Analysis and Applications, Vol. 13, No. 1 (2015) 23–43

DOI: 10.1142/S0219530514500055

Traveling wave solutions in n-dimensional delayed nonlocal diffusion system with mixed quasimonotonicity

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Received 25 January 2013
Accepted 21 June 2013
Published 30 January 2014

This paper is devoted to the study of an n-dimensional delayed system with nonlocal diffusion and mixed quasimonotonicity. By developing a new definition of upper–lower solutions and a new cross iteration scheme, we establish some existence results of traveling wave solutions. These results are applied to a nonlocal diffusion model which takes the four-species Lotka–Volterra model as its special case.

Keywords: Traveling wave solution; nonlocal diffusion; mixed quasimonotonicity.

Mathematics Subject Classification 2010: 34B40, 34C37, 92C45

1. Introduction

The theory of traveling wave solutions has attracted much attention due to its significant nature in biology, chemistry, epidemiology and physics (see, e.g. [16, 18, 6, 2, 1, 5, 10, 21, 24]). Schaaf [15] systematically studied two scalar reaction–diffusion equations with a single discrete delay for the so-called Huxley nonlinearity as well as Fisher nonlinearity by using the phase space analysis, the maximum principle for parabolic functional differential equations and the general theory for ordinary functional differential equations.

As a classical model in describing the spatial-temporal pattern, the delayed reaction–diffusion system

$$\frac{\partial u(x,t)}{\partial t} = D\frac{\partial^2 u(x,t)}{\partial x^2} + f(u_t(x)), \quad x \in \mathbb{R}, \ t \geq 0 \tag{1.1}$$

has attracted much attention, where $u(x,t) \in \mathbb{R}^n$ and D is a diagonal matrix of diffusion coefficients. Wu and Zou [19] established some existence results of traveling wave solutions for (1.1) by using upper–lower solutions and monotone iteration technique when the nonlinearity f satisfies the so-called quasimonotone (QM) condition or the exponential quasimonotone (QM*) condition. Ma [9] proved the existence of traveling wave solutions for (1.1) by using the Schauder's fixed point theorem under the assumption that the nonlinearity f satisfies (QM) condition. Since Ma [9] only considered delayed system with (QM) reaction terms, Huang and Zou [4] extended the results of Ma [9] to a class of delayed system with (QM*) reaction terms.

Murray [11] pointed out that the general reaction–diffusion system (1.1) has some shortcoming for modeling, in some cases, such as ecological and epidemiological models with spatial diffusion. One way to deal with these problems is to replace the term ${}^{(\partial^2 u(x,t))}/{}_{(\partial x^2)}$ with

$$(J * u)(x,t) - u(x,t) = \int_{\mathbb{R}} J(x-y)u(y,t)dy - u(x,t),$$

where $J(x)$ is an even and nonnegative function with $\int_{\mathbb{R}} J(x)dx = 1$. Recently, Pan *et al.* [12, 13] considered the traveling wavefronts of the following delayed nonlocal diffusion system

$$\begin{aligned}\frac{\partial u_i(x,t)}{\partial t} &= d_i[(J_i * u_i)(x,t) - u_i(x,t)] + f_i(u_t(x)),\\ &\qquad x \in \mathbb{R},\ \ t \geq 0,\ \ i = 1, \ldots, n,\end{aligned} \tag{1.2}$$

where the nonlinear reaction terms $f_i(i = 1, \ldots, n)$ satisfy the (QM) condition or the (QM*) condition.

It is quite common that the reaction terms in a virtual model may satisfy neither the (QM) condition nor the (QM*) condition, such as type-K Lotka–Volterra systems. There are many research works on the generalization of quasimonotonicity condition for two-dimensional delayed reaction–diffusion systems and a few for three-dimensional systems (see, e.g. [23, 25, 3, 17, 7, 22]). Wang and Zhou [8] considered the n-dimensional delayed reaction–diffusion system

$$\begin{cases}\dfrac{\partial u_1(x,t)}{\partial t} = d_1 \dfrac{\partial^2 u_1(x,t)}{\partial x^2} + f_1(u_1(x,t-\tau_{11}), \ldots, u_n(x,t-\tau_{1n})),\\ \vdots \\ \dfrac{\partial u_n(x,t)}{\partial t} = d_n \dfrac{\partial^2 u_n(x,t)}{\partial x^2} + f_n(u_1(x,t-\tau_{n1}), \ldots, u_n(x,t-\tau_{nn})),\end{cases} \tag{1.3}$$

where $d_i > 0$, $\tau_{ij} \geq 0$ denote the time delay and $f_i \in C(\mathbb{R}^n, \mathbb{R})$, $i, j = 1, \ldots, n$. By using the Schauder's fixed point theorem, they obtained some existence results of traveling wave solutions if the nonlinearity terms $f_i(i = 1, \ldots, n)$ satisfy the mixed quasimonotone (MQM) condition or the exponential mixed quasimonotone (MQM*) condition and the following two conditions:

(A1) $f_i(0, \ldots, 0) = f_i(k_1, \ldots, k_n) = 0$, k_i is a positive constant, $i = 1, \ldots, n$;

(A2) There exist n positive constants $L_1, \ldots, L_n$ such that

$$|f_i(\Phi) - f_i(\Psi)| \le L_i \|\Phi - \Psi\|, \quad i = 1, \ldots, n$$

for $\Phi = (\phi_1, \ldots, \phi_n)$, $\Psi = (\psi_1, \ldots, \psi_n) \in C([-\tau, 0], \mathbb{R}^n)$ with $0 \le \phi_i(s), \psi_i(s) \le M_i$, $s \in [-\tau, 0]$, $M_i \ge k_i$ is a positive constant, $i = 1, \ldots, n$. Here $\tau \triangleq \max\{\tau_{ij} \mid 1 \le i, j \le n\}$.

In [8] the condition (A2) plays an important role in proving the existence of traveling wave solutions. Thus, it is natural to ask whether a general reaction term $f = (f_1, \ldots, f_n)$ which does not satisfy the condition (A2) could lead to analogous conclusions. Pan [14] replaced (A2) with a weaker condition:

(A3) $|f(\Phi) - f(\Psi)| \to 0$ as $\|\Phi - \Psi\| \to 0$, for $\Phi = (\phi_1, \ldots, \phi_n)$, $\Psi = (\psi_1, \ldots, \psi_n) \in C([-\tau, 0], \mathbb{R}^n)$ with $0 \le \phi_i(s), \psi_i(s) \le M_i$, $s \in [-\tau, 0]$, $M_i \ge k_i$ is a positive constant, $i = 1, \ldots, n$. Here $\tau \triangleq \max\{\tau_{ij} \mid 1 \le i, j \le n\}$.

By a simple iteration process, she proved the existence of traveling wave solutions for a two-dimensional model. Similarly, for a three-dimensional nonlocal diffusion model satisfying (A1) and (A3), Xu and Weng [20] proved the existence of traveling waves.

In the present paper, we consider the following n-dimensional nonlocal diffusion system with time delays:

$$\begin{cases} \dfrac{\partial u_1(x,t)}{\partial t} = d_1[(J_1 * u_1)(x,t) - u_1(x,t)] + f_1(u_{1_t}(x), \ldots, u_{n_t}(x)), \\ \vdots \\ \dfrac{\partial u_n(x,t)}{\partial t} = d_n[(J_n * u_n)(x,t) - u_n(x,t)] + f_n(u_{1_t}(x), \ldots, u_{n_t}(x)), \end{cases} \tag{1.4}$$

where $t, x \in \mathbb{R}$, $d_i > 0$, $(J_i * u_i)(x,t) = \int_{\mathbb{R}} J_i(x-y) u_i(y,t) dy$, $f_i : C([-\tau, 0]; \mathbb{R}^n) \to \mathbb{R}$ is continuous and satisfies the (MQM*) condition which will be specified later, $i = 1, \ldots, n$, $u_{i_t}(x)$ is an element in $C([-\tau, 0]; \mathbb{R})$ parametrized by $x \in \mathbb{R}$ and given by

$$u_{i_t}(x)(s) = u_i(x, t+s), \quad s \in [-\tau, 0], \ i = 1, \ldots, n.$$

For convenience, we give the following hypotheses about f_i, J_i of (1.4) and all of them will be imposed throughout this paper.

(H1) $f_i(\hat{0}, \ldots, \hat{0}) = f_i(\hat{k}_1, \ldots, \hat{k}_n) = 0$, k_i is a positive constant, $i = 1, \ldots, n$, where $\hat{u} : [-\tau, 0] \to \mathbb{R}$ is the constant function with value u for all $s \in [-\tau, 0]$.

(H2) Let $f = (f_1, \ldots, f_n)$, $|\cdot|$ denote the Euclidean norm in $\mathbb{R}^n$ and $\|\cdot\|$ denote the supremum norm in $C([-\tau, 0], \mathbb{R}^n)$. $|f(\Phi) - f(\Psi)| \to 0$ as $\|\Phi - \Psi\| \to 0$, for $\Phi = (\phi_1, \ldots, \phi_n)$, $\Psi = (\psi_1, \ldots, \psi_n) \in C([-\tau, 0], \mathbb{R}^n)$ with $0 \le \phi_i(s), \psi_i(s) \le M_i$, $s \in [-\tau, 0]$, $M_i \ge k_i$ is a positive constant, $i = 1, \ldots, n$.

(H3) $J_i(x)$ is an even and nonnegative function with $\int_{\mathbb{R}} J_i(x)dx = 1$, $\int_{\mathbb{R}} J_i(x)e^{-\lambda x}dx < \infty$ for any $\lambda > 0$, $i = 1, \ldots, n$.

The rest of this paper is organized as follows. In Sec. 2, we reduce the existence of traveling wave solutions to the existence of fixed points of an operator. In Sec. 3, by the application of a new cross iteration scheme, we prove the existence of traveling wave solutions for the system (1.4) when the reaction terms satisfy the (MQM*) condition. In Sec. 4, we apply our main results to a four-dimensional delayed type-K Lotka–Volterra nonlocal diffusion system and prove the existence of traveling wave solutions.

2. Preliminaries

Throughout this paper, we employ the usual notations for the standard ordering in $\mathbb{R}^n$. That is, for $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$, we denote $u \le v$ if $u_i \le v_i$, $i = 1, \ldots, n$; $u < v$ if $u \le v$ but $u \ne v$; and $u \ll v$ if $u \le v$ but $u_i \ne v_i$, $i = 1, \ldots, n$.

A traveling wave solution of (1.4) is a special translation invariant solution of the form $u_1(x,t) = \varphi_1(x+ct), \ldots, u_n(x,t) = \varphi_n(x+ct)$, where $\varphi_1, \ldots, \varphi_n \in C^1(\mathbb{R}, \mathbb{R})$ are the profiles of the wave that propagates through the one-dimensional spatial domain at a constant speed $c > 0$. Substituting $u_1(x,t) = \varphi_1(x+ct), \ldots, u_n(x,t) = \varphi_n(x+ct)$ into (1.4) and denoting $x + ct$ by ξ, we find that (1.4) has a traveling wave solution if and only if the following wave equations

$$\begin{cases} d_1[(J_1 * \varphi_1)(\xi) - \varphi_1(\xi)] - c\varphi_1'(\xi) + f_1^c(\varphi_{1_\xi}, \ldots, \varphi_{n_\xi}) = 0, \\ \vdots \\ d_n[(J_n * \varphi_n)(\xi) - \varphi_n(\xi)] - c\varphi_n'(\xi) + f_n^c(\varphi_{1_\xi}, \ldots, \varphi_{n_\xi}) = 0 \end{cases} \tag{2.1}$$

with asymptotic boundary conditions

$$\lim_{\xi \to \pm\infty} \varphi_i(\xi) = \varphi_{i\pm}, \quad i = 1, \ldots, n \tag{2.2}$$

have a solution $(\varphi_1(\xi), \ldots, \varphi_n(\xi))$ on $\mathbb{R}$, where $f_i^c : C([-c\tau, 0], \mathbb{R}^n) \to \mathbb{R}$ is given by

$$f_i^c(\varphi_{1_\xi}, \ldots, \varphi_{n_\xi}) = f_i(\varphi_{1_\xi}^c, \ldots, \varphi_{n_\xi}^c),$$
$$\varphi_{i_\xi}^c(s) = \varphi_{i_\xi}(cs) = \varphi_i(\xi + cs), \quad s \in [-\tau, 0], \ \ i = 1, \ldots, n,$$

