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BLOW-UP SOLUTIONS FOR A CASE OF *b*-FAMILY EQUATIONS*

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Abstract In this article, we study the blow-up solutions for a case of b-family equations. Using the qualitative theory of differential equations and the bifurcation method of dynamical systems, we obtain five types of blow-up solutions: the hyperbolic blow-up solution, the fractional blow-up solution, the trigonometric blow-up solution, the first elliptic blow-up solution, and the second elliptic blow-up solution. Not only are the expressions of these blow-up solutions given, but also their relationships are discovered. In particular, it is found that two bounded solitary solutions are bifurcated from an elliptic blow-up solution.

Key words b-family equation; blow-up solutions; qualitative theory; bifurcation method

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1 Introduction

We consider the equation

$$u_t - u_{xxt} + u^2 u_x - u u_{xxx} = 0, \tag{1.1}$$

which is the case of $b = 0$ and $n = 2$ in the b-family equation

$$u_t - u_{xxt} + (b+1)u^n u_x - b u_x u_{xx} - u u_{xxx} = 0. \tag{1.2}$$

When $b = 2$ and $n = 1$, eq. (1.2) reduces to the famous CH equation established by Camassa and Holm [1]. When $b = 3$ and $n = 1$, eq. (1.2) changes to the DP equation built by Degasperis and Procesi [2]. Many authors have worked on these equations (for instance, Constantin and Strauss [3], Johnson [4], Lundmark and Szmigielski [5], Wu and Yin [6], Escher and Kolev [7], Zhang and Liu [8]), and because they have rich structures and properties, many pioneers have also been interested in more generalized forms of eq. (1.2). For when $n = 2$ and $b = 2$ or 3, for example, some authors have studied traveling waves as follows: Tian and Song [9] gave a physical explanation and got some peakons; Shen and Xu [10] investigated the existence of some traveling waves. For when the wave speed equals 1, Khuri [11] got a

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singular wave solution composed of triangular functions. For when the wave speed equals 1 or 2, Wazwaz [12] gave eleven explicit traveling wave solutions consisting of triangular functions or hyperbolic functions. He et al. [13] employed the bifurcation theory to get some solutions. Liu and Tang [14] investigated the bifurcations of periodic wave solutions. Liu and Liang [15] studied some nonlinear waves and their bifurcations. Liu [16, 17] studied the coexistence of multifarious exact solutions. For the case of when $b \neq 0, -1, -2$, Chen et al. [18] studied the periodic waves and their limit forms. For when $b > 1$, Yang et al. [19] studied the existence and bifurcation of peakons for eq. (1.2) with any arbitrary positive integer n. Li and Liu [20] studied the bifurcation and bounded exact solutions for eq. (1.1).

In view of the work of these pioneers, we see that the blow-up solutions of eq. (1.1) have been rarely investigated. Therefore, following the work in [20], we would like to study the blow-up solutions and the bifurcation of eq. (1.1) in this article. Many other authors have also studied the traveling wave solutions, such as Yang and Zhang [21], Zhang et al. [22].

The rest part of the paper is organized as follows: in Section 2, our main results are stated. In Section 3, the derivations of the main results are presented. A short conclusion is given in Section 4.

2 Main Results

In this section, we state our main results. For a given parameter g, we will utilize the following notations:

$$\sigma_1 = \left(3g - \sqrt{9g^2-4}\right)^{\frac{1}{3}}, \quad \text{when} \quad g > \frac{2}{3}, \tag{2.1}$$

$$\sigma_2 = \frac{\sqrt{4-9g^2}}{-3g}, \qquad \text{when} \qquad |g| \le \frac{2}{3}, \tag{2.2}$$

$$\sigma_3 = \left(-3g + \sqrt{9g^2-4}\right)^{\frac{1}{3}}, \quad \text{when} \quad g < -\frac{2}{3}, \tag{2.3}$$

