{\tau\to +\infty}\left|c_1^* - \frac{A}{\tau}\right| = 0, \tag{26}$$

where the positive constant A is given by

$$\begin{cases} (n+1)(n+3)bA^2 = 2B\left(1+\frac{B}{n+3}\right)^{n+2}, \\ B = \sqrt{1+(n+1)(n+3)(a-r)A^2}-1. \end{cases} \tag{27}$$

(2) *If $a = r$, then the critical wave speed c_1^* satisfies:*

$$0 \leq c_1^* \leq 2\sqrt{b},$$

and

$$\lim_{\tau\to 0^+} c_1^* = 2\sqrt{b}.$$

(3) *If $a < r$, then the critical wave speed c_1^* satisfies:*

$$2\sqrt{r-a} < c_1^* \leq 2\sqrt{r+b-a},$$

and

$$\lim_{\tau\to 0^+} c_1^* = 2\sqrt{r+b-a}.$$

Proof. (1) If $a > r$,

(i) Letting

$$F(c,\lambda) = \frac{b}{(1+\lambda c\tau)^{n+1}}, \tag{28}$$

$$G(c,\lambda) = c\lambda + (a-r) - \lambda^2,$$

then

$$\Delta_1(c,\lambda) = F(c,\lambda) - G(c,\lambda),$$

and the critical point (c_1^*, λ^*) (for convenience, we denote λ_1^* as λ^*) is the unique tangent point touched by the two surfaces $F(c,\lambda)$ and $G(c,\lambda)$. Obviously, $F(c_1^*,\lambda)$ is always above $G(c_1^*,\lambda)$ except the touched point λ^*, see Fig. 1.

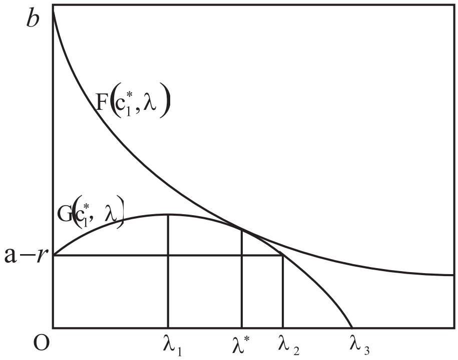

Fig. 1. The graphs of $F(c_1^*,\lambda)$ and $G(c_1^*,\lambda)$

Let

$$\lambda_1 = \frac{c_1^*}{2},$$
$$\lambda_2 = c_1^*,$$
$$\lambda_3 = \frac{c_1^* + \sqrt{c_1^{*2} + 4(a-r)}}{2},$$

where λ_1 is the point at which $G(c_1^*,\lambda)$ arrives the maximum

$$G(c_1^*,\lambda_1) = a - r + \frac{c_1^{*2}}{4},$$

λ_2 is the non-zero root of the equation $G(c_1^*,\lambda) = a - r$, and λ_3 is the positive root of the equation $G(c_1^*,\lambda) = 0$ (for the detail, we refer to Fig. 1). Since

$$F(c_1^*,\lambda_1) \geq G(c_1^*,\lambda_1),$$
$$F(c_1^*,\lambda_2) \geq G(c_1^*,\lambda_2),$$

namely,

$$\frac{b}{\left(1 + \frac{c_1^{*2}\tau}{2}\right)^{n+1}} \geq a - r + \frac{c_1^{*2}}{4},$$
$$\frac{b}{(1 + c_1^{*2}\tau)^{n+1}} \geq a - r.$$

This is equivalent to

$$\begin{aligned} c_1^{*2} &\leq 4\left(\frac{b}{\left(1+\frac{c_1^{*2}\tau}{2}\right)^{n+1}}-(a-r)\right) \leq 4(r+b-a), \quad (29)\\ c_1^{*2} &\leq \frac{1}{\tau}\left(\left(\frac{b}{a-r}\right)^{\frac{1}{n+1}}-1\right), \end{aligned}$$

which immediately imply the boundedness of c_1^* in (24):

$$0 \leq c_1^* \leq \min\left\{2\sqrt{r+b-a}, \sqrt{\frac{1}{\tau}\left(\left(\frac{b}{a-r}\right)^{\frac{1}{n+1}}-1\right)}\right\}.$$

(ii) To prove (25) as $\tau \to 0^+$, let

$$c_0^* = \lim_{\tau\to 0^+} c_1^*,$$

and

$$\lambda_0^* = \lim_{\tau\to 0^+} \lambda^*.$$

Since c_1^* and λ^* are bounded by

$$\begin{aligned} &0 \leq c_1^* \leq 2\sqrt{r+b-a},\\ &0 < \lambda^* < \lambda_3, \end{aligned}$$

respectively, and λ_3 is bounded by

$$\begin{aligned} \lambda_3 &= \frac{c_1^* + \sqrt{c_1^{*2}+4(a-r)}}{2}\\ &\leq \frac{2\sqrt{r+b-a}+\sqrt{4(r+b-a)+4(a-r)}}{2}\\ &= \sqrt{r+b-a}+\sqrt{b}, \end{aligned}$$

then c_0^* and λ_0^* are also bounded. Thus,

$$\lim_{\tau\to 0^+} \frac{1}{(1+\lambda^* c_1^*\tau)^{n+1}} = \frac{1}{(1+\lambda_0^*\cdot c_0^*\cdot 0)^{n+1}} = 1. \quad (30)$$