$(\varphi_{1-}, \ldots, \varphi_{n-})$ and $(\varphi_{1+}, \ldots, \varphi_{n+})$ are two equilibria of (2.1). Without loss of generality, we let $(\varphi_{1-}, \ldots, \varphi_{n-}) = (0, \ldots, 0)$ and $(\varphi_{1+}, \ldots, \varphi_{n+}) = (k_1, \ldots, k_n)$. The boundary conditions (2.2) become

$$\lim_{\xi \to -\infty} \varphi_i(\xi) = 0, \quad \lim_{\xi \to +\infty} \varphi_i(\xi) = k_i, \quad i = 1, \ldots, n. \tag{2.3}$$

For $\mathbf{M} = (M_1, \ldots, M_n)$, $M_i \ge k_i$ is a positive constant, $i = 1, \ldots, n$, let

$$C_{[\mathbf{0},\mathbf{M}]}(\mathbb{R}, \mathbb{R}^n) = \{(\varphi_1, \ldots, \varphi_n) \in C(\mathbb{R}, \mathbb{R}^n) : 0 \le \varphi_i(\xi) \le M_i, i = 1, \ldots, n, \xi \in \mathbb{R}\}.$$

For $(\varphi_1, \ldots, \varphi_n) \in C_{[\mathbf{0},\mathbf{M}]}(\mathbb{R}, \mathbb{R}^n)$ and constants $\beta_i > 0$, $i = 1, \ldots, n$, define $H = (H_1, \ldots, H_n) : C_{[\mathbf{0},\mathbf{M}]}(\mathbb{R}, \mathbb{R}^n) \to C(\mathbb{R}, \mathbb{R}^n)$ by

$$\begin{aligned} H_i(\varphi_1, \ldots, \varphi_n)(\xi) &= d_i(J_i * \varphi_i)(\xi) + (\beta_i - d_i)\varphi_i(\xi) \\ &\quad + f_i^c(\varphi_{1_\xi}, \ldots, \varphi_{n_\xi}), \quad i = 1, \ldots, n. \end{aligned} \tag{2.4}$$

Equation (2.1) can be rewritten as

$$\begin{cases} c\varphi_1'(\xi) = -\beta_1\varphi_1(\xi) + H_1(\varphi_1, \ldots, \varphi_n)(\xi), \\ \vdots \\ c\varphi_n'(\xi) = -\beta_n\varphi_n(\xi) + H_n(\varphi_1, \ldots, \varphi_n)(\xi). \end{cases} \tag{2.5}$$

Define $F = (F_1, \ldots, F_n) : C_{[\mathbf{0},\mathbf{M}]}(\mathbb{R}, \mathbb{R}^n) \to C(\mathbb{R}, \mathbb{R}^n)$ by

$$F_i(\varphi_1, \ldots, \varphi_n)(\xi) = \frac{1}{c} e^{-\frac{\beta_i}{c}\xi} \int_{-\infty}^{\xi} e^{\frac{\beta_i}{c}s} H_i(\varphi_1, \ldots, \varphi_n)(s)ds, \quad i = 1, \ldots, n. \tag{2.6}$$

Then it is clear that a fixed point of F is a traveling wave solution of (1.4) connecting $\mathbf{0} = (0, \ldots, 0)$ with $\mathbf{K} = (k_1, \ldots, k_n)$ if it satisfies (2.3).

3. Main Results

In this section, we consider the nonlocal diffusion system (1.4) with the following exponential mixed quasimonotonicity reaction terms:

(MQM*) There exist n positive constants $\beta_1, \ldots, \beta_n$ such that for $i = 1, \ldots, k$ and $j = k+1, \ldots, n$,

$$\begin{cases} f_i(\phi_1, \ldots, \phi_n) - f_i(\psi_1, \ldots, \psi_k, \phi_{k+1}, \ldots, \phi_n) + (\beta_i - d_i)(\phi_i(0) - \psi_i(0)) \geq 0, \\ f_i(\phi_1, \ldots, \phi_n) - f_i(\phi_1, \ldots, \phi_k, \psi_{k+1}, \ldots, \psi_n) \leq 0, \\ f_j(\phi_1, \ldots, \phi_n) - f_j(\phi_1, \ldots, \phi_k, \psi_{k+1}, \ldots, \psi_n) + (\beta_j - d_j)(\phi_j(0) - \psi_j(0)) \geq 0, \\ f_j(\phi_1, \ldots, \phi_n) - f_j(\psi_1, \ldots, \psi_k, \phi_{k+1}, \ldots, \phi_n) \leq 0 \end{cases}$$

for $\Phi = (\phi_1, \ldots, \phi_n)$, $\Psi = (\psi_1, \ldots, \psi_n) \in C([-\tau, 0], \mathbb{R}^n)$ with

$$\begin{cases} \text{(i) } 0 \leq \psi_l(s) \leq \phi_l(s) \leq M_l, \quad s \in [-\tau, 0], \quad l = 1, \ldots, n; \\ \text{(ii) } e^{\beta_l s}[\phi_l(s) - \psi_l(s)] \text{ is nondecreasing in } s \in [-\tau, 0], \quad l = 1, \ldots, n. \end{cases}$$

First, we give a lemma on the operators H and F.

Lemma 3.1. *Assume that* (MQM*) *holds. Then*

$$H_i(\phi_1, \ldots, \phi_k, \psi_{k+1}, \ldots, \psi_n)(\xi) \geq H_i(\psi_1, \ldots, \psi_k, \phi_{k+1}, \ldots, \phi_n)(\xi),$$

$$H_j(\phi_1, \ldots, \phi_k, \psi_{k+1}, \ldots, \psi_n)(\xi) \leq H_j(\psi_1, \ldots, \psi_k, \phi_{k+1}, \ldots, \phi_n)(\xi),$$

and

$$F_i(\phi_1, \ldots, \phi_k, \psi_{k+1}, \ldots, \psi_n)(\xi) \geq F_i(\psi_1, \ldots, \psi_k, \phi_{k+1}, \ldots, \phi_n)(\xi),$$

$$F_j(\phi_1, \ldots, \phi_k, \psi_{k+1}, \ldots, \psi_n)(\xi) \leq F_j(\psi_1, \ldots, \psi_k, \phi_{k+1}, \ldots, \phi_n)(\xi),$$

for $\Phi = (\phi_1, \ldots, \phi_n)$, $\Psi = (\psi_1, \ldots, \psi_n) \in C(\mathbb{R}, \mathbb{R}^n)$ *with*

$$\begin{cases} \text{(i) } \mathbf{0} \le \Psi \le \Phi \le \mathbf{M}; \\ \text{(ii) } e^{\beta_l \xi}[\phi_l(\xi) - \psi_l(\xi)] \text{ is nondecreasing in } \xi \in \mathbb{R}, \quad l = 1, \ldots, n, \end{cases}$$

where $i = 1, \ldots, k$ *and* $j = k+1, \ldots, n$.

Proof. By (MQM*) and the definition of operator H, for $i = 1, \ldots, k$ and $j = k+1, \ldots, n$, we have

$$\begin{aligned} &H_i(\phi_1, \ldots, \phi_k, \psi_{k+1}, \ldots, \psi_n)(\xi) - H_i(\psi_1, \ldots, \psi_k, \phi_{k+1}, \ldots, \phi_n)(\xi) \\ &\quad = d_i(J_i * (\phi_i - \psi_i))(\xi) + (\beta_i - d_i)(\phi_i(\xi) - \psi_i(\xi)) \\ &\qquad + f_i^c(\phi_{1_\xi}, \ldots, \phi_{k_\xi}, \psi_{k+1_\xi}, \ldots, \psi_{n_\xi}) - f_i^c(\psi_{1_\xi}, \ldots, \psi_{k_\xi}, \phi_{k+1_\xi}, \ldots, \phi_{n_\xi}) \\ &\quad \ge (\beta_i - d_i)(\phi_i(\xi) - \psi_i(\xi)) + f_i^c(\phi_{1_\xi}, \ldots, \phi_{n_\xi}) - f_i^c(\psi_{1_\xi}, \ldots, \psi_{k_\xi}, \phi_{k+1_\xi}, \ldots, \phi_{n_\xi}) \\ &\qquad + f_i^c(\phi_{1_\xi}, \ldots, \phi_{k_\xi}, \psi_{k+1_\xi}, \ldots, \psi_{n_\xi}) - f_i^c(\phi_{1_\xi}, \ldots, \phi_{n_\xi}) \\ &\quad \ge 0, \\ &H_j(\phi_1, \ldots, \phi_k, \psi_{k+1}, \ldots, \psi_n)(\xi) - H_j(\psi_1, \ldots, \psi_k, \phi_{k+1}, \ldots, \phi_n)(\xi) \\ &\quad = d_j(J_j * (\psi_j - \phi_j))(\xi) + (\beta_j - d_j)(\psi_j(\xi) - \phi_j(\xi)) \\ &\qquad + f_j^c(\phi_{1_\xi}, \ldots, \phi_{k_\xi}, \psi_{k+1_\xi}, \ldots, \psi_{n_\xi}) - f_j^c(\psi_{1_\xi}, \ldots, \psi_{k_\xi}, \phi_{k+1_\xi}, \ldots, \phi_{n_\xi}) \\ &\quad \le (\beta_j - d_j)(\psi_j(\xi) - \phi_j(\xi)) + f_j^c(\phi_{1_\xi}, \ldots, \phi_{k_\xi}, \psi_{k+1_\xi}, \ldots, \psi_{n_\xi}) - f_j^c(\phi_{1_\xi}, \ldots, \phi_{n_\xi}) \\ &\qquad + f_j^c(\phi_{1_\xi}, \ldots, \phi_{n_\xi}) - f_j^c(\psi_{1_\xi}, \ldots, \psi_{k_\xi}, \phi_{k+1_\xi}, \ldots, \phi_{n_\xi}) \\ &\quad \le 0. \end{aligned}$$

From the definition of F_i in (2.6), the rest of the lemma is obvious. □

Now, we are in a position to give the new definition of a pair of upper and lower solutions of (2.1).

Definition 3.2. A pair of continuous functions $\overline{\Phi}(\xi) = (\overline{\varphi}_1(\xi), \ldots, \overline{\varphi}_n(\xi))$ and $\underline{\Phi}(\xi) = (\underline{\varphi}_1(\xi), \ldots, \underline{\varphi}_n(\xi))$ are called an upper solution and a lower solution of (2.1), respectively, if $\overline{\Phi}(\xi)$, $\underline{\Phi}(\xi)$ are differentiable on $\mathbb{R}\backslash\Upsilon$ and satisfy

$$\begin{cases} c\overline{\varphi}_i'(\xi) - d_i[(J_i * \overline{\varphi}_i)(\xi) - \overline{\varphi}_i(\xi)] - f_i^c(\overline{\varphi}_{1_\xi}, \ldots, \overline{\varphi}_{k_\xi}, \underline{\varphi}_{k+1_\xi}, \ldots, \underline{\varphi}_{n_\xi}) \ge 0, \\ c\overline{\varphi}_j'(\xi) - d_j[(J_j * \overline{\varphi}_j)(\xi) - \overline{\varphi}_j(\xi)] - f_j^c(\underline{\varphi}_{1_\xi}, \ldots, \underline{\varphi}_{k_\xi}, \overline{\varphi}_{k+1_\xi}, \ldots, \overline{\varphi}_{n_\xi}) \ge 0, \\ c\underline{\varphi}_i'(\xi) - d_i[(J_i * \underline{\varphi}_i)(\xi) - \underline{\varphi}_i(\xi)] - f_i^c(\underline{\varphi}_{1_\xi}, \ldots, \underline{\varphi}_{k_\xi}, \overline{\varphi}_{k+1_\xi}, \ldots, \overline{\varphi}_{n_\xi}) \le 0, \\ c\underline{\varphi}_j'(\xi) - d_j[(J_j * \underline{\varphi}_j)(\xi) - \underline{\varphi}_j(\xi)] - f_j^c(\overline{\varphi}_{1_\xi}, \ldots, \overline{\varphi}_{k_\xi}, \underline{\varphi}_{k+1_\xi}, \ldots, \underline{\varphi}_{n_\xi}) \le 0, \end{cases} \tag{3.1}$$

for $\xi \in \mathbb{R}\backslash\Upsilon$, $i = 1,\ldots,k$ and $j = k+1,\ldots,n$, where $\Upsilon = \{\xi_1, \xi_2, \ldots, \xi_r\}$ is a finite set of points with $\xi_1 < \xi_2 < \cdots < \xi_r$.