$$\alpha = \begin{cases} 2\cos\left[\frac{1}{3}(\pi + \arctan\sigma_2)\right], & \text{when} \ \ 0 < g \le \frac{2}{3}, \\ \sqrt{3}, & \text{when} \ \ g = 0, \\ 2\cos\left[\frac{1}{3}\arctan\sigma_2\right], & \text{when} \ \ -\frac{2}{3} \le g < 0, \\ \frac{\sqrt[3]{2}}{\sigma_3} + \frac{\sigma_3}{\sqrt[3]{2}}, & \text{when} \ \ g < -\frac{2}{3}. \end{cases} \tag{2.4}$$

$$\beta = \frac{1}{2}\left(-\alpha + \sqrt{3(4-\alpha^2)}\right), \quad \text{when} \quad |g| < \frac{2}{3}, \tag{2.5}$$

$$\gamma = \begin{cases} \frac{-\sqrt[3]{2}}{\sigma_1} - \frac{\sigma_1}{\sqrt[3]{2}}, & \text{when} \ \ g > \frac{2}{3}, \\ \frac{1}{2}\left(-\alpha - \sqrt{3(4-\alpha^2)}\right), & \text{when} \ \ -|g| \le \frac{2}{3}, \end{cases} \tag{2.6}$$

$$\alpha_* = -2\alpha - 3; \tag{2.7}$$

$$\beta_* = -2\beta - 3; \tag{2.8}$$

and

$$\xi = x - t. \tag{2.9}$$

Remark 2.1 From above notations, we have the following facts:

(1) When $g > \frac{2}{3}$, γ is real, and α, β are complex;

(2) When $g = \frac{2}{3}$, $\alpha = \beta = 1$, $\gamma = -2$;

(3) When $-\frac{2}{3} < g < \frac{2}{3}$, α, β, γ, α_*, β_* are real and

$$\alpha_* < \beta_* < \gamma < \beta < \alpha; \tag{2.10}$$

(4) When $g = -\frac{2}{3}$, $\alpha = 2$, $\beta = \gamma = -1$;

(5) When $g < -\frac{2}{3}$, α is real, β, γ are complex.

Using above notations, we have the following proposition:

Proposition Eq.(1.1) has the following blow-up solutions:

(1) Hyperbolic blow-up solutions

If we let

$$u_\alpha(\xi, g) = \alpha + 3(\alpha + 1)\,\mathrm{csch}^2\left(\frac{\sqrt{\alpha+1}}{2}\,\xi\right), \tag{2.11}$$

and

$$u_\beta(\xi, g) = \beta + 3(\beta + 1)\,\mathrm{csch}^2\left(\frac{\sqrt{\beta+1}}{2}\,\xi\right), \tag{2.12}$$

then as g varies, eq. (1.1) has the following hyperbolic blow-up solutions:

$(1)_\mathrm{a}$ When $g = \frac{2}{3}$, there exists a hyperbolic blow-up solution

$$u_1(\xi) = 1 + 6\,\mathrm{csch}^2\left(\frac{\sqrt{2}}{2}\,\xi\right). \tag{2.13}$$

$(1)_\mathrm{b}$ When $|g| < \frac{2}{3}$, there exist two hyperbolic blow-up solutions: $u_\alpha(\xi, g)$ and $u_\beta(\xi, g)$.

$(1)_\mathrm{c}$ When $g = -\frac{2}{3}$, there exists a hyperbolic blow-up solution

$$u_2(\xi) = 2 + 9\,\mathrm{csch}^2\left(\frac{\sqrt{3}}{2}\,\xi\right). \tag{2.14}$$

$(1)_\mathrm{d}$ When $g \le -\frac{2}{3}$, there exists a hyperbolic blow-up solution $u_\alpha(\xi, g)$.