Noting that (c_1^*, λ^*) satisfies the equations in Lemma 4(i), we have,

$$\begin{aligned} \frac{b}{(1+\lambda^* c_1^*\tau)^{n+1}} &= c_1^*\lambda^* + (a-r) - \lambda^{*2}, \quad (31)\\ -\frac{(n+1)bc_1^*\tau}{(1+\lambda^* c_1^*\tau)^{n+2}} &= c_1^* - 2\lambda^*. \quad (32) \end{aligned}$$

Taking limits of the above equations as $\tau \to 0^+$, and applying (30), we have

$$\begin{aligned} b &= c_0^*\lambda_0^* + (a-r) - \lambda_0^{*2}, \\ 0 &= c_0^* - 2\lambda_0^*, \end{aligned}$$

which gives

$$\begin{aligned} \lambda_0^* &= \frac{c_0^*}{2}, \\ c_0^* &= 2\sqrt{r+b-a}, \end{aligned}$$

i.e.,

$$\lim_{\tau\to 0^+} c_1^* = 2\sqrt{r+b-a}.$$

This completes the proof of (25).

Now, we are going to prove the asymptotic behavior (26) as $\tau \to +\infty$. From

$$0 \le c_1^* \le \sqrt{\frac{1}{\tau}\left(\left(\frac{b}{a-r}\right)^{\frac{1}{n+1}} - 1\right)},$$

we have

$$c_1^* = \mathcal{O}(\tau^{-\alpha}) \to 0$$

as $\tau \to +\infty$, with $\alpha \ge \dfrac{1}{2}$. In what follows, we shall determine that $\alpha = 1$.

From (31), (32), we can obtain

$$-\frac{(n+1)c_1^*\tau}{1+\lambda^* c_1^*\tau}(c_1^*\lambda^* + (a-r) - \lambda^{*2}) = c_1^* - 2\lambda^*,$$

which can be solved in λ^* as

$$\begin{aligned} \lambda^* = {} & \frac{(n+2)c_1^*}{2(n+3)} - \frac{1}{(n+3)c_1^*\tau} \\ & + \frac{1}{2}\sqrt{\left(\frac{(n+2)c_1^*}{n+3} - \frac{2}{(n+3)c_1^*\tau}\right)^2 + \frac{4(n+1)(a-r)}{n+3} + \frac{4}{(n+3)\tau}}. \end{aligned}$$

Note that $c_1^* = \mathcal{O}(\tau^{-\alpha})$ as $\tau \to +\infty$. Then the above equation for λ^* is reduced to

$$\begin{aligned} \lambda^* &\approx \mathcal{O}(\tau^{-\alpha}) - \mathcal{O}(\tau^{-(1-\alpha)}) \\ &\quad + \frac{1}{2}\sqrt{\left[\mathcal{O}(\tau^{-\alpha}) - \mathcal{O}(\tau^{-(1-\alpha)})\right]^2 + \frac{4(n+1)(a-r)}{n+3} + \frac{4}{(n+3)\tau}} \\ &\approx \mathcal{O}(1) \quad \text{for} \quad \alpha = 1, \text{ or} \\ &\quad\ \mathcal{O}(\tau^{\alpha-1}) \quad \text{for} \quad \alpha > 1, \text{ or} \\ &\quad\ \mathcal{O}(1) \quad \text{for} \quad 1/2 \le \alpha < 1, \text{ as} \quad \tau \to +\infty. \end{aligned}$$

It is also verified that

$$c_1^*\tau \approx \begin{cases} \mathcal{O}(1) & \text{for} \quad \alpha = 1, \text{ or} \\ \mathcal{O}(\tau^{-(\alpha-1)}) & \text{for} \quad \alpha > 1, \text{ or} \\ \mathcal{O}(\tau^{1-\alpha}) & \text{for} \quad 1/2 \leq \alpha < 1, \text{ as} \quad \tau \to +\infty, \end{cases}$$

and

$$\lambda^* c_1^*\tau \approx \begin{cases} \mathcal{O}(1) & \text{for} \quad \alpha = 1, \text{ or} \\ \mathcal{O}(1) & \text{for} \quad \alpha > 1, \text{ or} \\ \mathcal{O}(\tau^{1-\alpha}) & \text{for} \quad 1/2 \leq \alpha < 1, \text{ as} \quad \tau \to +\infty. \end{cases}$$

Now, when $\alpha > 1$, letting $\tau \to +\infty$ and applying the above equations to (32), we obtain

$$\left| \frac{(n+1)bc_1^*\tau}{(1+\lambda^* c_1^*\tau)^{n+2}} \right| \approx \mathcal{O}(\tau^{-(\alpha-1)}) \to 0, \text{ as} \quad \tau \to +\infty,$$

and

$$|c_1^* - 2\lambda^*| \approx \mathcal{O}(\tau^{\alpha-1}) \to +\infty, \text{ as} \quad \tau \to +\infty.$$

This implies that (32) does not match the order of τ for both the left and right hand sides, so we can not have $\alpha > 1$.