In what follows, we assume that there exist an upper solution $\overline{\Phi}(\xi) = (\overline{\varphi}_1(\xi),\ldots,\overline{\varphi}_n(\xi))$ and a lower solution $\underline{\Phi}(\xi) = (\underline{\varphi}_1(\xi),\ldots,\underline{\varphi}_n(\xi))$ of (2.1) satisfying the following properties

(P1) $\mathbf{0} \le \underline{\Phi}(\xi) \le \overline{\Phi}(\xi) \le \mathbf{M}$, $\xi \in \mathbb{R}$,
(P2) $\lim_{\xi\to-\infty} \overline{\Phi}(\xi) = \mathbf{0}$, $\lim_{\xi\to+\infty} \underline{\Phi}(\xi) = \lim_{\xi\to+\infty} \overline{\Phi}(\xi) = \mathbf{K}$,
(P3) $e^{\beta_l \xi}[\overline{\varphi}_l(\xi) - \underline{\varphi}_l(\xi)]$ is nondecreasing in $\xi \in \mathbb{R}$, $l = 1,\ldots,n$.

Define the following profile set:

$$\Gamma = \left\{ \Phi \in C_{[\mathbf{0},\mathbf{M}]}(\mathbb{R},\mathbb{R}^n) : \begin{array}{c} \text{(i) } \underline{\Phi}(\xi) \le \Phi(\xi) \le \overline{\Phi}(\xi),\ \xi \in \mathbb{R}; \\ \text{(ii) } e^{\beta_l\xi}[\overline{\varphi}_l(\xi) - \varphi_l(\xi)] \text{ and } e^{\beta_l\xi}[\varphi_l(\xi) - \underline{\varphi}_l(\xi)] \\ \text{are nondecreasing in } \xi \in \mathbb{R},\ l = 1,\ldots,n \end{array} \right\}.$$

Clearly, $\overline{\Phi} \in \Gamma$, $\underline{\Phi} \in \Gamma$ by (P1) and (P3). Now we start the iteration scheme with $\overline{\Phi}$ and $\underline{\Phi}$. Define

$$\begin{cases} \overline{\varphi}_i^{(m)}(\xi) = F_i(\overline{\varphi}_1^{(m-1)},\ldots,\overline{\varphi}_k^{(m-1)},\underline{\varphi}_{k+1}^{(m-1)},\ldots,\underline{\varphi}_n^{(m-1)})(\xi), \\ \overline{\varphi}_j^{(m)}(\xi) = F_j(\underline{\varphi}_1^{(m-1)},\ldots,\underline{\varphi}_k^{(m-1)},\overline{\varphi}_{k+1}^{(m-1)},\ldots,\overline{\varphi}_n^{(m-1)})(\xi), \\ \underline{\varphi}_i^{(m)}(\xi) = F_i(\underline{\varphi}_1^{(m-1)},\ldots,\underline{\varphi}_k^{(m-1)},\overline{\varphi}_{k+1}^{(m-1)},\ldots,\overline{\varphi}_n^{(m-1)})(\xi), \\ \underline{\varphi}_j^{(m)}(\xi) = F_j(\overline{\varphi}_1^{(m-1)},\ldots,\overline{\varphi}_k^{(m-1)},\underline{\varphi}_{k+1}^{(m-1)},\ldots,\underline{\varphi}_n^{(m-1)})(\xi), \\ (\overline{\varphi}_1^{(0)}(\xi),\ldots,\overline{\varphi}_n^{(0)}(\xi)) = (\overline{\varphi}_1(\xi),\ldots,\overline{\varphi}_n(\xi)), \\ (\underline{\varphi}_1^{(0)}(\xi),\ldots,\underline{\varphi}_n^{(0)}(\xi)) = (\underline{\varphi}_1(\xi),\ldots,\underline{\varphi}_n(\xi)), \end{cases} \tag{3.2}$$

where $\xi \in \mathbb{R}$, $i = 1,\ldots,k$ and $j = k+1,\ldots,n$; $m = 1,2\ldots$. Then we have the following result.

Lemma 3.3. *Assume that* (MQM*) *holds. If* $c \ge 1$, *then the functions* $\overline{\Phi}^{(m)}(\xi) = (\overline{\varphi}_1^{(m)}(\xi),\ldots,\overline{\varphi}_n^{(m)}(\xi))$ *and* $\underline{\Phi}^{(m)}(\xi) = (\underline{\varphi}_1^{(m)}(\xi),\ldots,\underline{\varphi}_n^{(m)}(\xi))$ $(m = 1,2,\ldots)$ *defined by* (3.2) *satisfy the following*:

(i) $\underline{\Phi}^{(m-1)}(\xi) \le \underline{\Phi}^{(m)}(\xi) \le \overline{\Phi}^{(m)}(\xi) \le \overline{\Phi}^{(m-1)}(\xi)$;
(ii) $\overline{\Phi}^{(m)} \in \Gamma$, $\underline{\Phi}^{(m)} \in \Gamma$;
(iii) $\overline{\Phi}^{(m)}(\xi)$ *and* $\underline{\Phi}^{(m)}(\xi)$ *are upper and lower solutions of* (2.1), *respectively.*

Proof. We first show that the conclusion is true for $m = 1$. Since $\overline{\Phi}(\xi)$ and $\underline{\Phi}(\xi)$ are upper and lower solutions of (2.1), respectively, for $i = 1,\ldots,k$, we have

$$\begin{aligned} &H_i(\underline{\varphi}_1,\ldots,\underline{\varphi}_k,\overline{\varphi}_{k+1},\ldots,\overline{\varphi}_n)(\xi) \\ &\quad = f_i^c(\underline{\varphi}_{1_\xi},\ldots,\underline{\varphi}_{k_\xi},\overline{\varphi}_{k+1_\xi},\ldots,\overline{\varphi}_{n_\xi}) + (\beta_i - d_i)\underline{\varphi}_i(\xi) + d_i(J_i * \underline{\varphi}_i)(\xi) \\ &\quad \ge c\underline{\varphi}_i'(\xi) + \beta_i\underline{\varphi}_i(\xi), \quad \text{for } \xi \in \mathbb{R}\backslash\Upsilon. \end{aligned}$$

Let $\xi_0 = -\infty$ and $\xi_{r+1} = +\infty$; then for $\xi_{q-1} < \xi < \xi_q$ with $q = 1, 2, \ldots, r+1$, we have

$$\begin{aligned}
\underline{\varphi}_i^{(1)}(\xi) &= F_i(\underline{\varphi}_1, \ldots, \underline{\varphi}_k, \overline{\varphi}_{k+1}, \ldots, \overline{\varphi}_n)(\xi) \\
&= \frac{1}{c} e^{-\frac{\beta_i}{c}\xi} \int_{-\infty}^{\xi} e^{\frac{\beta_i}{c}s} H_i(\underline{\varphi}_1, \ldots, \underline{\varphi}_k, \overline{\varphi}_{k+1}, \ldots, \overline{\varphi}_n)(s) ds \\
&\geq \frac{1}{c} e^{-\frac{\beta_i}{c}\xi} \left\{ \left(\sum_{j=1}^{q-1} \int_{\xi_{j-1}}^{\xi_j} + \int_{\xi_{q-1}}^{\xi} \right) e^{\frac{\beta_i}{c}s} [c\underline{\varphi}_i'(s) + \beta_i \underline{\varphi}_i(s)] ds \right\} \\
&= \underline{\varphi}_i(\xi), \quad \text{for } \xi \in \mathbb{R} \backslash \Upsilon.
\end{aligned}$$

By the continuity of $\underline{\varphi}_i^{(1)}(\xi)$ and $\underline{\varphi}_i(\xi)$, we get that $\underline{\varphi}_i(\xi) \leq \underline{\varphi}_i^{(1)}(\xi)$ for $\xi \in \mathbb{R}$.

Similarly, we can show that

$$\begin{aligned}
(\underline{\varphi}_1(\xi), \ldots, \underline{\varphi}_n(\xi)) &\leq (\underline{\varphi}_1^{(1)}(\xi), \ldots, \underline{\varphi}_n^{(1)}(\xi)), \\
(\overline{\varphi}_1^{(1)}(\xi), \ldots, \overline{\varphi}_n^{(1)}(\xi)) &\leq (\overline{\varphi}_1(\xi), \ldots, \overline{\varphi}_n(\xi))
\end{aligned}$$

for $\xi \in \mathbb{R}$. By Lemma 3.1, it is easy to see that

$$(\underline{\varphi}_1^{(1)}(\xi), \ldots, \underline{\varphi}_n^{(1)}(\xi)) \leq (\overline{\varphi}_1^{(1)}(\xi), \ldots, \overline{\varphi}_n^{(1)}(\xi))$$

for $\xi \in \mathbb{R}$. This completes the proof of (i) for $m = 1$.

Now we prove the conclusion (ii) for $m = 1$. Firstly, (i) indicates that $\underline{\Phi}^{(1)}(\xi) \leq \Phi(\xi) \leq \overline{\Phi}^{(1)}(\xi), \xi \in \mathbb{R}$. Now, noting the fact that

$$\phi(\xi) = \frac{1}{c} e^{-\frac{\beta}{c}\xi} \int_{-\infty}^{\xi} e^{\frac{\beta}{c}s} [c\phi'(s) + \beta\phi(s)] ds$$

for any $c, \beta \in \mathbb{R}$, $\phi \in C^1(\mathbb{R}, \mathbb{R})$, we have from a direct calculation that for $i = 1, \ldots, k$

$$\begin{aligned}
&e^{\beta_i \xi}[\underline{\varphi}_i^{(1)}(\xi) - \underline{\varphi}_i(\xi)] \\
&= \frac{1}{c} e^{(\beta_i - \frac{\beta_i}{c})\xi} \int_{-\infty}^{\xi} e^{\frac{\beta_i}{c}s} [c[\underline{\varphi}_i^{(1)}]'(s) + \beta_i \underline{\varphi}_i^{(1)}(s) - (c\underline{\varphi}_i'(s) + \beta_i \underline{\varphi}_i(s))] ds \\
&= \frac{1}{c} e^{(\beta_i - \frac{\beta_i}{c})\xi} \int_{-\infty}^{\xi} e^{\frac{\beta_i}{c}s} [c[\underline{\varphi}_i^{(1)}]'(s) + \beta_i \underline{\varphi}_i^{(1)}(s) - H_i(\underline{\varphi}_1, \ldots, \underline{\varphi}_k, \overline{\varphi}_{k+1}, \ldots, \overline{\varphi}_n)(s) \\
&\quad - (c\underline{\varphi}_i'(s) + \beta_i \underline{\varphi}_i(s) - H_i(\underline{\varphi}_1, \ldots, \underline{\varphi}_k, \overline{\varphi}_{k+1}, \ldots, \overline{\varphi}_n)(s))] ds \\
&= -\frac{1}{c} e^{(\beta_i - \frac{\beta_i}{c})\xi} \int_{-\infty}^{\xi} e^{\frac{\beta_i}{c}s} [c\underline{\varphi}_i'(s) + \beta_i \underline{\varphi}_i(s) - H_i(\underline{\varphi}_1, \ldots, \underline{\varphi}_k, \overline{\varphi}_{k+1}, \ldots, \overline{\varphi}_n)(s)] ds
\end{aligned}$$

and

$$\begin{aligned}
&\frac{d}{d\xi}[e^{\beta_i\xi}[\underline{\varphi}_i^{(1)}(\xi) - \underline{\varphi}_i(\xi)]] \\
&\quad = -\frac{1}{c}\left(\beta_i - \frac{\beta_i}{c}\right)e^{(\beta_i - \frac{\beta_i}{c})\xi} \\
&\qquad \times \int_{-\infty}^{\xi} e^{\frac{\beta_i}{c}s}[c\underline{\varphi}_i'(s) + \beta_i\underline{\varphi}_i(s) - H_i(\underline{\varphi}_1,\ldots,\underline{\varphi}_k,\overline{\varphi}_{k+1},\ldots,\overline{\varphi}_n)(s)]ds \\
&\qquad - \frac{1}{c}e^{\beta_i\xi}[c\underline{\varphi}_i'(\xi) + \beta_i\underline{\varphi}_i(\xi) - H_i(\underline{\varphi}_1,\ldots,\underline{\varphi}_k,\overline{\varphi}_{k+1},\ldots,\overline{\varphi}_n)(\xi)] \\
&\quad \geq 0 \quad \text{if } c \geq 1.
\end{aligned}$$

Similarly, we can show that $\overline{\Phi}^{(1)}(\xi)$ and $\underline{\Phi}^{(1)}(\xi)$ satisfy (ii) of Γ. This completes the proof of (ii) for $m = 1$.