The solutions $u_\alpha(\xi, g)$ and $u_\beta(\xi, g)$ have the following limit properties:

$$\lim_{g\to\frac{2}{3}-0} u_\alpha(\xi, g) = \lim_{g\to\frac{2}{3}-0} u_\beta(\xi, g) = u_1(\xi) \quad \text{(see (2.13))}, \tag{2.15}$$

and

$$\lim_{g\to-\frac{2}{3}+0} u_\alpha(\xi, g) = \lim_{g\to-\frac{2}{3}+0} u_\beta(\xi, g) = u_2(\xi) \quad \text{(see (2.14))}. \tag{2.16}$$

(2) Fractional blow-up solution

When $g = -\frac{2}{3}$, eq. (1.1) has the fractional blow-up solution

$$u_3(\xi) = \frac{12 - \xi^2}{\xi^2}. \tag{2.17}$$

(3) Trigonometric blow-up solution

When $g > -\frac{2}{3}$, eq. (1.1) has the trigonometric blow-up solution

$$u_\gamma(\xi, g) = \gamma - \frac{3(1+\gamma)}{\sin^2\left(\frac{\sqrt{|1+\gamma|}}{2}\ \xi\right)}, \tag{2.18}$$

which possesses the limit

$$\lim_{g \to -\frac{2}{3}+0} u_\gamma(\xi, g) = u_3(\xi) \quad \text{(see (2.17))}. \tag{2.19}$$

This indicates that when $g > -\frac{2}{3}$ and tends to $-\frac{2}{3}$, the trigonometric blow-up solution reduces to the fractional blow-up solution.

(4) The first elliptic blow-up solutions

As g varies, let

$$I_1 = \begin{cases} (-\infty, \gamma), & \text{when} \quad g > \dfrac{2}{3}, \\ (-\infty, \gamma) \quad \text{and } c \neq -5, & \text{when} \quad g = \dfrac{2}{3}, \\ (-\infty, \alpha_*) \ \text{ or } (\beta_*, \gamma), & \text{when} \quad |g| < \dfrac{2}{3}, \\ (-\infty, \alpha_*), & \text{when} \quad g \leq -\dfrac{2}{3}. \end{cases} \tag{2.20}$$

When $c \in I_1$, there exists the first elliptic blow-up solution

$$u_c(\xi, g) = c + \frac{a-c}{\mathrm{sn}^2\left(\eta_c \xi, k_c\right)}, \tag{2.21}$$

where

$$a = \frac{1}{2\sqrt{c-1}}\left((-3-c)\sqrt{c-1} + \sqrt{-9+15c+9c^2+c^3-24g}\right), \tag{2.22}$$

$$b = \frac{1}{2\sqrt{c-1}}\left(3-2c-c^2-\sqrt{c-1}\sqrt{-9+15c+9c^2+c^3-24g}\right), \tag{2.23}$$

$$k_c = \frac{b-c}{a-c}, \tag{2.24}$$

and

$$\eta_c = \frac{\sqrt{a-c}}{12}. \tag{2.25}$$

The solution $u_c(\xi, g)$ has the following limit properties:

$(4)_\mathrm{a}$ When $g > -\frac{2}{3}$ and c closes to γ, it follows that

$$\lim_{c \to \gamma - 0} u_c(\xi, g) = u_\gamma(\xi, g) \quad \text{(see (2.18))}, \tag{2.26}$$

and

$$\lim_{c \to -\frac{2}{3}} u_c(\xi, g) = u_3(\xi) \quad \text{(see (2.17))}. \tag{2.27}$$

$(4)_\mathrm{b}$ When $g = \frac{2}{3}$ and $c \neq -5$, we have that

$$\lim_{c \to -5} u_c(\xi, g) = u_1(\xi) \quad \text{(see (2.13))}. \tag{2.28}$$

$(4)_\mathrm{c}$ When $-\frac{2}{3} < g < \frac{2}{3}$ and $c \in (-\infty, \alpha_*)$, it follows that

$$\lim_{c \to \alpha_* - 0} u_c(\xi, g) = u_\alpha(\xi, g) \quad \text{(see (2.11))}. \tag{2.29}$$