Similarly, if $1/2 \leq \alpha < 1$, then

$$\left| \frac{(n+1)bc_1^*\tau}{(1+\lambda^* c_1^*\tau)^{n+2}} \right| \approx \mathcal{O}(\tau^{-(1-\alpha)(n+1)}) \to 0, \text{ as} \quad \tau \to +\infty,$$

and

$$|c_1^* - 2\lambda^*| \approx \mathcal{O}(1), \text{ as} \quad \tau \to +\infty.$$

This also shows that the orders of τ as $\tau \to +\infty$ in both the left and right hand sides of (32) do not match. So, we can not allow $1/2 \leq \alpha < 1$. Therefore, it follows that the unique possibility for α is $\alpha = 1$.

From the discussion above, we obtain $c_1^* = \mathcal{O}(\tau^{-1})$ and $\lambda^* = \mathcal{O}(1)$ as $\tau \to +\infty$. Let us assume

$$\lim_{\tau \to +\infty} \left| c_1^* - \frac{A}{\tau} \right| = 0,$$
$$\lim_{\tau \to +\infty} \lambda^* = C,$$

for some positive constants A and C. Now, we are going to determine A and C.

As $\tau \to +\infty$, taking limits of (31), (32), and using $\lim_{\tau \to +\infty} c_1^*\tau = A$, $\lim_{\tau \to +\infty} \lambda^* = C$, we obtain

$$\frac{b}{(1+AC)^{n+1}} = (a-r) - C^2,$$
$$-\frac{(n+1)bA}{(1+AC)^{n+2}} = -2C.$$

Solving the above equations gives

$$C = \frac{-1 + \sqrt{1 + (n+1)(n+3)(a-r)A^2}}{(n+3)A},$$

and A is given by

$$\begin{cases} (n+1)(n+3)bA^2 = 2B\left(1 + \frac{B}{n+3}\right)^{n+2}, \\ B = \sqrt{1 + (n+1)(n+3)(a-r)A^2} - 1. \end{cases}$$

(2) If $a = r$, the proof is similar to (1). From (29)–(31), we obtain

$$0 \le c_1^* \le 2\sqrt{b},$$

and

$$\lim_{\tau \to 0^+} c_1^* = 2\sqrt{b}.$$

(3) If $a < r$, the proof is also similar to (1). From (29)–(31), we obtain

$$0 \le c_1^* \le 2\sqrt{r + b - a},$$

and

$$\lim_{\tau \to 0^+} c_1^* = 2\sqrt{r + b - a}.$$

Since $G(c^*, \lambda_1) = a - r + \dfrac{c_1^{*2}}{4} > 0$, we have

$$c_1^* > 2\sqrt{r - a}.$$

The proof is completed. □

Remark 15. Asymptotics (25) implies that $c_1^* = 2\sqrt{r + b - a}$ is the critical wave speed for the corresponding reaction-diffusion equation (8) without time delay (i.e., $\tau = 0$).

Similarly, we have the following theorem.

Theorem 16. *Consider Eq.* (16).
(1) *If $a > r$, then the critical wave speed c_2^* satisfies:*
(i) *Upper and lower bounds of c_2^*:*

$$0 \le c_2^* \le \min\left\{2\sqrt{r + b - a}, \sqrt{\frac{1}{\tau}E}\right\}, \tag{33}$$

where the positive constant E is given by

$$E = \frac{b - 2(a-r) + \sqrt{(b - 2(a-r))^2 + 8(a-r)(r+b-a)}}{2(a-r)}. \tag{34}$$

(ii) *Asymptotic behavior of c_2^* with respect to the time delay τ:*
Let τ be free, and the other parameters a, b, r be fixed, then

$$\lim_{\tau \to 0^+} c_2^* = 2\sqrt{r + b - a}. \tag{35}$$

(2) *If* $a = r$, *then the critical wave speed* c_2^* *satisfies:*

$$0 \le c_2^* \le 2\sqrt{b},$$

and

$$\lim_{\tau \to 0^+} c_2^* = 2\sqrt{b}.$$

(3)*If* $a < r$, *then the critical wave speed* c_2^* *satisfies:*

$$2\sqrt{r-a} < c_2^* \le 2\sqrt{r+b-a},$$

and

$$\lim_{\tau \to 0^+} c_2^* = 2\sqrt{r+b-a}.$$

Proof. The proof is similar to that of Theorem 14, so we omit the details here. ☐

Acknowledgements. The authors are grateful to the anonymous reviewer who noted some important papers in reference to our work. Research is supported by the Fundamental Research Funds for the Central Universities (No. 2012ZM0057) and the National Natural Science Foundation of China (No. 11171115).

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(Received June 20 2011)

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