Now we prove the conclusion (iii) for $m = 1$. In fact, we have from (i) and Lemma 3.1 that for $i = 1,\ldots,k$,

$$\begin{aligned}
&c[\underline{\varphi}_i^{(1)}]'(\xi) - d_i[(J_i * \underline{\varphi}_i^{(1)})(\xi) - \underline{\varphi}_i^{(1)}(\xi)] - f_i^c(\underline{\varphi}_{1_\xi}^{(1)},\ldots,\underline{\varphi}_{k_\xi}^{(1)},\overline{\varphi}_{k+1_\xi}^{(1)},\ldots,\overline{\varphi}_{n_\xi}^{(1)}) \\
&\quad = c[\underline{\varphi}_i^{(1)}]'(\xi) + \beta_i\underline{\varphi}_i^{(1)}(\xi) - H_i(\underline{\varphi}_1^{(1)},\ldots,\underline{\varphi}_k^{(1)},\overline{\varphi}_{k+1}^{(1)},\ldots,\overline{\varphi}_n^{(1)})(\xi) \\
&\quad \leq c[\underline{\varphi}_i^{(1)}]'(\xi) + \beta_i\underline{\varphi}_i^{(1)}(\xi) - H_i(\underline{\varphi}_1,\ldots,\underline{\varphi}_k,\overline{\varphi}_{k+1},\ldots,\overline{\varphi}_n)(\xi) \\
&\quad = 0 \quad \text{for } \xi \in \mathbb{R}.
\end{aligned}$$

In a similar way, we can get that for $i = 1,\ldots,k$ and $j = k+1,\ldots,n$,

$$c[\underline{\varphi}_j^{(1)}]'(\xi) - d_j[(J_j * \underline{\varphi}_j^{(1)})(\xi) - \underline{\varphi}_j^{(1)}(\xi)] - f_j^c(\overline{\varphi}_{1_\xi}^{(1)},\ldots,\overline{\varphi}_{k_\xi}^{(1)},\underline{\varphi}_{k+1_\xi}^{(1)},\ldots,\underline{\varphi}_{n_\xi}^{(1)}) \leq 0,$$

$$c[\overline{\varphi}_i^{(1)}]'(\xi) - d_i[(J_i * \overline{\varphi}_i^{(1)})(\xi) - \overline{\varphi}_i^{(1)}(\xi)] - f_i^c(\overline{\varphi}_{1_\xi}^{(1)},\ldots,\overline{\varphi}_{k_\xi}^{(1)},\underline{\varphi}_{k+1_\xi}^{(1)},\ldots,\underline{\varphi}_{n_\xi}^{(1)}) \geq 0,$$

$$c[\overline{\varphi}_j^{(1)}]'(\xi) - d_j[(J_j * \overline{\varphi}_j^{(1)})(\xi) - \overline{\varphi}_j^{(1)}(\xi)] - f_j^c(\underline{\varphi}_{1_\xi}^{(1)},\ldots,\underline{\varphi}_{k_\xi}^{(1)},\overline{\varphi}_{k+1_\xi}^{(1)},\ldots,\overline{\varphi}_{n_\xi}^{(1)}) \geq 0.$$

By the above argument, we know that the conclusion of Lemma 3.3 is true for $m = 1$. Suppose the conclusion is true for $m = p$. Then by induction, it is easy to see that the conclusion holds for $m = p+1$. Therefore, the conclusion holds for any positive integer m. The proof is completed. □

Lemma 3.4. *There exist* $\overline{\Phi}^* = (\overline{\varphi}_1^*,\ldots,\overline{\varphi}_n^*) \in \Gamma$, $\underline{\Phi}^* = (\underline{\varphi}_1^*,\ldots,\underline{\varphi}_n^*) \in \Gamma$ *such that*

$$\lim_{m\to\infty} \overline{\Phi}^{(m)}(\xi) = \overline{\Phi}^*(\xi), \quad \lim_{m\to\infty} \underline{\Phi}^{(m)}(\xi) = \underline{\Phi}^*(\xi),$$

and the convergence is uniform with respect to the supremum norm.

Proof. Note that Γ is a closed set with respect to the supremum norm in $C(\mathbb{R},\mathbb{R}^n)$. By the monotonicity and boundedness of $\overline{\Phi}^{(m)}$ and $\underline{\Phi}^{(m)}$, there exist continuous functions $\overline{\Phi}^* = (\overline{\varphi}_1^*, \ldots, \overline{\varphi}_n^*) \in \Gamma$, $\underline{\Phi}^* = (\underline{\varphi}_1^*, \ldots, \underline{\varphi}_n^*) \in \Gamma$ such that

$$\lim_{m\to\infty} \overline{\Phi}^{(m)}(\xi) = \overline{\Phi}^*(\xi), \quad \lim_{m\to\infty} \underline{\Phi}^{(m)}(\xi) = \underline{\Phi}^*(\xi),$$

and the convergence is pointwise. We now prove that the convergence is uniform in $\xi \in \mathbb{R}$.

By (P2) and Lemma 3.3, for any given $\varepsilon > 0$, there exists $T = T(\varepsilon) > 0$ such that

$$\sup_{\xi<-T} |\overline{\Phi}^{(m)}(\xi)| < \frac{\varepsilon}{2}, \quad \sup_{\xi>T} |\overline{\Phi}^{(m)}(\xi) - \mathbf{K}| < \frac{\varepsilon}{2},$$

$$\sup_{\xi<-T} |\underline{\Phi}^{(m)}(\xi)| < \frac{\varepsilon}{2}, \quad \sup_{\xi>T} |\underline{\Phi}^{(m)}(\xi) - \mathbf{K}| < \frac{\varepsilon}{2}$$

for all $m \in \mathbb{N}$, and this also implies that

$$\sup_{\xi<-T} |\overline{\Phi}^*(\xi)| < \frac{\varepsilon}{2}, \quad \sup_{\xi>T} |\overline{\Phi}^*(\xi) - \mathbf{K}| < \frac{\varepsilon}{2},$$

$$\sup_{\xi<-T} |\underline{\Phi}^*(\xi)| < \frac{\varepsilon}{2}, \quad \sup_{\xi>T} |\underline{\Phi}^*(\xi) - \mathbf{K}| < \frac{\varepsilon}{2}.$$

Thus, for all $m \in \mathbb{N}$,

$$\sup_{|\xi|>T} |\overline{\Phi}^{(m)}(\xi) - \overline{\Phi}^*(\xi)| < \varepsilon,$$

$$\sup_{|\xi|>T} |\underline{\Phi}^{(m)}(\xi) - \underline{\Phi}^*(\xi)| < \varepsilon.$$

We now consider the sequences $\overline{\Phi}^{(m)}(\xi)|_{\xi\in[-T,T]}$, $\underline{\Phi}^{(m)}(\xi)|_{\xi\in[-T,T]}$ for $m \geq 1$. It is clear that $\overline{\Phi}^{(m)}(\xi)|_{\xi\in[-T,T]}$, $\underline{\Phi}^{(m)}(\xi)|_{\xi\in[-T,T]}$, $[\overline{\Phi}^{(m)}]'(\xi)|_{\xi\in[-T,T]}$, $[\underline{\Phi}^{(m)}]' \times (\xi)|_{\xi\in[-T,T]}$, $m \geq 1$ are uniformly bounded. Then the uniform continuity implies that there exist subsequences of $\overline{\Phi}^{(m)}(\xi)|_{\xi\in[-T,T]}$ and $\underline{\Phi}^{(m)}(\xi)|_{\xi\in[-T,T]}$ which are uniformly convergent with respect to the supremum norm of space $C([-T,T],\mathbb{R}^n)$ according to the Ascoli–Arzela lemma. Combining this with the monotonicity of $\overline{\Phi}^{(m)}(\xi)|_{\xi\in[-T,T]}$, $\underline{\Phi}^{(m)}(\xi)|_{\xi\in[-T,T]}$ with respect to m, there exists a positive integer N^* such that

$$\sup_{\xi\in[-T,T]} |\overline{\Phi}^{(m)}(\xi) - \overline{\Phi}^*(\xi)| < \varepsilon,$$

$$\sup_{\xi\in[-T,T]} |\underline{\Phi}^{(m)}(\xi) - \underline{\Phi}^*(\xi)| < \varepsilon$$

for all $m > N^*$, which further indicates that

$$\sup_{\xi\in\mathbb{R}} |\overline{\Phi}^{(m)}(\xi) - \overline{\Phi}^*(\xi)| < \varepsilon,$$

$$\sup_{\xi\in\mathbb{R}} |\underline{\Phi}^{(m)}(\xi) - \underline{\Phi}^*(\xi)| < \varepsilon$$

for all $m > N^*$. The proof is completed. □

Now we state our main theorem.

Theorem 3.5. *Assume that* (H1)–(H3) *and* (MQM*) *hold. If* (2.1) *has an upper solution* $\overline{\Phi}(\xi) = (\overline{\varphi}_1(\xi), \ldots, \overline{\varphi}_n(\xi))$ *and a lower solution* $\underline{\Phi}(\xi) = (\underline{\varphi}_1(\xi), \ldots, \underline{\varphi}_n(\xi))$ *satisfying* (P1)–(P3), *then* (1.4) *has at least a traveling wave solution satisfying* (2.3) *for any* $c \geq 1$.

Proof. It is clear that

$$\begin{aligned}&(\overline{\varphi}_1^*(\xi), \ldots, \overline{\varphi}_k^*(\xi), \underline{\varphi}_{k+1}^*(\xi), \ldots, \underline{\varphi}_n^*(\xi))\\ &\quad = (F_1(\overline{\varphi}_1^*, \ldots, \overline{\varphi}_k^*, \underline{\varphi}_{k+1}^*, \ldots, \underline{\varphi}_n^*)(\xi), \ldots, F_n(\overline{\varphi}_1^*, \ldots, \overline{\varphi}_k^*, \underline{\varphi}_{k+1}^*, \ldots, \underline{\varphi}_n^*)(\xi))\end{aligned}$$

and

$$\begin{aligned}&(\underline{\varphi}_1^*(\xi), \ldots, \underline{\varphi}_k^*(\xi), \overline{\varphi}_{k+1}^*(\xi), \ldots, \overline{\varphi}_n^*(\xi))\\ &\quad = (F_1(\underline{\varphi}_1^*, \ldots, \underline{\varphi}_k^*, \overline{\varphi}_{k+1}^*, \ldots, \overline{\varphi}_n^*)(\xi), \ldots, F_n(\underline{\varphi}_1^*, \ldots, \underline{\varphi}_k^*, \overline{\varphi}_{k+1}^*, \ldots, \overline{\varphi}_n^*)(\xi))\end{aligned}$$

for $\xi \in \mathbb{R}$ by the Lebesgues dominated convergence theorem and (3.2). Then $(\overline{\varphi}_1^*, \ldots, \overline{\varphi}_k^*, \underline{\varphi}_{k+1}^*, \ldots, \underline{\varphi}_n^*)$ and $(\underline{\varphi}_1^*, \ldots, \underline{\varphi}_k^*, \overline{\varphi}_{k+1}^*, \ldots, \overline{\varphi}_n^*)$ are two fixed points of F, which also satisfy (2.1). Therefore, (P2) indicates that $(\overline{\varphi}_1^*(\xi), \ldots, \overline{\varphi}_k^*(\xi), \underline{\varphi}_{k+1}^*(\xi), \ldots, \underline{\varphi}_n^*(\xi))$ and $(\underline{\varphi}_1^*(\xi), \ldots, \underline{\varphi}_k^*(\xi), \overline{\varphi}_{k+1}^*(\xi), \ldots, \overline{\varphi}_n^*(\xi))$ are two traveling wave solutions of (1.4), which may be the same. The proof is completed. □

Corollary 3.6. *Assume that* $\Phi = (\varphi_1, \ldots, \varphi_n) \in \Gamma$ *is a traveling wave solution of* (1.4) *satisfying* (2.3). *Then we have*

$$\underline{\Phi}^*(\xi) \leq \Phi(\xi) \leq \overline{\Phi}^*(\xi), \quad \xi \in \mathbb{R}. \tag{3.3}$$

Proof. Since $\Phi = (\varphi_1, \ldots, \varphi_n) \in \Gamma$ is a traveling wave solution of (1.4) satisfying (2.3), we have

$$\varphi_i(\xi) = F_i(\varphi_1, \ldots, \varphi_n)(\xi), \quad i = 1, \ldots, n.$$