$(4)_{\rm d}$ When $-\frac{2}{3}<g<\frac{2}{3}$ and $c\in(\beta_*,\gamma)$, we have that

$$\lim_{c\to\beta_*+0} u_c(\xi,g)=u_\beta(\xi,g) \quad (\text{see} \quad (2.12)). \tag{2.30}$$

$(4)_{\rm e}$ When $g\le -\frac{2}{3}$ and $c\in(-\infty,\alpha_*)$, we have the limit

$$\lim_{c\to\alpha_*-0} u_c(\xi,g)=u_\alpha(\xi,g) \quad (\text{see} \quad (2.11)). \tag{2.31}$$

(5) The second elliptic blow-up solutions

As g varies, denote I_2 as

$$I_2=\begin{cases}(\alpha_*,\beta_*), & \text{when} \quad -\dfrac{2}{3}<g<\dfrac{2}{3},\\ (-7,-1) & \text{when } g=-\dfrac{2}{3},\\ (\alpha_*,1), & \text{when} \quad g\le-\dfrac{2}{3}.\end{cases} \tag{2.32}$$

When $d\in I_2$, there exists the second elliptic blow-up solution

$$u_d(\xi,g)=\frac{A_d+d+(d-A_d)\,\mathrm{cn}\left(\sqrt{\frac{A_d}{3}}\,(x-t),k_d\right)}{1+\mathrm{cn}\left(\sqrt{\frac{A_d}{3}}\,(x-t),k_d\right)}, \tag{2.33}$$

where

$$A_d=\sqrt{(d-d_0)(d-d_1)}, \tag{2.34}$$

$$k_d=\frac{2A_d+d_0+d_1-2d}{4A_d}, \tag{2.35}$$

$$d_1=\frac{1}{2}\left(-3-d+\sqrt{45-6d-3d^2-4h_d}\right), \tag{2.36}$$

$$d_0=\frac{1}{2}\left(-3-d-\sqrt{45-6d-3d^2-4h_d}\right), \tag{2.37}$$

and

$$h_d=\frac{(d-1)^3+6(d-1)^2+4-6g}{1-d}. \tag{2.38}$$

The solution $u_d(\xi,g)$ possesses the following limit properties:

$(5)_{\rm a}$ When $-\frac{2}{3}<g<\frac{2}{3}$, it follows that

$$\lim_{d\to\alpha_*+0} u_d(\xi,g)=u^*_\alpha(\xi,g) \tag{2.39}$$

and

$$\lim_{d\to\beta_*-0} u_d(\xi,g)=u^*_\beta(\xi,g), \tag{2.40}$$

where

$$u^*_\alpha(\xi,g)=\alpha-3(1+\alpha)\,\mathrm{sech}^2\frac{\sqrt{1+\alpha}}{2}\,\xi \tag{2.41}$$

and

$$u^*_\beta(\xi,g)=\beta-3(1+\beta)\,\mathrm{sech}^2\frac{\sqrt{1+\beta}}{2}\,\xi. \tag{2.42}$$

Note that $u^*_\alpha(\xi,g)$ and $u^*_\beta(\xi,g)$ are two bounded solitary solutions of eq. (1.1). Hence, the two limits in (2.39) and (2.40) explain that two bounded solitary solutions can be bifurcated from an elliptic blow-up solution.

(5)$_\mathrm{b}$ When $g \to \frac{2}{3} - 0$, we have

$$\lim_{d \to \frac{2}{3} - 0} u_d(\xi, g) = u_1^*(\xi), \tag{2.43}$$

where

$$u_1^*(\xi) = 1 - 6\ \mathrm{sech}^2 \frac{\sqrt{2}\xi}{2}. \tag{2.44}$$

(5)$_\mathrm{c}$ When $g = -\frac{2}{3}$, we have

$$\lim_{d \to -7+0} u_d(\xi, g) = u_2^*(\xi) \tag{2.45}$$

and

$$\lim_{d \to -1-0} u_d(\xi, g) = -1, \tag{2.46}$$

where

$$u_2^*(\xi) = 2 - 9\ \mathrm{sech}^2 \frac{\sqrt{3}}{2}\ \xi. \tag{2.47}$$

Note that $u_1^*(\xi)$ and $u_2^*(\xi)$ are bounded solitary solutions of eq. (1.1). This states that two bounded solitary solutions are bifurcated from an elliptic blow-up solution.

3 Derivations for the Proposition

In this section, we give derivations for the proposition.

Letting

$$\xi = x - t, \tag{3.1}$$

and substituting $u = \varphi(\xi)$ into eq. (1.5), it follows that

$$-\varphi' + \varphi''' + \varphi^2\varphi' - \varphi\varphi''' = 0. \tag{3.2}$$

Integrating (3.2) for once, we have

$$\varphi''(\varphi - 1) = f(\varphi) + \frac{1}{2}(\varphi')^2, \tag{3.3}$$

where

$$f(\varphi) = g - \varphi + \frac{1}{3}(\varphi)^3, \tag{3.4}$$

and g is an integral constant.

When $\varphi \neq 1$, via (3.3), we have

$$\varphi'' = \frac{f(\varphi) + \frac{1}{2}(\varphi')^2}{\varphi - 1}, \tag{3.5}$$

which yields the planar system

$$\begin{cases} \dfrac{\mathrm{d}\varphi}{\mathrm{d}\xi} = y, \\ \dfrac{\mathrm{d}y}{\mathrm{d}\xi} = \dfrac{f(\varphi) + \frac{1}{2}(\varphi')^2}{\varphi - 1}. \end{cases} \tag{3.6}$$

Under the transformation

$$\mathrm{d}\tau = \frac{\mathrm{d}\xi}{\varphi - 1}, \tag{3.7}$$

system (3.5) becomes

$$\begin{cases} \dfrac{\mathrm{d}\varphi}{\mathrm{d}\tau} = y(\varphi - 1), \\ \dfrac{\mathrm{d}y}{\mathrm{d}\tau} = f(\varphi) + \dfrac{1}{2}y^2. \end{cases} \tag{3.8}$$

Note that both systems (3.6) and (3.8) own the same first integration:

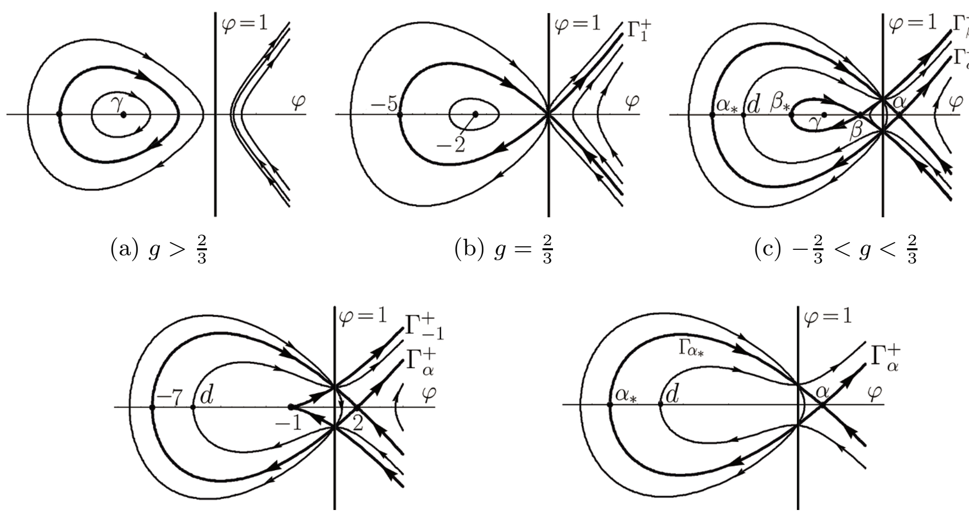
$$H(\varphi, y) = h, \tag{3.9}$$

where h is another integral constant and

$$H(\varphi, y) = 3(\varphi - 1)^{-1}y^2 - (\varphi - 1)^2 - 6(\varphi - 1) + (6g - 4)(\varphi - 1)^{-1}. \tag{3.10}$$