We obtain from Lemma 3.1 that

$$\underline{\varphi}_i^{(m)}(\xi) \leq \varphi_i(\xi) \leq \overline{\varphi}_i^{(m)}(\xi), \quad i = 1, \ldots, n, \ \xi \in \mathbb{R}, \ m \in \mathbb{N},$$

which leads to (3.3). □

4. Applications

In this section, we shall apply our results to a specific model. Consider the existence of traveling wave solutions for four-species Lotka–Volterra nonlocal diffusion system

$$\begin{cases} \dfrac{\partial u_1(x,t)}{\partial t} = d_1[(J_1 * u_1)(x,t) - u_1(x,t)] + r_1 u_1(x,t)[1 - a_{11}u_1(x,t-\tau_{11}) \\ \qquad\qquad + a_{12}u_2^{\frac{1}{2}}(x,t-\tau_{12}) - a_{13}u_3^{\frac{1}{2}}(x,t-\tau_{13}) - a_{14}u_4^{\frac{1}{2}}(x,t-\tau_{14})], \\ \dfrac{\partial u_2(x,t)}{\partial t} = d_2[(J_2 * u_2)(x,t) - u_2(x,t)] + r_2 u_2(x,t)[1 + a_{21}u_1^{\frac{1}{2}}(x,t-\tau_{21}) \\ \qquad\qquad - a_{22}u_2(x,t-\tau_{22}) - a_{23}u_3^{\frac{1}{2}}(x,t-\tau_{23}) - a_{24}u_4^{\frac{1}{2}}(x,t-\tau_{24})], \\ \dfrac{\partial u_3(x,t)}{\partial t} = d_3[(J_3 * u_3)(x,t) - u_3(x,t)] + r_3 u_3(x,t)[1 - a_{31}u_1^{\frac{1}{2}}(x,t-\tau_{31}) \\ \qquad\qquad - a_{32}u_2^{\frac{1}{2}}(x,t-\tau_{32}) - a_{33}u_3(x,t-\tau_{33}) + a_{34}u_4^{\frac{1}{2}}(x,t-\tau_{34})], \\ \dfrac{\partial u_4(x,t)}{\partial t} = d_4[(J_4 * u_4)(x,t) - u_4(x,t)] + r_4 u_4(x,t)[1 - a_{41}u_1^{\frac{1}{2}}(x,t-\tau_{41}) \\ \qquad\qquad - a_{42}u_2^{\frac{1}{2}}(x,t-\tau_{42}) + a_{43}u_3^{\frac{1}{2}}(x,t-\tau_{43}) - a_{44}u_4(x,t-\tau_{44})], \end{cases} \tag{4.1}$$

where $d_i > 0$, $r_i > 0$, $a_{ij} \geq 0$, $\tau_{ij} > 0$ denote the time delay, $J_i : \mathbb{R} \to \mathbb{R}$ satisfies (H3), $i, j = 1, 2, 3, 4$. We are interested in the co-existence of species, so we require that the coefficients a_{ij} $(i, j = 1, 2, 3, 4)$ be given such that (4.1) has a positive equilibrium (k_1, k_2, k_3, k_4). By normalization, we now assume that

$$\begin{cases} 1 - a_{11} + a_{12} - a_{13} - a_{14} = 0, \\ 1 + a_{21} - a_{22} - a_{23} - a_{24} = 0, \\ 1 - a_{31} - a_{32} - a_{33} + a_{34} = 0, \\ 1 - a_{41} - a_{42} + a_{43} - a_{44} = 0, \end{cases} \tag{4.2}$$

such that $(k_1, k_2, k_3, k_4) = (1, 1, 1, 1)$ is a positive equilibrium of (4.1). In particular, we also assume that

$$a_{ii} > \sum_{1 \leq j \leq 4, j \neq i} a_{ij}, \quad i = 1, 2, 3, 4. \tag{4.3}$$

We shall investigate the traveling wave solutions connecting $(0, 0, 0, 0)$ and $(1, 1, 1, 1)$. Clearly, the wave equations corresponding to (4.1) are

$$\begin{cases} c\varphi_1'(\xi) = d_1[(J_1 * \varphi_1)(\xi) - \varphi_1(\xi)] + r_1\varphi_1(\xi)[1 - a_{11}\varphi_1(\xi - c\tau_{11}) \\ \qquad\qquad + a_{12}\varphi_2^{\frac{1}{2}}(\xi - c\tau_{12}) - a_{13}\varphi_3^{\frac{1}{2}}(\xi - c\tau_{13}) - a_{14}\varphi_4^{\frac{1}{2}}(\xi - c\tau_{14})], \\ c\varphi_2'(\xi) = d_2[(J_2 * \varphi_2)(\xi) - \varphi_2(\xi)] + r_2\varphi_2(\xi)[1 + a_{21}\varphi_1^{\frac{1}{2}}(\xi - c\tau_{21}) \\ \qquad\qquad - a_{22}\varphi_2(\xi - c\tau_{22}) - a_{23}\varphi_3^{\frac{1}{2}}(\xi - c\tau_{23}) - a_{24}\varphi_4^{\frac{1}{2}}(\xi - c\tau_{24})], \\ c\varphi_3'(\xi) = d_3[(J_3 * \varphi_3)(\xi) - \varphi_3(\xi)] + r_3\varphi_3(\xi)[1 - a_{31}\varphi_1^{\frac{1}{2}}(\xi - c\tau_{31}) \\ \qquad\qquad - a_{32}\varphi_2^{\frac{1}{2}}(\xi - c\tau_{32}) - a_{33}\varphi_3(\xi - c\tau_{33}) + a_{34}\varphi_4^{\frac{1}{2}}(\xi - c\tau_{34})], \\ c\varphi_4'(\xi) = d_4[(J_4 * \varphi_4)(\xi) - \varphi_4(\xi)] + r_4\varphi_4(\xi)[1 - a_{41}\varphi_1^{\frac{1}{2}}(\xi - c\tau_{41}) \\ \qquad\qquad - a_{42}\varphi_2^{\frac{1}{2}}(\xi - c\tau_{42}) + a_{43}\varphi_3^{\frac{1}{2}}(\xi - c\tau_{43}) - a_{44}\varphi_4(\xi - c\tau_{44})]. \end{cases} \tag{4.4}$$

Let $\tau = \max_{1\le i,j\le 4}\{\tau_{ij}\}$. For any $(\varphi_1,\varphi_2,\varphi_3,\varphi_4) \in C([-\tau,0],\mathbb{R}^4)$, denote

$$\begin{cases} f_1(\varphi_1,\varphi_2,\varphi_3,\varphi_4) \\ \quad = r_1\varphi_1(0)[1 - a_{11}\varphi_1(-\tau_{11}) + a_{12}\varphi_2^{\frac{1}{2}}(-\tau_{12}) - a_{13}\varphi_3^{\frac{1}{2}}(-\tau_{13}) - a_{14}\varphi_4^{\frac{1}{2}}(-\tau_{14})], \\ f_2(\varphi_1,\varphi_2,\varphi_3,\varphi_4) \\ \quad = r_2\varphi_2(0)[1 + a_{21}\varphi_1^{\frac{1}{2}}(-\tau_{21}) - a_{22}\varphi_2(-\tau_{22}) - a_{23}\varphi_3^{\frac{1}{2}}(-\tau_{23}) - a_{24}\varphi_4^{\frac{1}{2}}(-\tau_{24})], \\ f_3(\varphi_1,\varphi_2,\varphi_3,\varphi_4) \\ \quad = r_3\varphi_3(0)[1 - a_{31}\varphi_1^{\frac{1}{2}}(-\tau_{31}) - a_{32}\varphi_2^{\frac{1}{2}}(-\tau_{32}) - a_{33}\varphi_3(-\tau_{33}) + a_{34}\varphi_4^{\frac{1}{2}}(-\tau_{34})], \\ f_4(\varphi_1,\varphi_2,\varphi_3,\varphi_4) \\ \quad = r_4\varphi_4(0)[1 - a_{41}\varphi_1^{\frac{1}{2}}(-\tau_{41}) - a_{42}\varphi_2^{\frac{1}{2}}(-\tau_{42}) + a_{43}\varphi_3^{\frac{1}{2}}(-\tau_{43}) - a_{44}\varphi_4(-\tau_{44})]. \end{cases} \tag{4.5}$$

Clearly, $f = (f_1, f_2, f_3, f_4)$ satisfies (H1)–(H2) for any $\mathbf{M} = (M_1, M_2, M_3, M_4)$ with $M_i \ge 1, i = 1, 2, 3, 4$. Next, we verify that $f = (f_1, f_2, f_3, f_4)$ satisfies (MQM*).

Lemma 4.1. *If* $\tau_{ii}, i = 1, 2, 3, 4$ *are small enough, then* $f = (f_1, f_2, f_3, f_4)$ *satisfies* (MQM*) *defined in Sec.* 3.

Proof. Let $\mathbf{M} = (M_1, M_2, M_3, M_4)$ with $M_i \ge 1, i = 1, 2, 3, 4$, and $\beta_1, \beta_2, \beta_3, \beta_4$ are positive constants with

$$\begin{aligned} \beta_1 &> d_1 - r_1(1 - a_{13}M_3^{\frac{1}{2}} - a_{14}M_4^{\frac{1}{2}} - 2a_{11}M_1), \\ \beta_2 &> d_2 - r_2(1 - a_{23}M_3^{\frac{1}{2}} - a_{24}M_4^{\frac{1}{2}} - 2a_{22}M_2), \\ \beta_3 &> d_3 - r_3(1 - a_{31}M_1^{\frac{1}{2}} - a_{32}M_2^{\frac{1}{2}} - 2a_{33}M_3), \\ \beta_4 &> d_4 - r_4(1 - a_{41}M_1^{\frac{1}{2}} - a_{42}M_2^{\frac{1}{2}} - 2a_{44}M_4). \end{aligned} \tag{4.6}$$

Let $\Phi = (\phi_1, \phi_2, \phi_3, \phi_4)$, $\Psi = (\psi_1, \psi_2, \psi_3, \psi_4) \in C([-\tau, 0], \mathbb{R}^4)$ satisfy the following:

(i) $0 \le \psi_l(s) \le \phi_l(s) \le M_l$, $s \in [-\tau, 0]$, $l = 1, 2, 3, 4$;
(ii) $e^{\beta_l s}[\phi_l(s) - \psi_l(s)]$ is nondecreasing in $s \in [-\tau, 0]$, $l = 1, 2, 3, 4$.

Then

$$\begin{aligned} &f_1(\phi_1,\phi_2,\phi_3,\phi_4) - f_1(\psi_1,\psi_2,\phi_3,\phi_4) \\ &\quad = r_1\phi_1(0)[1 - a_{11}\phi_1(-\tau_{11}) + a_{12}\phi_2^{\frac{1}{2}}(-\tau_{12}) - a_{13}\phi_3^{\frac{1}{2}}(-\tau_{13}) - a_{14}\phi_4^{\frac{1}{2}}(-\tau_{14})] \\ &\qquad - r_1\psi_1(0)[1 - a_{11}\psi_1(-\tau_{11}) + a_{12}\psi_2^{\frac{1}{2}}(-\tau_{12}) - a_{13}\phi_3^{\frac{1}{2}}(-\tau_{13}) - a_{14}\phi_4^{\frac{1}{2}}(-\tau_{14})] \\ &\quad = r_1[\phi_1(0) - \psi_1(0)] - r_1a_{13}\phi_3^{\frac{1}{2}}(-\tau_{13})[\phi_1(0) - \psi_1(0)] \\ &\qquad - r_1a_{14}\phi_4^{\frac{1}{2}}(-\tau_{14})[\phi_1(0) - \psi_1(0)] - r_1a_{11}[\phi_1(0)\phi_1(-\tau_{11}) - \psi_1(0)\psi_1(-\tau_{11})] \\ &\qquad + r_1a_{12}[\phi_1(0)\phi_2^{\frac{1}{2}}(-\tau_{12}) - \psi_1(0)\psi_2^{\frac{1}{2}}(-\tau_{12})] \end{aligned}$$

$$
\begin{aligned}
&\geq r_1(1 - a_{13}M_3^{\frac{1}{2}} - a_{14}M_4^{\frac{1}{2}})[\phi_1(0) - \psi_1(0)] \\
&\quad - r_1 a_{11}[\phi_1(0)\phi_1(-\tau_{11}) - \psi_1(0)\phi_1(-\tau_{11}) + \psi_1(0)\phi_1(-\tau_{11}) - \psi_1(0)\psi_1(-\tau_{11})] \\
&\geq r_1(1 - a_{13}M_3^{\frac{1}{2}} - a_{14}M_4^{\frac{1}{2}} - a_{11}M_1)[\phi_1(0) - \psi_1(0)] \\
&\quad - r_1 a_{11}\psi_1(0)[\phi_1(-\tau_{11}) - \psi_1(-\tau_{11})] \\
&\geq r_1(1 - a_{13}M_3^{\frac{1}{2}} - a_{14}M_4^{\frac{1}{2}} - a_{11}M_1 - a_{11}M_1 e^{\beta_1\tau_{11}})[\phi_1(0) - \psi_1(0)].
\end{aligned}
$$

By (4.6), if τ_{11} is small, we have

$$
\beta_1 > d_1 - r_1(1 - a_{13}M_3^{\frac{1}{2}} - a_{14}M_4^{\frac{1}{2}} - a_{11}M_1 - a_{11}M_1 e^{\beta_1\tau_{11}}).
$$

Thus,

$$
f_1(\phi_1, \phi_2, \phi_3, \phi_4) - f_1(\psi_1, \psi_2, \phi_3, \phi_4) + (\beta_1 - d_1)[\phi_1(0) - \psi_1(0)] \geq 0.
$$

Furthermore,

$$
\begin{aligned}
&f_1(\phi_1, \phi_2, \phi_3, \phi_4) - f_1(\phi_1, \phi_2, \psi_3, \psi_4) \\
&\quad = r_1\phi_1(0)[1 - a_{11}\phi_1(-\tau_{11}) + a_{12}\phi_2^{\frac{1}{2}}(-\tau_{12}) - a_{13}\phi_3^{\frac{1}{2}}(-\tau_{13}) - a_{14}\phi_4^{\frac{1}{2}}(-\tau_{14})] \\
&\qquad - r_1\phi_1(0)[1 - a_{11}\phi_1(-\tau_{11}) + a_{12}\phi_2^{\frac{1}{2}}(-\tau_{12}) - a_{13}\psi_3^{\frac{1}{2}}(-\tau_{13}) - a_{14}\psi_4^{\frac{1}{2}}(-\tau_{14})] \\
&\quad = -r_1 a_{13}\phi_1(0)(\phi_3^{\frac{1}{2}}(-\tau_{13}) - \psi_3^{\frac{1}{2}}(-\tau_{13})) - r_1 a_{14}\phi_1(0)(\phi_4^{\frac{1}{2}}(-\tau_{14}) - \psi_4^{\frac{1}{2}}(-\tau_{14})) \\
&\quad \leq 0.
\end{aligned}
$$

By the above argument, we know that f_1 satisfies the (MQM*) condition. In a similar way, we can show that f_2, f_3, f_4 also satisfy the (MQM*) condition if $\tau_{22}, \tau_{33}, \tau_{44}$ are small. The proof is completed. □

Now, we are in a position to construct an upper and a lower solution of (4.4). Define functions

$$
\begin{cases}
\Delta_1(\lambda, c) = c\lambda - d_1 \displaystyle\int_{\mathbb{R}} J_1(s)e^{-\lambda s}ds + d_1 - r_1(1 + a_{12}M_2^{\frac{1}{2}}), \\
\Delta_2(\lambda, c) = c\lambda - d_2 \displaystyle\int_{\mathbb{R}} J_2(s)e^{-\lambda s}ds + d_2 - r_2(1 + a_{21}M_1^{\frac{1}{2}}), \\
\Delta_3(\lambda, c) = c\lambda - d_3 \displaystyle\int_{\mathbb{R}} J_3(s)e^{-\lambda s}ds + d_3 - r_3(1 + a_{34}M_4^{\frac{1}{2}}), \\
\Delta_4(\lambda, c) = c\lambda - d_4 \displaystyle\int_{\mathbb{R}} J_4(s)e^{-\lambda s}ds + d_4 - r_4(1 + a_{43}M_3^{\frac{1}{2}}).
\end{cases}
$$

It is easily seen that the following observations hold.

Lemma 4.2. *There exists $c^* > 0$ such that*

(i) *if $c > c^*$, then the equation $\Delta_i(\lambda, c) = 0$ has two distinct positive roots $\lambda_{i1}(c)$, $\lambda_{i2}(c)$ with $0 < \lambda_{i1}(c) < \lambda_{i2}(c)$, $i = 1, 2, 3, 4$;*
(ii) *if $c \leq c^*$, then at least one of the equations $\Delta_1(\lambda, c) = 0$, $\Delta_2(\lambda, c) = 0$, $\Delta_3(\lambda, c) = 0$, $\Delta_4(\lambda, c) = 0$ have no more than one real root.*

Assume that $c > \max\{c^*, 1\}$, for brief, we rewrite $\lambda_{i1}(c)$, $\lambda_{i2}(c)$ given in Lemma 4.2 as λ_{i1}, λ_{i2}. In view of (4.3), there exist constants $\varepsilon_j > 0$, $j = 0, 1, \ldots, 8$ such that $\varepsilon_2, \varepsilon_4, \varepsilon_6, \varepsilon_8 < 1$ and

$$\begin{cases} a_{11}\varepsilon_1 - a_{12}\varepsilon_3 - a_{13}\varepsilon_6 - a_{14}\varepsilon_8 > \varepsilon_0, \\ a_{22}\varepsilon_3 - a_{21}\varepsilon_1 - a_{23}\varepsilon_6 - a_{24}\varepsilon_8 > \varepsilon_0, \\ a_{33}\varepsilon_5 - a_{31}\varepsilon_2 - a_{32}\varepsilon_4 - a_{34}\varepsilon_7 > \varepsilon_0, \\ a_{44}\varepsilon_7 - a_{41}\varepsilon_2 - a_{42}\varepsilon_4 - a_{43}\varepsilon_5 > \varepsilon_0, \\ a_{11}\varepsilon_2 - a_{12}\varepsilon_4 - a_{13}\varepsilon_5 - a_{14}\varepsilon_7 > \varepsilon_0, \\ a_{22}\varepsilon_4 - a_{21}\varepsilon_2 - a_{23}\varepsilon_5 - a_{24}\varepsilon_7 > \varepsilon_0, \\ a_{33}\varepsilon_6 - a_{31}\varepsilon_1 - a_{32}\varepsilon_3 - a_{34}\varepsilon_8 > \varepsilon_0, \\ a_{44}\varepsilon_8 - a_{41}\varepsilon_1 - a_{42}\varepsilon_3 - a_{43}\varepsilon_6 > \varepsilon_0. \end{cases} \tag{4.7}$$

Using the above constants λ_{i1}, ε_j, $i = 1, 2, 3, 4$, $j = 1, \ldots, 8$ and a small enough $\lambda = \lambda(c)$, we define the following continuous functions:

$$\underline{\varphi}_1(\xi) = \begin{cases} 0, & \xi \leq \xi_1, \\ 1 - \varepsilon_2 e^{-\lambda\xi}, & \xi > \xi_1, \end{cases} \qquad \overline{\varphi}_1(\xi) = \begin{cases} e^{\lambda_{11}\xi}, & \xi \leq \xi_5, \\ 1 + \varepsilon_1 e^{-\lambda\xi}, & \xi > \xi_5, \end{cases}$$

$$\underline{\varphi}_2(\xi) = \begin{cases} 0, & \xi \leq \xi_2, \\ 1 - \varepsilon_4 e^{-\lambda\xi}, & \xi > \xi_2, \end{cases} \qquad \overline{\varphi}_2(\xi) = \begin{cases} e^{\lambda_{21}\xi}, & \xi \leq \xi_6, \\ 1 + \varepsilon_3 e^{-\lambda\xi}, & \xi > \xi_6, \end{cases}$$

$$\underline{\varphi}_3(\xi) = \begin{cases} 0, & \xi \leq \xi_3, \\ 1 - \varepsilon_6 e^{-\lambda\xi}, & \xi > \xi_3, \end{cases} \qquad \overline{\varphi}_3(\xi) = \begin{cases} e^{\lambda_{31}\xi}, & \xi \leq \xi_7, \\ 1 + \varepsilon_5 e^{-\lambda\xi}, & \xi > \xi_7, \end{cases}$$

$$\underline{\varphi}_4(\xi) = \begin{cases} 0, & \xi \leq \xi_4, \\ 1 - \varepsilon_8 e^{-\lambda\xi}, & \xi > \xi_4, \end{cases} \qquad \overline{\varphi}_4(\xi) = \begin{cases} e^{\lambda_{41}\xi}, & \xi \leq \xi_8, \\ 1 + \varepsilon_7 e^{-\lambda\xi}, & \xi > \xi_8. \end{cases}$$

It is easy to see that $1 < \sup_{\xi\in\mathbb{R}} \overline{\varphi}_i(\xi) = M_i$, $\overline{\varphi}_i(\xi)$, $\underline{\varphi}_i(\xi)$ satisfy (P1)–(P3), $i = 1, 2, 3, 4$, and $\min\{\xi_5, \xi_6, \xi_7, \xi_8\} \geq \max\{\xi_1, \xi_2, \xi_3, \xi_4\} + c\tau$ for a small enough $\lambda > 0$. In fact, we have $\xi_i = \frac{1}{\lambda}\ln\varepsilon_{2i} < 0$ for $i = 1, 2, 3, 4$, and $\xi_i > 0$ for $i = 5, 6, 7, 8$. For any given $c > 0$, one can choose $\lambda = \lambda(c) > 0$ small enough so that $\min\{\xi_5, \xi_6, \xi_7, \xi_8\} \geq \max\{\xi_1, \xi_2, \xi_3, \xi_4\} + c\tau$. We shall show that $\overline{\Phi}(\xi) = (\overline{\varphi}_1(\xi), \overline{\varphi}_2(\xi), \overline{\varphi}_3(\xi), \overline{\varphi}_4(\xi))$ is an upper solution of (4.4) and $\underline{\Phi}(\xi) = (\underline{\varphi}_1(\xi), \underline{\varphi}_2(\xi), \underline{\varphi}_3(\xi), \underline{\varphi}_4(\xi))$ is a lower solution of (4.4).

Lemma 4.3. *If $\lambda > 0$ is small enough, then $\overline{\Phi}(\xi) = (\overline{\varphi}_1(\xi), \overline{\varphi}_2(\xi), \overline{\varphi}_3(\xi), \overline{\varphi}_4(\xi))$ is an upper solution of* (4.4) *and $\underline{\Phi}(\xi) = (\underline{\varphi}_1(\xi), \underline{\varphi}_2(\xi), \underline{\varphi}_3(\xi), \underline{\varphi}_4(\xi))$ is a lower solution of* (4.4).