Therefore these two systems have the same topological phase portraits except for the line $\varphi = 1$. This implies that from the phase portrait of system (3.8) we can understand that of system (3.6). There are a number of other interesting facts that we can identify regarding (3.6) and (3.8) as well. For instance, on the right side of the line $\varphi = 1$, the direction of the vector fields defined by (3.6) and (3.8) are identical, both are clockwise. On the left side of the line $\varphi = 1$, the directions of the two vector fields are inverse, that is, the direction of the vector fields defined by (3.6) is clockwise, and the direction of the vector fields defined by (3.8) is anti-clockwise.

Finally, according to the qualitative theory of differential equations and the above information, we obtain the bifurcation phase portraits of system (3.6) as displayed in Fig.1.



(a) $g > \frac{2}{3}$ (b) $g = \frac{2}{3}$ (c) $-\frac{2}{3} < g < \frac{2}{3}$

(d) $g = -\frac{2}{3}$ (e) $g < -\frac{2}{3}$

Fig.1 The bifurcation phase portraits of system (3.6)

From (3.9) and (3.10), we have

$$y^2 = \frac{1}{3}\left((\varphi - 1)^3 + 6(\varphi - 1)^2 + h(\varphi - 1) + 4 - 6g\right). \tag{3.11}$$

Using some orbits in Fig.1 (a)–(e) and (3.11), we derive the blow-up solutions as follows:

(1) The derivations of hyperbolic blow-up solutions

$(1)_{\rm a}$ When $g=\frac{2}{3}$, from Fig.1 (b) we see that there exist two open orbits connecting a homoclinic orbit at $(1,0)$. Let Γ_1^+ denote the upward branch of the two open orbits, and note that the homoclinic orbit passes through $(-5,0)$. Therefore, on the $\varphi-y$ plane, Γ_1^+ has the expression

$$y=\frac{1}{\sqrt{3}}(\varphi-1)\sqrt{\varphi+5},\quad \text{where}\quad -5<\varphi<\infty. \tag{3.12}$$

Substituting the expression of Γ_1^+ into $\frac{{\rm d}\varphi}{y}={\rm d}\xi$ and integrating along Γ_1^+, we get

$$\int_\varphi^\infty \frac{1}{(s-1)\sqrt{s+5}}{\rm d}s=\frac{1}{\sqrt{3}}\int_\xi^0 {\rm d}s. \tag{3.13}$$

Completing the integrations in (3.13) and solving the equation for φ, and noting that $u=\varphi$, we obtain the hyperbolic blow-up solution $u_1(\xi)$ as (2.13).

$(1)_{\rm b}$ When $-\frac{2}{3}<g<\frac{2}{3}$, from Fig.1 (c) we see that $(\alpha,0)$ and $(\beta,0)$ are two saddles.

Denote Γ_α^+ as the upward branch of the two open orbits connecting with $(\alpha,0)$, and Γ_β^+ as the upward branch of the two open orbits connecting with $(\beta,0)$. Substituting $h=H(\alpha,0)$ and $h=H(\beta,0)$ into (3.11), on the $\varphi-y$ plane, we obtain the expression of the open orbits Γ_α^+ and Γ_β^+ as

$$\Gamma_\alpha^+:\quad y=\frac{1}{\sqrt{3}}(\varphi-\alpha)\sqrt{\varphi-\alpha_*},\quad \text{where}\quad \alpha<\varphi<\infty, \tag{3.14}$$

and

$$\Gamma_\beta^+:\quad y=\frac{1}{\sqrt{3}}(\varphi-\beta)\sqrt{\varphi-\beta_*},\quad \text{where}\quad \beta<\varphi<\infty. \tag{3.15}$$