Proof. For $\xi \le \xi_5$, we have $\overline{\varphi}_1(\xi) = e^{\lambda_{11}\xi}$ and

$$\begin{aligned}
&c\overline{\varphi}_1'(\xi) - d_1[(J_1 * \overline{\varphi}_1)(\xi) - \overline{\varphi}_1(\xi)] - r_1\overline{\varphi}_1(\xi)[1 - a_{11}\overline{\varphi}_1(\xi - c\tau_{11}) \\
&\qquad + a_{12}\overline{\varphi}_2^{\frac{1}{2}}(\xi - c\tau_{12}) - a_{13}\underline{\varphi}_3^{\frac{1}{2}}(\xi - c\tau_{13}) - a_{14}\underline{\varphi}_4^{\frac{1}{2}}(\xi - c\tau_{14})] \\
&\quad \ge c\overline{\varphi}_1'(\xi) - d_1(J_1 * \overline{\varphi}_1)(\xi) + d_1\overline{\varphi}_1(\xi) - r_1(1 + a_{12}M_2^{\frac{1}{2}})\overline{\varphi}_1(\xi) \\
&\quad \ge e^{\lambda_{11}\xi}\left(c\lambda_{11} - d_1\int_{\mathbb{R}} J_1(s)e^{-\lambda_{11}s}ds + d_1 - r_1(1 + a_{12}M_2^{\frac{1}{2}})\right) \\
&\quad = e^{\lambda_{11}\xi}\Delta_1(\lambda_{11}, c) \\
&\quad = 0.
\end{aligned}$$

For $\xi_5 < \xi \le \xi_5 + c\tau_{11}$, we have

$$\begin{aligned}
\overline{\varphi}_1(\xi) &= 1 + \varepsilon_1 e^{-\lambda\xi}, \\
\overline{\varphi}_1(\xi - c\tau_{11}) &= e^{\lambda_{11}(\xi - c\tau_{11})}, \\
\overline{\varphi}_2(\xi - c\tau_{12}) &\le 1 + \varepsilon_3 e^{-\lambda(\xi - c\tau_{12})}, \\
\underline{\varphi}_3(\xi - c\tau_{13}) &= 1 - \varepsilon_6 e^{-\lambda(\xi - c\tau_{13})}, \\
\underline{\varphi}_4(\xi - c\tau_{14}) &= 1 - \varepsilon_8 e^{-\lambda(\xi - c\tau_{14})}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&c\overline{\varphi}_1'(\xi) - d_1[(J_1 * \overline{\varphi}_1)(\xi) - \overline{\varphi}_1(\xi)] - r_1\overline{\varphi}_1(\xi)[1 - a_{11}\overline{\varphi}_1(\xi - c\tau_{11}) \\
&\qquad + a_{12}\overline{\varphi}_2^{\frac{1}{2}}(\xi - c\tau_{12}) - a_{13}\underline{\varphi}_3^{\frac{1}{2}}(\xi - c\tau_{13}) - a_{14}\underline{\varphi}_4^{\frac{1}{2}}(\xi - c\tau_{14})] \\
&\quad \ge -c\varepsilon_1\lambda e^{-\lambda\xi} - d_1\int_{\mathbb{R}} J_1(s)(1 + \varepsilon_1 e^{-\lambda(\xi - s)})ds + d_1(1 + \varepsilon_1 e^{-\lambda\xi}) \\
&\qquad - r_1(1 + \varepsilon_1 e^{-\lambda\xi})(1 - a_{11}e^{\lambda_{11}(\xi_5 - c\tau_{11})} + a_{12}(1 + \varepsilon_3 e^{-\lambda(\xi - c\tau_{12})})^{\frac{1}{2}} \\
&\qquad - a_{13}(1 - \varepsilon_6 e^{-\lambda(\xi - c\tau_{13})})^{\frac{1}{2}} - a_{14}(1 - \varepsilon_8 e^{-\lambda(\xi - c\tau_{14})})^{\frac{1}{2}}) \\
&\quad \ge -c\varepsilon_1\lambda e^{-\lambda\xi} - d_1\varepsilon_1 e^{-\lambda\xi}\int_{\mathbb{R}} J_1(s)e^{\lambda s}ds + d_1\varepsilon_1 e^{-\lambda\xi} - r_1(1 + \varepsilon_1 e^{-\lambda\xi}) \\
&\qquad \times (1 - a_{11}e^{-c\lambda_{11}\tau_{11}}(1 + \varepsilon_1 e^{-\lambda\xi_5}) + a_{12}(1 + \varepsilon_3 e^{-\lambda(\xi - c\tau_{12})} \\
&\qquad - a_{13}(1 - \varepsilon_6 e^{-\lambda(\xi - c\tau_{13})}) - a_{14}(1 - \varepsilon_8 e^{-\lambda(\xi - c\tau_{14})})) \\
&\quad = -c\varepsilon_1\lambda\varrho - d_1\varepsilon_1\varrho\int_{\mathbb{R}} J_1(s)e^{\lambda s}ds + d_1\varepsilon_1\varrho - r_1(1 + \varepsilon_1\varrho) \\
&\qquad \times (1 - a_{11}e^{-c\lambda_{11}\tau_{11}}(1 + \varepsilon_1 e^{-\lambda\xi_5}) + a_{12}(1 + \varepsilon_3\varrho e^{\lambda c\tau_{12}} \\
&\qquad - a_{13}(1 - \varepsilon_6\varrho e^{\lambda c\tau_{13}}) - a_{14}(1 - \varepsilon_8\varrho e^{\lambda c\tau_{14}})) \\
&\quad =: I_1(\lambda, \varrho),
\end{aligned}$$

where $\varrho := e^{-\lambda\xi} \in (0, e^{-\lambda\xi_5}) \subset (0,1)$ for $\xi \in (\xi_5, \xi_5 + c\tau_{11}]$ and $\lambda > 0$. Note $1 - a_{11} + a_{12} - a_{13} - a_{14} = 0$. It follows from (4.7) that for any $\varrho \in (0,1)$,

$$\begin{aligned} I_1(0,\varrho) &= -r_1(1+\varepsilon_1\varrho)(1 - a_{11}e^{-c\lambda_{11}\tau_{11}} + a_{12} - a_{13} - a_{14} \\ &\quad - a_{11}\varepsilon_1 e^{-c\lambda_{11}\tau_{11}} + a_{12}\varepsilon_3\varrho + a_{13}\varepsilon_6\varrho + a_{14}\varepsilon_8\varrho) \\ &> -r_1(1+\varepsilon_1\varrho)(a_{11}(1 - e^{-c\lambda_{11}\tau_{11}}) - a_{11}\varepsilon_1 e^{-c\lambda_{11}\tau_{11}} \\ &\quad + a_{12}\varepsilon_3 + a_{13}\varepsilon_6 + a_{14}\varepsilon_8) \\ &> 0 \end{aligned}$$

for small enough τ_{11}. Therefore, there exists small enough $\lambda_1 > 0$ such that $I_1(\lambda,\varrho) > 0$ for any $\lambda \in (0,\lambda_1)$, $\varrho \in (0,1)$.

For $\xi > \xi_5 + c\tau_{11}$, we have $\overline{\varphi}_1(\xi - c\tau_{11}) = 1 + \varepsilon_1 e^{-\lambda(\xi - c\tau_{11})}$ and

$$\begin{aligned} & c\overline{\varphi}_1'(\xi) - d_1[(J_1 * \overline{\varphi}_1)(\xi) - \overline{\varphi}_1(\xi)] - r_1\overline{\varphi}_1(\xi)[1 - a_{11}\overline{\varphi}_1(\xi - c\tau_{11}) \\ &\qquad + a_{12}\overline{\varphi}_2^{\frac{1}{2}}(\xi - c\tau_{12}) - a_{13}\underline{\varphi}_3^{\frac{1}{2}}(\xi - c\tau_{13}) - a_{14}\underline{\varphi}_4^{\frac{1}{2}}(\xi - c\tau_{14})] \\ &\quad \geq -c\varepsilon_1\lambda e^{-\lambda\xi} - d_1\varepsilon_1 e^{-\lambda\xi}\int_{\mathbb{R}} J_1(s)e^{\lambda s}ds + d_1\varepsilon_1 e^{-\lambda\xi} - r_1(1 + \varepsilon_1 e^{-\lambda\xi}) \\ &\qquad \times (1 - a_{11}(1 + \varepsilon_1 e^{-\lambda(\xi - c\tau_{11})}) + a_{12}(1 + \varepsilon_3 e^{-\lambda(\xi - c\tau_{12})}) \\ &\qquad - a_{13}(1 - \varepsilon_6 e^{-\lambda(\xi - c\tau_{13})}) - a_{14}(1 - \varepsilon_8 e^{-\lambda(\xi - c\tau_{14})})) \\ &\quad = -c\varepsilon_1\lambda\varrho - d_1\varepsilon_1\varrho\int_{\mathbb{R}} J_1(s)e^{\lambda s}ds + d_1\varepsilon_1\varrho - r_1(1+\varepsilon_1\varrho) \\ &\qquad \times (-a_{11}\varepsilon_1\varrho e^{\lambda c\tau_{11}} + a_{12}\varepsilon_3\varrho e^{\lambda c\tau_{12}} + a_{13}\varepsilon_6\varrho e^{\lambda c\tau_{13}} + a_{14}\varepsilon_8\varrho e^{\lambda c\tau_{14}}) \\ &\quad =: I_2(\lambda,\varrho), \end{aligned}$$

where $\varrho = e^{-\lambda\xi}$ is defined as above. It follows from (4.7) that

$$I_2(0,\varrho) = -r_1(1+\varepsilon_1\varrho)\varrho(-a_{11}\varepsilon_1 + a_{12}\varepsilon_3 + a_{13}\varepsilon_6 + a_{14}\varepsilon_8) > 0.$$

Therefore, there exists small enough $\lambda_2 > 0$ such that $I_2(\lambda,\varrho) > 0$ for any $\lambda \in (0,\lambda_2)$, $\varrho \in (0,1)$. Let $\tilde{\lambda}_1 = \min\{\lambda_1,\lambda_2\}$; then we have

$$\begin{aligned} & c\overline{\varphi}_1'(\xi) - d_1[(J_1 * \overline{\varphi}_1)(\xi) - \overline{\varphi}_1(\xi)] - r_1\overline{\varphi}_1(\xi)[1 - a_{11}\overline{\varphi}_1(\xi - c\tau_{11}) \\ &\qquad + a_{12}\overline{\varphi}_2^{\frac{1}{2}}(\xi - c\tau_{12}) - a_{13}\underline{\varphi}_3^{\frac{1}{2}}(\xi - c\tau_{13}) - a_{14}\underline{\varphi}_4^{\frac{1}{2}}(\xi - c\tau_{14})] \\ &\quad \geq 0 \quad \text{for } \lambda \in (0,\tilde{\lambda}_1). \end{aligned}$$

Similarly, there exist $\tilde{\lambda}_2$, $\tilde{\lambda}_3$, $\tilde{\lambda}_4$ such that

$$\begin{aligned} & c\overline{\varphi}_2'(\xi) - d_2[(J_2 * \overline{\varphi}_2)(\xi) - \overline{\varphi}_2(\xi)] - r_2\overline{\varphi}_2(\xi)[1 + a_{21}\overline{\varphi}_1^{\frac{1}{2}}(\xi - c\tau_{21}) \\ &\qquad - a_{22}\overline{\varphi}_2(\xi - c\tau_{22}) - a_{23}\underline{\varphi}_3^{\frac{1}{2}}(\xi - c\tau_{23}) - a_{24}\underline{\varphi}_4^{\frac{1}{2}}(\xi - c\tau_{24})] \\ &\quad \geq 0 \quad \text{for } \lambda \in (0,\tilde{\lambda}_2), \end{aligned}$$

$$\begin{aligned}
&c\overline{\varphi}_3'(\xi) - d_3[(J_3 * \overline{\varphi}_3)(\xi) - \overline{\varphi}_3(\xi)] - r_3\overline{\varphi}_3(\xi)[1 - a_{31}\underline{\varphi}_1^{\frac{1}{2}}(\xi - c\tau_{31}) \\
&\qquad - a_{32}\underline{\varphi}_2^{\frac{1}{2}}(\xi - c\tau_{32}) - a_{33}\overline{\varphi}_3(\xi - c\tau_{33}) + a_{34}\overline{\varphi}_4^{\frac{1}{2}}(\xi - c\tau_{34})] \\
&\quad \geq 0 \quad \text{for } \lambda \in (0, \tilde{\lambda}_3), \\
&c\overline{\varphi}_4'(\xi) - d_4[(J_4 * \overline{\varphi}_4)(\xi) - \overline{\varphi}_4(\xi)] - r_4\overline{\varphi}_4(\xi)[1 - a_{41}\underline{\varphi}_1^{\frac{1}{2}}(\xi - c\tau_{41}) \\
&\qquad - a_{42}\underline{\varphi}_2^{\frac{1}{2}}(\xi - c\tau_{42}) + a_{43}\overline{\varphi}_3^{\frac{1}{2}}(\xi - c\tau_{43}) - a_{44}\overline{\varphi}_4(\xi - c\tau_{44})] \\
&\quad \geq 0 \quad \text{for } \lambda \in (0, \tilde{\lambda}_4),
\end{aligned}$$

respectively.

By the above argument, we know that $\overline{\Phi}(\xi) = (\overline{\varphi}_1(\xi), \overline{\varphi}_2(\xi), \overline{\varphi}_3(\xi), \overline{\varphi}_4(\xi))$ is an upper solution of (4.4) for a small enough $\lambda > 0$.

Now we show that $\underline{\Phi}(\xi) = (\underline{\varphi}_1(\xi), \underline{\varphi}_2(\xi), \underline{\varphi}_3(\xi), \underline{\varphi}_4(\xi))$ is a lower solution of (4.4).