Substituting the expressions of Γ_α^+ and Γ_β^+ into $\frac{{\rm d}\varphi}{y}={\rm d}\xi$ and integrating them along Γ_α^+ and Γ_β^+, respectively, we get

$$\int_\varphi^\infty \frac{1}{(s-\alpha)\sqrt{s-\alpha_*}}{\rm d}s=\frac{1}{\sqrt{3}}\int_\xi^0 {\rm d}s \tag{3.16}$$

and

$$\int_\varphi^\infty \frac{1}{(s-\beta)\sqrt{s-\beta_*}}{\rm d}s=\frac{1}{\sqrt{3}}\int_\xi^0 {\rm d}s. \tag{3.17}$$

Completing these integrations in (3.16)–(3.17), solving the equations for φ, and noting that $u=\varphi$, we obtain hyperbolic blow-up solutions $u_\alpha(\xi,g)$ and $u_\beta(\xi,g)$ as (2.11) and (2.12).

$(1)_{\rm c}$ When $g=-\frac{2}{3}$, from Fig.1 (d) we see that there are two open orbits connecting the saddle $(2,0)$ and located on the right side of this saddle. Denoting Γ_2^+ as the upward branch of the two open orbits, and substituting $h=H(2,0)$ into (3.11), we get the expression of Γ_2^+ as

$$\Gamma_2^+:\quad y=\frac{1}{\sqrt{3}}(\varphi+7)\sqrt{\varphi-2},\quad \text{where}\quad 2<\varphi<\infty. \tag{3.18}$$

In a fashion similar to the derivation of $u_1(\xi)$, using (3.18), we obtain $u_2(\xi)$ as (2.14).

$(1)_{\rm d}$ When $g\le-\frac{2}{3}$, from Fig.1 (e) we see that two open orbits connect the saddle $(\alpha,0)$ and are located on the right side of this saddle. The upward branch of the two open orbits is of the same expression as (3.14). Consequently, we obtain the same solution $u_\alpha(\xi,g)$ as (2.11).

Utilizing the expressions (2.11) and (2.12), we have the limits of $u_\alpha(\xi,g)$ and $u_\beta(\xi,g)$ as (2.15) and (2.16).

(2) The derivations of the Fractional blow-up solution

When $g=-\frac{2}{3}$, via Fig.1 (d) we see that there are two open orbits connecting the degenerative singular point $(-1,0)$ and located on the right side of this point. The upward branch of the two open orbits is expressed as

$$y=\frac{1}{\sqrt{3}}(\varphi+1)^{\frac{3}{2}},\quad \text{where}\quad -1<\varphi<\infty. \tag{3.19}$$

Using (3.19) and similarly to the derivation of $u_1(\xi)$, we obtain $u_3(\xi)$ as (2.17).

(3) The derivations of the trigonometric blow-up solution

When $g>-\frac{2}{3}$, from Fig.1 (a)–(c), it is seen that $(\gamma,0)$ is a center. Substituting $h=H(\gamma,0)$ into (3.11), it follows that

$$y^2=\frac{1}{3}(\varphi+2\gamma+3)(\varphi-\gamma)^2. \tag{3.20}$$

Note that when $g>-\frac{2}{3}$, we have $\gamma<-2\gamma-3$. Therefore, when $-(2\gamma+3)\le\varphi<\infty$, in (3.20) the expression is that of an open orbit passing the point $(-2\gamma-3,0)$, and the open orbit is located on the right side of point $(-2\gamma-3,0)$. Thus, using (3.20), and similarly to the derivation of $u_1(\xi)$, we obtain $u_\gamma(\xi,g)$ as (2.18). From (2.18), we get the limit of $u_\gamma(\xi,g)$ as (2.19).

(4) The derivations of the first elliptic blow-up solutions

As g varies, let I_1 be the interval given in (2.20). When $c\in I_1$, substituting $h=H(c,0)$ into (3.11), it follows that

$$y^2=\frac{1}{3}(\varphi-a)(\varphi-b)(\varphi-c), \tag{3.21}$$

where a and b are given in (2.22) and (2.23). It is easy to see that a, b and c satisfy the inequality

$$c<b<a. \tag{3.22}$$

Note that when $a\le\varphi<\infty$, the expression in (3.20) is that of an open orbit passing the point $(a,0)$, and the open orbit is located on the right side of point $(a,0)$. Thus, similarly to the derivation of $u_1(\xi)$, we have

$$\int_\varphi^\infty\frac{1}{\sqrt{(s-a)(s-b)(s-c)}}\mathrm{d}s=\frac{1}{\sqrt{3}}\int_\xi^0\mathrm{d}s. \tag{3.23}$$

Completing the integrations in (3.23), solving the equations for φ, and noting that $u=\varphi$, we obtain the first elliptic blow-up solution $u_c(\xi,g)$ as (2.21). Via the expression of $u_c(\xi,g)$ we get its six limits as (2.26)–(2.31).

(5) The derivations of the second elliptic blow-up solutions

As g varies, let I_2 be the interval given in (2.32). When $d\in I_2$, substituting $h=H(d,0)$ into (3.11), it follows that

$$y^2=\frac{1}{3}(\varphi-d)((\varphi-b_1)^2+a_1^2), \tag{3.24}$$

where

$$a_1^2=-\frac{(d_1-d_0)^2}{4}, \tag{3.25}$$

$$b_1=\frac{d_1+d_0}{2}, \tag{3.26}$$

and d_1, d_0 are given in (2.36) and (2.37).

Note that when $d \leq \varphi < \infty$, the expression in (3.24) is that of an open orbit passing the point $(d, 0)$, and the open orbit is located on the right side of the point $(d, 0)$.

Hence, in a fashion similar to the derivation of $u_1(\xi)$, we have

$$\int_d^{\infty} \frac{1}{\sqrt{(s-d)((s-b_1)+a_1^2)}} \mathrm{d}s = \frac{1}{\sqrt{3}} \int_0^{\xi} \mathrm{d}s. \tag{3.27}$$

Completing the integrations in (3.27), solving the equations for φ, and noting that $u = \varphi$, we obtain the second elliptic blow-up solution $u_d(\xi, g)$ as (2.33). Via the expression of $u_d(\xi, g)$ we get its six limits as (2.39)–(2.40) and (2.43)–(2.46). Thus, we have completed the derivations of the Proposition.

4 Conclusion

In this article, we have studied the blow-up solutions of eq. (1.1). We have obtained five types of blow-up solutions: the hyperbolic blow-up solution (see (2.11)–(2.14)), the fractional blow-up solution (see (2.17)), the trigonometric blow-up solution (see (2.18)), the first elliptic blow-up solution (see (2.21)), and the second elliptic blow-up solution (see (2.33)).

Furthermore, we have discovered the following relationships between these solutions:

(i) From (2.17)–(2.19) it is seen that a fractional blow-up solution can be bifurcated from a trigonometric blow-up solution, while (2.17), (2.21) and (2.27) show that a fractional blow-up solution also can be bifurcated from an elliptic blow-up solution.

(ii) From (2.18), (2.21) and (2.26) it is shown that a trigonometric blow-up solution can be bifurcated from an elliptic blow-up solution.

(iii) Via (2.21) and (2.28)–(2.31), we see that a hyperbolic blow-up solution can be bifurcated from an elliptic blow-up solution.

(iv) From (2.33), (2.39)–(2.42), (2.43)–(2.45) and (2.47), it is explained that two bounded solitary solutions can be bifurcated from an elliptic blow-up solution.

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