For $\xi \leq \xi_1$, we have $\underline{\varphi}_1(\xi) = 0$ and

$$\begin{aligned}
&c\underline{\varphi}_1'(\xi) - d_1[(J_1 * \underline{\varphi}_1)(\xi) - \underline{\varphi}_1(\xi)] - r_1\underline{\varphi}_1(\xi)[1 - a_{11}\underline{\varphi}_1(\xi - c\tau_{11}) \\
&\qquad + a_{12}\underline{\varphi}_2^{\frac{1}{2}}(\xi - c\tau_{12}) - a_{13}\overline{\varphi}_3^{\frac{1}{2}}(\xi - c\tau_{13}) - a_{14}\overline{\varphi}_4^{\frac{1}{2}}(\xi - c\tau_{14})] \\
&\quad \leq 0.
\end{aligned}$$

For $\xi_1 < \xi \leq \xi_1 + c\tau_{11}$, we have

$$\begin{aligned}
\underline{\varphi}_1(\xi) &= 1 - \varepsilon_2 e^{-\lambda\xi}, \\
\underline{\varphi}_1(\xi - c\tau_{11}) &= 0, \\
\underline{\varphi}_2(\xi - c\tau_{12}) &\geq 1 - \varepsilon_4 e^{-\lambda(\xi - c\tau_{12})}, \\
\overline{\varphi}_3(\xi - c\tau_{13}) &\leq 1 + \varepsilon_5 e^{-\lambda(\xi - c\tau_{13})}, \\
\overline{\varphi}_4(\xi - c\tau_{14}) &\leq 1 + \varepsilon_7 e^{-\lambda(\xi - c\tau_{14})}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&c\underline{\varphi}_1'(\xi) - d_1[(J_1 * \underline{\varphi}_1)(\xi) - \underline{\varphi}_1(\xi)] - r_1\underline{\varphi}_1(\xi)[1 - a_{11}\underline{\varphi}_1(\xi - c\tau_{11}) \\
&\qquad + a_{12}\underline{\varphi}_2^{\frac{1}{2}}(\xi - c\tau_{12}) - a_{13}\overline{\varphi}_3^{\frac{1}{2}}(\xi - c\tau_{13}) - a_{14}\overline{\varphi}_4^{\frac{1}{2}}(\xi - c\tau_{14})] \\
&\quad \leq c\varepsilon_2\lambda e^{-\lambda\xi} - d_1\int_{\mathbb{R}} J_1(s)(1 - \varepsilon_2 e^{-\lambda(\xi - s)})ds \\
&\qquad + d_1(1 - \varepsilon_2 e^{-\lambda\xi}) - r_1(1 - \varepsilon_2 e^{-\lambda\xi})(1 + a_{12}(1 - \varepsilon_4 e^{-\lambda(\xi - c\tau_{12})})^{\frac{1}{2}} \\
&\qquad - a_{13}(1 + \varepsilon_5 e^{-\lambda(\xi - c\tau_{13})})^{\frac{1}{2}} - a_{14}(1 + \varepsilon_7 e^{-\lambda(\xi - c\tau_{14})})^{\frac{1}{2}}) \\
&\quad \leq c\varepsilon_2\lambda e^{-\lambda\xi} + d_1\varepsilon_2 e^{-\lambda\xi}\int_{\mathbb{R}} J_1(s)e^{\lambda s}ds - d_1\varepsilon_2 e^{-\lambda\xi} - r_1(1 - \varepsilon_2 e^{-\lambda\xi}) \\
&\qquad \times (1 + a_{12}(1 - \varepsilon_4 e^{-\lambda(\xi - c\tau_{12})}) - a_{13}(1 + \varepsilon_5 e^{-\lambda(\xi - c\tau_{13})}) \\
&\qquad - a_{14}(1 + \varepsilon_7 e^{-\lambda(\xi - c\tau_{14})}))
\end{aligned}$$

$$= c\varepsilon_2\lambda\varrho + d_1\varepsilon_2\varrho\int_{\mathbb{R}} J_1(s)e^{\lambda s}ds - d_1\varepsilon_2\varrho - r_1(1-\varepsilon_2\varrho)$$
$$\times (a_{11} - a_{12}\varepsilon_4\varrho e^{\lambda c\tau_{12}} - a_{13}\varepsilon_5\varrho e^{\lambda c\tau_{13}} - a_{14}\varepsilon_7\varrho e^{\lambda c\tau_{14}})$$
$$=: I_3(\lambda,\varrho).$$

Note $\varrho = e^{-\lambda\xi}$ and $\xi_1 = \frac{1}{\lambda}\ln\varepsilon_2$. Thus $\varrho \in (0, e^{-\lambda\xi_1}) = (0, \frac{1}{\varepsilon_2})$. It follows from (4.7) that for $\varrho \in (0, \frac{1}{\varepsilon_2})$,

$$I_3(0,\varrho) = -r_1(1-\varepsilon_2\varrho)(a_{11} - a_{12}\varepsilon_4\varrho - a_{13}\varepsilon_5\varrho - a_{14}\varepsilon_7\varrho)$$
$$< -r_1\left(\frac{1}{\varepsilon_2} - \varrho\right)(a_{11}\varepsilon_2 - a_{12}\varepsilon_4 - a_{13}\varepsilon_5 - a_{14}\varepsilon_7) < 0.$$

Therefore, there exists small enough $\lambda_3 > 0$ such that $I_3(\lambda,\varrho) < 0$ for any $\lambda \in (0,\lambda_3)$, $\varrho \in (0, \frac{1}{\varepsilon_2})$.

For $\xi > \xi_1 + c\tau_{11}$, we have $\underline{\varphi}_1(\xi - c\tau_{11}) = 1 - \varepsilon_2 e^{-\lambda(\xi - c\tau_{11})}$ and

$$c\underline{\varphi}_1'(\xi) - d_1[(J_1 * \underline{\varphi}_1)(\xi) - \underline{\varphi}_1(\xi)] - r_1\underline{\varphi}_1(\xi)[1 - a_{11}\underline{\varphi}_1(\xi - c\tau_{11})$$
$$+ a_{12}\underline{\varphi}_2^{\frac{1}{2}}(\xi - c\tau_{12}) - a_{13}\overline{\varphi}_3^{\frac{1}{2}}(\xi - c\tau_{13}) - a_{14}\overline{\varphi}_4^{\frac{1}{2}}(\xi - c\tau_{14})]$$
$$\le c\varepsilon_2\lambda e^{-\lambda\xi} + d_1\varepsilon_2 e^{-\lambda\xi}\int_{\mathbb{R}} J_1(s)e^{\lambda s}ds - d_1\varepsilon_2 e^{-\lambda\xi} - r_1(1 - \varepsilon_2 e^{-\lambda\xi})$$
$$\times (1 - a_{11}(1 - \varepsilon_2 e^{-\lambda(\xi - c\tau_{11})}) + a_{12}(1 - \varepsilon_4 e^{-\lambda(\xi - c\tau_{12})})$$
$$- a_{13}(1 + \varepsilon_5 e^{-\lambda(\xi - c\tau_{13})}) - a_{14}(1 + \varepsilon_7 e^{-\lambda(\xi - c\tau_{14})}))$$
$$= c\varepsilon_2\lambda\varrho + d_1\varepsilon_2\varrho\int_{\mathbb{R}} J_1(s)e^{\lambda s}ds - d_1\varepsilon_2\varrho - r_1(1-\varepsilon_2\varrho)$$
$$\times (a_{11}\varepsilon_2\varrho e^{\lambda c\tau_{11}} - a_{12}\varepsilon_4\varrho e^{\lambda c\tau_{12}} - a_{13}\varepsilon_5\varrho e^{\lambda c\tau_{13}} - a_{14}\varepsilon_7\varrho e^{\lambda c\tau_{14}})$$
$$=: I_4(\lambda,\varrho).$$

It follows from (4.7) that for $\varrho \in (0, \frac{1}{\varepsilon_2})$,

$$I_4(0,\varrho) = -r_1(1-\varepsilon_2\varrho)\varrho(a_{11}\varepsilon_2 - a_{12}\varepsilon_4 - a_{13}\varepsilon_5 - a_{14}\varepsilon_7) < 0.$$

Therefore, there exists small enough $\lambda_4 > 0$ such that $I_4(\lambda,\varrho) < 0$ for any $\lambda \in (0,\lambda_4)$, $\varrho \in (0, \frac{1}{\varepsilon_2})$. Let $\tilde{\lambda}_5 = \min\{\lambda_3, \lambda_4\}$; then we have

$$c\underline{\varphi}_1'(\xi) - d_1[(J_1 * \underline{\varphi}_1)(\xi) - \underline{\varphi}_1(\xi)] - r_1\underline{\varphi}_1(\xi)[1 - a_{11}\underline{\varphi}_1(\xi - c\tau_{11})$$
$$+ a_{12}\underline{\varphi}_2^{\frac{1}{2}}(\xi - c\tau_{12}) - a_{13}\overline{\varphi}_3^{\frac{1}{2}}(\xi - c\tau_{13}) - a_{14}\overline{\varphi}_4^{\frac{1}{2}}(\xi - c\tau_{14})]$$
$$\le 0 \quad \text{for } \lambda \in (0, \tilde{\lambda}_5).$$

Similarly, there exist $\tilde{\lambda}_6$, $\tilde{\lambda}_7$, $\tilde{\lambda}_8$ such that

$$\begin{aligned}
&c\underline{\varphi}_2'(\xi) - d_2[(J_2 * \underline{\varphi}_2)(\xi) - \underline{\varphi}_2(\xi)] - r_2\underline{\varphi}_2(\xi)[1 + a_{21}\underline{\varphi}_1^{\frac{1}{2}}(\xi - c\tau_{21}) \\
&\quad - a_{22}\underline{\varphi}_2(\xi - c\tau_{22}) - a_{23}\overline{\varphi}_3^{\frac{1}{2}}(\xi - c\tau_{23}) - a_{24}\overline{\varphi}_4^{\frac{1}{2}}(\xi - c\tau_{24})] \\
&\le 0 \quad \text{for } \lambda \in (0, \tilde{\lambda}_6),
\end{aligned}$$

$$\begin{aligned}
&c\underline{\varphi}_3'(\xi) - d_3[(J_3 * \underline{\varphi}_3)(\xi) - \underline{\varphi}_3(\xi)] - r_3\underline{\varphi}_3(\xi)[1 - a_{31}\overline{\varphi}_1^{\frac{1}{2}}(\xi - c\tau_{11}) \\
&\quad - a_{32}\overline{\varphi}_2^{\frac{1}{2}}(\xi - c\tau_{32}) - a_{33}\underline{\varphi}_3(\xi - c\tau_{13}) + a_{34}\underline{\varphi}_4^{\frac{1}{2}}(\xi - c\tau_{34})] \\
&\le 0 \quad \text{for } \lambda \in (0, \tilde{\lambda}_7),
\end{aligned}$$

$$\begin{aligned}
&c\underline{\varphi}_4'(\xi) - d_4[(J_4 * \underline{\varphi}_4)(\xi) - \underline{\varphi}_4(\xi)] - r_4\underline{\varphi}_4(\xi)[1 - a_{41}\overline{\varphi}_1^{\frac{1}{2}}(\xi - c\tau_{41}) \\
&\quad - a_{42}\overline{\varphi}_2^{\frac{1}{2}}(\xi - c\tau_{42}) + a_{43}\underline{\varphi}_3^{\frac{1}{2}}(\xi - c\tau_{43}) - a_{44}\underline{\varphi}_4(\xi - c\tau_{44})] \\
&\le 0 \quad \text{for } \lambda \in (0, \tilde{\lambda}_8),
\end{aligned}$$

respectively.

By the above argument, we know that $\underline{\Phi}(\xi) = (\underline{\varphi}_1(\xi), \underline{\varphi}_2(\xi), \underline{\varphi}_3(\xi), \underline{\varphi}_4(\xi))$ is a lower solution of (4.4) for a small enough $\lambda > 0$. The proof is completed. □

As the direct consequence of Theorem 3.5, we have the following result.

Theorem 4.4. *If* τ_{ii}, $i = 1, 2, 3, 4$ *are small enough, then for* $c > \max\{c^*, 1\}$, *Eq.* (4.1) *admits a traveling wave solution* $\Phi(\xi) = (\varphi_1(\xi), \varphi_2(\xi), \varphi_3(\xi), \varphi_4(\xi))$ *satisfying* $\Phi(-\infty) = (0, 0, 0, 0)$ *and* $\Phi(+\infty) = (1, 1, 1, 1)$.

Acknowledgments

We are very grateful to the anonymous referees for careful reading and helpful suggestions which led to an improvement of our original manuscript. Research is supported by the Fundamental Research Funds for the Central Universities (No. 2014ZZ0064), the National Science Foundation of Guangdong Province (No. S2012040007959) and the National Natural Science Foundation of China (No. 11241002).

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